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STRATIFIED SYSTEMS OF LOGIC

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The suffixes used in logic to indicate differences of type may be regarded either as belonging to the formalism itself, or as being part of the machinery for deciding which rows of symbols (without suffixes) are to be admitted as significant. The two different attitudes do not necessarily lead to different formalisms, but when types are regarded as only one way of regulating the calculus it is natural to consider other possible ways, in particular the direct characterization of the significant formulae. Direct criteria for stratification were given by Quine, in his 'New Foundations for Mathematical Logic'(7). In the corresponding typed form of this theory ordinary integers are adequate as type-suffixes, and the direct description is correspondingly simple, but in other theories, including that recently proposed by Church(4), a partially ordered set of types must be used. In the present paper criteria, equivalent to the existence of a correct typing, are given for a general class of formalisms, which includes Church's system, several systems proposed by Quine, and (with some slight modifications, given in the last paragraph) *Principia Mathematica*. (The discussion has been given this general form rather with a view to clarity than to comprehensiveness.)

The effect of stratification on the rules of procedure is not discussed in this paper, except in so far as all formulae occurring are required to be stratified;† and the question of possible relaxations of the stratification conditions is therefore also not considered. The object is rather, by showing how existing type-systems could be axiomatically treated, to provide a convenient machinery for such generalizations.

1. Stratification can be defined for any kind of 'scheme' in which a finite number of identifiable places are filled by letters (e.g. formulae, matrices, sets of equations), provided that for every scheme, $\mathscr{S}$, of the system the incidence of two relations between letters, '$\mathbf{X}\,\eta\,\mathbf{Y}$ in $\mathscr{S}$' and '$\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$', is determined in such a way that

(A 1) η is symmetrical,

(A 2) η and γ are preserved under any homomorphic change of letters.‡

(The conditions are satisfied, e.g., if the schemes are rows of letters, and '$\mathbf{X}\,\eta\,\mathbf{Y}$ in $\mathscr{S}$' means 'an X occurs next to a Y in $\mathscr{S}$', and '$\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$' means 'an X followed by a Y occurs in $\mathscr{S}$'; but (A 2) is not satisfied if '$\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$' means 'every X in $\mathscr{S}$ is followed by a Y'.)

† On this point cf. *P.M.* vol. 3, p. 75 and *256·66; and Quine(10), p. 136.

‡ 'Letter' always means 'kind of letter', not a particular occurrence.

A change of letters is *homomorphic* if places in $\mathscr{S}$ which contained the same letter before the change continue to do so after it; it is *isomorphic* if it and its inverse are homomorphic. A relation $\mathbf{X}\,\mathrm{R}\,\mathbf{Y}$ is 'preserved under homomorphic changes of letters' if $\mathbf{X}\,\mathrm{R}\,\mathbf{Y}$ in $\mathscr{S}$ implies $\mathbf{X}'\,\mathrm{R}\,\mathbf{Y}'$ in $\mathscr{S}'$, where $\mathbf{X}'$ and $\mathbf{Y}'$ replace X and Y in any homomorphic change that turns $\mathscr{S}$ into $\mathscr{S}'$

An equivalence relation, 'level in $\mathscr{S}$', is derived from η as follows. If $\mathbf{X}\,\eta\,\mathbf{Y}$ in $\mathscr{S}$, and $\mathbf{X}$ and $\mathbf{Y}$ are different letters, it is an *η-reduction* to replace $\mathbf{Y}$ everywhere by $\mathbf{X}$. This may give rise to some new η-relations among the remaining letters, but since the number of different letters is diminished, repeated η-reductions lead finally to an η-irreducible scheme, $\mathscr{S}_f$. '$\mathbf{X}$ is level with $\mathbf{Y}$ in $\mathscr{S}$' shall mean that $\mathbf{X}$ and $\mathbf{Y}$ are replaced by the same letter in $\mathscr{S}_f$. This equivalence relation divides the letters of $\mathscr{S}$ into *level-classes*; that containing $\mathbf{X}$ is denoted by $\{\mathbf{X}\}$.

There may be some freedom of choice in the order of η-reductions, but *the level-classes are independent of the order of reduction.* For if not it must happen that, in the series of schemes $\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2, \ldots$, leading by η-reduction to an irreducible scheme $\mathscr{S}_f$, a scheme $\mathscr{S}_n$ is changed to $\mathscr{S}_{n+1}$ by a substitution of $\mathbf{U}$ for $\mathbf{V}$ that puts $\mathbf{U}$ into two places, p and q, filled by different letters in $\mathscr{S}'_f$, another irreducible form of $\mathscr{S}$. It may be assumed that $(\mathscr{S}_n, \mathscr{S}_{n+1})$ is the first such η-reduction in the series. But then $\mathscr{S}'_f$ is obtained from $\mathscr{S}_n$ by a homomorphic change of letters; and since $\mathbf{U}\,\eta\,\mathbf{V}$ in $\mathscr{S}_n$, $\mathbf{U}'\,\eta\,\mathbf{V}'$ in $\mathscr{S}'_f$, where $\mathbf{U}'$ and $\mathbf{V}'$ occupy the places p and q, and hence by hypothesis are different. Therefore $\mathscr{S}'_f$ is not η-irreducible, contrary to hypothesis.

It follows from this result that all η-irreducible forms of $\mathscr{S}$ are alphabetically isomorphic.

A relation Γ among level-classes is next determined by the rules

(Γ_1) if $\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$, $\{\mathbf{X}\}\,\Gamma\,\{\mathbf{Y}\}$ in $\mathscr{S}$;

(Γ_2) Γ is transitive.

The relation '$\{\mathbf{X}\}\,\Gamma\,\{\mathbf{Y}\}$ in $\mathscr{S}$' holds only if it is deducible from (Γ_1) and (Γ_2); and '$\mathbf{X}\,\Gamma\,\mathbf{Y}$ in $\mathscr{S}$' has the same meaning. Clearly a necessary and sufficient condition that $\mathbf{X}\,\Gamma\,\mathbf{Y}$ in $\mathscr{S}$ is that either $\mathbf{X}_f\,\gamma\,\mathbf{Y}_f$, or $\mathbf{X}_f\,\gamma\,\mathbf{X}_1\,\gamma\,\mathbf{X}_2\,\gamma \ldots \gamma\,\mathbf{X}_k\,\gamma\,\mathbf{Y}_f$ in $\mathscr{S}_f$ for some $\mathbf{X}_i$'s, where $\mathbf{X}_f$ and $\mathbf{Y}_f$ replace $\mathbf{X}$ and $\mathbf{Y}$ in $\mathscr{S}_f$.

Finally, the scheme $\mathscr{S}$ is *stratified* if $\mathbf{X}\,\Gamma\,\mathbf{X}$ holds for no $\mathbf{X}$, i.e. if Γ is a partial ordering of the letters in $\mathscr{S}$.†

2. The definitions of the preceding paragraph will be applied to logical formalisms, not directly, but through their 'defining equations', an expression which will now be explained.

In most symbolisms that are used in mathematical logic, the formulae are built up step by step from certain minimal formulae. A single step consists in placing a number of formulae already constructed in a row, say $\mathfrak{X}_1\,\mathfrak{X}_2 \ldots \mathfrak{X}_k$, and indicating, by adding an 'operational symbol' and appropriate brackets, which 'function' of them is wanted. (In some formalisms mere juxtaposition, or inclusion of one formula in a pair of brackets, is a method of construction.) If $\mathfrak{X}$ is the new formula, $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_k$ will be called its *factors*; and factors, factors of factors, and so on, are the *segments* of $\mathfrak{X}$. (A formula is not a segment of itself.) As an example, in Church's untyped 'λ-K-calculus' [Church (5)], if $\mathfrak{A}$ and $\mathfrak{B}$ denote formulae and $\mathfrak{x}$ a variable, we can construct the new formulae $\mathfrak{X} = (\mathfrak{A}\mathfrak{B})$ and $\mathfrak{Y} = (\lambda\mathfrak{x}\mathfrak{A})$. To give a complete specification of $\mathfrak{X}$ it would be necessary to give similar equations for the factors $\mathfrak{A}$ and $\mathfrak{B}$, say $\mathfrak{A} = (\lambda\mathfrak{y}\mathfrak{C})$, $\mathfrak{B} = (\mathfrak{C}\mathfrak{C})$, and so on, the process terminating when all the factors on the right-hand

† In the main application to be made, 'level' will correspond roughly to 'having the same type', 'Γ' to 'having higher type'.

side are minimal formulae. Any formula can be specified in this way by giving the defining equations leading back step by step to the minimal formulae of which it is composed. For example, if $\mathfrak{X}$ is the formula $(a(a)(b(ab)))$ of the λ-calculus, $\mathfrak{X}$ is seen to be determined by the equations

$$\mathfrak{X} = (a\mathfrak{Y}\mathfrak{Z}), \quad \mathfrak{Y} = (a), \quad \mathfrak{Z} = (b\mathfrak{U}), \quad \mathfrak{U} = (ab),$$

in all of which only single letters appear as 'factors' inside the brackets on the right-hand side.

It is such *sets of equations*, $\mathscr{E}$, that play the part of the schemes $\mathscr{S}$ in paragraph 1. The letters occurring in equations are therefore object letters, not syntactical names, in our discussion. The italic capitals U, V, W, X, Y, Z, with or without suffixes, will accordingly be used from now on for the letters in equations, German letters being reserved for the names of the formulae which, in certain instances, are the solutions of the equations. Clarendon capitals, $\mathbf{U}$, $\mathbf{V}$, ... are used as syntactical names for single letters in equations.

For the purposes of the general discussion it may be supposed that the 'operational' symbols are $\Phi, \Phi_1, \Phi_2, \ldots$, and are placed at the beginning of a formula, mere juxtaposition and bracketing being excluded (though in discussing particular formalisms the symbols in ordinary use will be retained). Our theory is therefore concerned with sets of formal equations such as $X = \Phi_1 YZ$, $Y = \Phi_2 Y$, called 'Φ-equations'. For the present the Φ-equations are restricted only by the condition that each Φ_i shall always be followed by the same number of letters, and a *Φ-system* is set up by specifying this number for a finite number of Φ_i's. No condition of 'solubility' is imposed on the sets of equations. (N.B. The symbols Φ_i themselves are not 'letters' in our theory.)

3. A preliminary application of the concepts of paragraph 1 leads immediately to a formal theory of 'equality' and 'solubility' for such sets of equations. Let '$\mathbf{X} e \mathbf{Y}$ in $\mathscr{E}$' mean that the set of equations $\mathscr{E}$ contains a pair of equations $\mathbf{X} = \Phi \mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_k$ and $\mathbf{Y} = \Phi \mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_k$, with identical right-hand sides. Then *equality* in $\mathscr{E}$ ('$\mathbf{X} = \mathbf{Y}$ in $\mathscr{E}$') is, by definition, the level-relation derived as in paragraph 1, on taking η to be e. The relation e has the required properties of symmetry and of being preserved under a homomorphic change of letters, and letters that are 'equal' according to this definition are in fact those which can be proved equal by means of the equations $\mathscr{E}$.†

We now take the relation '$\mathbf{X} \gamma \mathbf{Y}$ in $\mathscr{E}$' of paragraph 1 to be '$\mathscr{E}$ contains an equation $\mathbf{X} = \Phi \mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_k$, where $\mathbf{Y}$ is one of the letters $\mathbf{X}_i$'. This also is a relation preserved in homomorphic changes. The Γ-relation derived from it, in combination with e as 'η', will be denoted by $\succ$, ('$\mathbf{X} \succ \mathbf{Y}$' will also be written '$\mathbf{Y} \prec \mathbf{X}$'). An e-irreducible set of equations is *soluble* if (1) *it is stratified relative to* $\succ$, and (2) *no letter stands on the left of more than one equation*; and any set $\mathscr{E}$ is soluble if its e-reduced form is soluble. There must in this case be *minimal* letters for the partial ordering $\succ$,‡ and the system can in fact be 'solved' in terms of them for the remaining letters, by repeatedly substituting the right-hand sides of equations, enclosed in brackets, for their left-hand sides. If there is a single maximal letter, $\mathbf{X}$, $\mathscr{E}$ is called a *defining set* for $\mathbf{X}$.

A *formalism* is derived from a Φ-system by, first, specifying certain 'minimal formulae'

† '$\mathbf{X} = \mathbf{Y}$' is never used between equation-letters to mean '$\mathbf{X}$ and $\mathbf{Y}$ are the same letter'.

‡ Letters, $\mathbf{X}$, such that $\mathbf{X} \succ \mathbf{Y}$ in $\mathscr{E}$ for no $\mathbf{Y}$.

(which may be any symbols whatever), the substitution of which for the minimal letters in the solutions of defining sets gives the *formulae* of the system, and secondly, stating (possibly) other conditions to be satisfied by a formula, or its defining equations, in order that it may be a significant, or *well-formed*, formula of the system. If the minimal formulae are substituted for the minimal letters in the equations themselves of a soluble set, we obtain a *prepared set*, whose solutions are formulae of the system. The 'values' obtained for letters other than **X**, when a prepared defining set for **X** is solved, are the segments, as above defined, of the value of **X** itself; and the relation '$\prec$', carried over from letters **X** to their values, becomes 'is a segment of'.

4. The stratification proper of logical formulae is, of course, entirely distinct from the 'solubility' just defined. It depends on relations γ and η which must be specified separately for each Φ-system, and are subject to no restriction, in the first place, beyond those imposed in paragraph 1.

Example 1. In the system developed in Quine's 'New foundations for mathematical logic'(7), there are three kinds of equation,

$$\mathbf{X} = (\mathbf{Y} \in \mathbf{Z}), \quad \mathbf{X} = (\mathbf{U} \downarrow \mathbf{V}), \quad \mathbf{X} = (\mathbf{P})\,\mathbf{Q}.$$

For the stricter, 'classical', stratification proposed in (7) '$\mathbf{X}\gamma\mathbf{Y}$ in $\mathscr{E}$' must be defined to mean '$\mathscr{E}$ contains an equation $\mathbf{Z} = (\mathbf{Y} \in \mathbf{X})$'; and '$\mathbf{X}\eta\mathbf{Y}$' to mean '$\mathbf{X}e\mathbf{Y}$, or $\mathbf{X}\gamma\mathbf{U}$ and $\mathbf{Y}\gamma\mathbf{U}$ for some **U**, or $\mathbf{V}\gamma\mathbf{X}$ and $\mathbf{V}\gamma\mathbf{Y}$ for some **V**'.

The minimal formulae are small italic letters, and for a 'significant' formula, in addition to the condition of stratification, the equations must be such that those letters, and only those, are minimal that occur on the right of equations $\mathbf{X} = (\mathbf{Y} \in \mathbf{Z})$. Hence the test for stratification, applied directly to formulae occurring in the theory, is as follows. If *either* $\mathbf{x} \in \mathbf{u}$ and $\mathbf{y} \in \mathbf{u}$, *or* $\mathbf{v} \in \mathbf{x}$ and $\mathbf{v} \in \mathbf{y}$, occur in the formula, replace **y** throughout by **x**, and continue this process as long as possible (making also any possible e-reductions). If in the final result there is no cycle $\mathbf{x}_1 \in \mathbf{x}_2 \in \ldots \in \mathbf{x}_k \in \mathbf{x}_1$, the original formula is stratified.†

5. In order to bring stratification into correlation with types some restriction on the relations η and γ is necessary. A form of theory sufficiently general to cover most extant formalisms, including that of Church(4), will now be described. Some generalizations, enabling *Principia Mathematica* and some other systems to be brought within the scope of the theory, are briefly described at the end of the paper.

The assumptions made in the previous paragraph, about the existence and properties of the relations γ and η, are now replaced by the assumption that any equation $\mathbf{X}_0 = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$ determines a number of *positional relations*, $\mathbf{X}_m\gamma_i\mathbf{X}_n$, i.e. relations that hold if, and only if, certain places in the equation are filled by $\mathbf{X}_m$ and $\mathbf{X}_n$. There are to be a finite number, r, of the relations γ_i, and they are to satisfy

(A 3) if the relation $\mathbf{X}_m\gamma_i\mathbf{X}_n$ follows from $\mathbf{X}_0 = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$, then $\mathbf{X}_m\gamma_h\mathbf{X}_{n_h}$ also follows, for $h = 1, 2, \ldots, r$ and some $\mathbf{X}_{n_h}$ $(1 \leqslant n_h \leqslant k)$.

† A simpler test given by Quine in (7), p. 78, is not quite correct. See Bernays's review of the paper (1).

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A (Φ, γ)-*system* is set up by specifying, for a number of operators Φ_j, how many variables each takes, and what γ_i-relations follow from any equation $\mathbf{X} = \Phi_j \mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_k$; provided that the relations are positional and satisfy (A 3).

The relations η and γ are now defined in terms of the γ_i: '$\mathbf{X}\gamma\mathbf{Y}$ in $\mathscr{E}$' means that $\mathbf{X}\gamma_i\mathbf{Y}$ in $\mathscr{E}$ for some i, and '$\mathbf{X}\eta\mathbf{Y}$ in $\mathscr{E}$' that one of the following holds:

(η_1) $\mathbf{X}e\mathbf{Y}$;

(η_2) $\mathbf{U}\gamma_i\mathbf{X}$ and $\mathbf{U}\gamma_i\mathbf{Y}$ in $\mathscr{E}$ for some $\mathbf{U}$ and i;

(η_3) $\mathbf{X}\gamma_i\mathbf{U}_i$ and $\mathbf{Y}\gamma_i\mathbf{U}_i$ in $\mathscr{E}$ for $i = 1, 2, \ldots, r$, and some $\mathbf{U}_i$.

Condition (A 1) is clearly satisfied, and from the positional character of the γ_i it follows that η and γ satisfy (A 2). Hence the definitions and results of the previous paragraph may be carried over to any (Φ, γ)-system, and to any formalism derived from it. (Note that η is *not* a positional relation.)

Example 2. Returning to the formalism of Example 1, we see that there is only one γ_i-relation, γ, and that η is defined in accordance with (η_1), (η_2) and (η_3).

Example 3. In Church's λ-K-calculus [Church (5)] there are two kinds of defining equations, $\mathbf{Z} = (\mathbf{XY})$ and $\mathbf{Z} = (\lambda\mathbf{XY})$. The stratification introduced in Church (4) involves two γ_i-relations: the equation $\mathbf{Z} = (\mathbf{XY})$ gives $\mathbf{X}\gamma_1\mathbf{Z}$ and $\mathbf{X}\gamma_2\mathbf{Y}$, and $\mathbf{Z} = (\lambda\mathbf{XY})$ gives $\mathbf{Z}\gamma_1\mathbf{Y}$ and $\mathbf{Z}\gamma_2\mathbf{X}$. As an illustration consider first $(f(fx))$. Its equations are $X = (fx)$, $Y = (fX)$. By (η_2), $x\eta X$, whence, putting x for X, the two equations give $x = Y$. Putting x for Y we obtain two copies of the equation $x = (fx)$, an irreducible set with no γ-cycle: the original formula is stratified. Secondly, consider $(x(f(fx)))$. The defining equations are

$$X = (fx), \quad Y = (fX), \quad Z = (xY),$$

in which x, X, Y are again level. The η-reduced form of the last equation is $Z = (xx)$, giving $x\gamma_2 x$. The formula is therefore not stratified.

6. *Types* are the formulae of a system in which the minimal formulae are small Greek letters, and the only principle of construction is to enclose a row of given formulae in a pair of round brackets. Types in general will be denoted by heavy small Greek letters. A type is *primitive* if it consists of a single letter, not in brackets, and, in accordance with our previous notations, if $\boldsymbol{\alpha}$ is $(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 \ldots \boldsymbol{\alpha}_k)$, the $\boldsymbol{\alpha}_i$ are the *factors* of $\boldsymbol{\alpha}$.

A formula is *typed* by attaching types in a random way to it and its segments. 'Correct' typing is defined only if the formalism is derived from a (Φ, γ)-system, and should then mean that segments of a given formula are so matched with types that the relations $\succ$ for types and Γ for segments correspond. Since $\succ$ is a partial ordering of the types, this will ensure that Γ is a partial ordering of the segments, and hence that the formula is stratified. The object of using types, however, is to avoid the process of determining the level-classes. The criteria should therefore, while aiming at the above correlation, be expressed directly in terms of the relations γ_i.

We consider first the attaching of types to the letters, or rather to the places, in Φ-equations. It is convenient to regard a place to which no type is attached as having null type.

An *r-fold type* is defined inductively for positive integral r to be either a primitive type, or a type with r factors, each of which is an r-fold type.

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The following are the conditions to be satisfied for a *correct typing*:

(t_1) Each letter has the same type (or none) at each of its occurrences. When this condition is satisfied, we denote the type of $\mathbf{X}$ by $\tau(\mathbf{X})$, and write $\tau(\mathbf{X}) = 0$ if the type is null. If $\tau(\mathbf{X})$ is neither null nor primitive, $\tau_i(\mathbf{X})$ denotes its ith factor.

(t_2) $\tau(\mathbf{X})$ is either null or an r-fold type (where r is the number of γ_i-relations).

(t_3) If $\mathbf{X}\gamma_i\mathbf{Y}$, $\tau(\mathbf{Y})$ is $\tau_i(\mathbf{X})$.

(t_4) A letter satisfying no γ_i-relation has null type.

Condition (t_3) implies that, if $\mathbf{X}\gamma_i\mathbf{Y}$, neither letter has null type.

A *formula*, in a formalism derived from a (Φ, γ)-system, is correctly typed by assigning to its segments the types given to the corresponding letters in a correct typing (if such exists) of a prepared set of defining equations.

Example 4. In the formalism of Examples 1 and 2 a single primitive type suffices, and the other types are obtained by enclosing it in any number of pairs of brackets. We may therefore use the positive integers instead. The letters with non-null types are those that appear on the right of equations $\mathbf{X} = (\mathbf{Y} \in \mathbf{Z})$, and by (t_3) the type of $\mathbf{Z}$ is 1 higher than that of $\mathbf{Y}$.

In the typed λ-K-calculus (without logical constants, Example 3) our rules state that if $\mathbf{Z} = (\mathbf{XY})$, and $\mathbf{Z}$ and $\mathbf{Y}$ have types $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\mathbf{X}$ has the type $(\boldsymbol{\alpha\beta})$; if $\mathbf{Z} = (\lambda\mathbf{XY})$, and $\mathbf{X}$ and $\mathbf{Y}$ have the types $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\mathbf{Z}$ has the type $(\boldsymbol{\beta\alpha})$. These are the rules given by Church (4), save that there may now be any number of different primitive types.

7. Difficulties may arise in connexion with letters of the kind occurring in (t_4). Consider, as an illustration, the (Φ, γ)-system, $\mathbf{\Phi}$, with three kinds of equation:

$$\mathbf{X} = \Phi_1\mathbf{Y}, \quad \mathbf{X} = \Phi_2\mathbf{UV}, \quad \mathbf{W} = \Phi_3\mathbf{XYZ},$$

and a single relation γ_1. Let the first equation imply no γ_1-relation, the second $\mathbf{X}\gamma_1\mathbf{U}$ and $\mathbf{X}\gamma_1\mathbf{V}$, the third $\mathbf{Y}\gamma_1\mathbf{Z}$. In this system consider the equations

$$Y = \Phi_1 U, \quad Z = \Phi_1 V, \quad X = \Phi_2 UV, \quad W = \Phi_3 XYZ \tag{1}$$

(a defining set). The following typing satisfies (t_1) to (t_4):

$$Y_{(\beta)} = \Phi_1 U_\alpha, \quad Z_\beta = \Phi_1 V_\alpha, \quad X_{(\alpha)} = \Phi_2 U_\alpha V_\alpha, \quad W = \Phi_3 X_{(\alpha)} Y_{(\beta)} Z_\beta,$$

but the set of equations is not stratified. For the relations $X\gamma_1 U$ and $X\gamma_1 V$ give $U\eta V$. On putting U for V the first two equations give $Y = Z$, and this, combined with $Y\gamma_1 Z$ (from the last equation), gives a γ-cycle. Moreover, in the solution

$$W = \Phi_3((\Phi_2 UV)(\Phi_1 U)(\Phi_1 V))$$

of the equations, the types, (β) and β, of the last two factors depend not merely on their own structure, but on that of the first factor, a situation which seems to be at variance with the idea of a logical type.

Both these difficulties could be overcome by requiring, in place of (t_4), that any letter occurring on the left of an isolating equation in $\mathscr{E}$ shall have null type; where $\mathbf{X}_0 = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$ is an *isolating equation* if it implies no γ_i-relation involving $\mathbf{X}_0$. The letters Y and Z are then required to be untyped in (1), and no correct typing is possible. But apart from being rather unnatural, this stronger condition would merely

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displace the difficulty, for with the new rule the last three equations of (1), which form a stratified set, would be untypeable.

It seems that this is a point where the ideas of stratification and typing diverge. The system Φ admits a satisfactory definition of stratification, but not a satisfactory system of types, if all sets of equations are admitted. We shall therefore introduce a restriction which excludes such sets as (1).

8. A letter occurring in a set of equations, but satisfying no γ_i-relation, is said to be *isolated*. A set of equations is *non-singular* if all letters that appear on the left of isolating equations are isolated.

THEOREM 1. *The equation* $X = \Phi X_1 X_2 \ldots X_k$ *determines the type of* X *absolutely in non-singular sets, in terms of the types of* $X_1, X_2, \ldots, X_k$.

The meaning is that if the equation $Y = \Phi Y_1 Y_2 \ldots Y_k$ occurs in the same or another correctly typed non-singular set, and if† $\tau(X_i) = \tau(Y_i)$, then $\tau(X) = \tau(Y)$.

If the equation is an isolating one, both X and Y are isolated, and by (t_4) have null type. If not, either, for some i and h, $X_h \gamma_i X$ follows from the equation, and then $Y_h \gamma_i Y$, giving $\tau(X) = \tau_i(X_h) = \tau_i(Y_h) = \tau(Y)$; or, by (A 3), $X \gamma_i X_{n_i}$, for $i = 1, 2, \ldots, r$ and some n_i's; and then $Y \gamma_i Y_{n_i}$, and

$$\begin{aligned}\tau(X) &= (\tau(X_{n_1})\,\tau(X_{n_2}) \ldots \tau(X_{n_r}))\\ &= (\tau(Y_{n_1})\,\tau(Y_{n_2}) \ldots \tau(Y_{n_r})) = \tau(Y).\end{aligned}$$

THEOREM 2. *If* $\tau(X) = \tau(Y)$ *in a correct typing of any set* $\mathscr{E}$, *and if* $\mathscr{E}'$ *is the result of substituting* X *everywhere for* Y, *the same typing‡ of* $\mathscr{E}'$ *is correct.*

Condition (t_1) is satisfied since $\tau(X) = \tau(Y)$, (t_2) and (t_3) by their positional character. As regards (t_4): if a letter, Z, other than X is isolated in $\mathscr{E}'$ it is isolated in $\mathscr{E}$, and $\tau(Z) = 0$; and if X is isolated in $\mathscr{E}'$ it is *a fortiori* isolated in $\mathscr{E}$.

THEOREM 3. *A level-class in a non-singular set either consists entirely of isolated letters, or else contains none.*

The theorem is certainly true of an η-irreducible set, since each level-class then contains only one letter. Hence if it is false for a set $\mathscr{E}$, the paired letters in one of the η-reductions leading from $\mathscr{E}$ to $\mathscr{E}_f$ must be an isolated letter, X_0, and a non-isolated letter, Y_0, with $X_0 \eta Y_0$. This relation η cannot follow from (η_2) or (η_3), since X_0 satisfies no γ_i-relation. It therefore follows from (η_1), say

$$X_0 = \Phi X_1 X_2 \ldots X_k \quad \text{and} \quad Y_0 = \Phi X_1 X_2 \ldots X_k.$$

Since Y_0 is a non-isolated letter these are not isolating equations. Hence the first of them implies a γ_i-relation involving X_0, contrary to the assumption that X_0 is isolated.

In view of this result we may speak of 'isolated' and 'non-isolated level-classes' in non-singular sets of equations. All the members of an isolated level-class in $\mathscr{E}$ are equal in $\mathscr{E}$.

† The use of '=' between types to mean 'is identical with' can hardly be confused with the formal '=' between equation-letters.

‡ I.e. the same distribution of types among the *places* in $\mathscr{E}'$.

Corollary 1. *In a non-singular set the isolated level-classes are all lowest level-classes.*†

Corollary 2. *If $\mathscr{E}$ is non-singular, $\mathscr{E}_f$ is non-singular.* Let $X_0 = \Phi X_1 X_2 \ldots X_k$ be an isolating equation in $\mathscr{E}_f$. This corresponds to at least one equation $Y_0 = \Phi Y_1 Y_2 \ldots Y_k$ in $\mathscr{E}$, where X_i is $(Y_i)_f$. Since the Y-equation is also isolating Y_0 is isolated in $\mathscr{E}$, and therefore all members of $\{Y_0\}$ are isolated. But if X_0 satisfied a γ_i-relation in $\mathscr{E}_f$, some original of X_0 in $\mathscr{E}$, i.e. some member of $\{Y_0\}$, would satisfy a γ_i-relation in $\mathscr{E}$. Therefore X_0 is isolated in $\mathscr{E}_f$.

Theorem 4. *A necessary and sufficient condition that a non-singular set of equations admit a correct typing is that it be stratified.*

Necessary. Let the typing be τ. It will be shown that if an η-reduction turns $\mathscr{E}$ into $\mathscr{E}'$, the same typing of $\mathscr{E}'$ is correct. In view of Theorem 2 it is sufficient to show that if $X \eta Y$ in $\mathscr{E}$, $\tau(X) = \tau(Y)$:

(i) if $X \eta Y$ by (η_1), say $X = \Phi X_1 X_2 \ldots X_k$ and $Y = \Phi X_1 X_2 \ldots X_k$, $\tau(X) = \tau(Y)$ by Theorem 1;

(ii) if $X \eta Y$ by (η_2), say $U \gamma_i X$ and $U \gamma_i Y$, $\tau(X) = \tau_i(U) = \tau(Y)$ by (t_3);

(iii) if $X \eta Y$ by (η_3), say $X \gamma_i U_i$ and $Y \gamma_i U_i$ for $i = 1, 2, \ldots, r$,

$$\tau(X) = (\tau(U_1)\tau(U_2)\ldots\tau(U_r)) = \tau(Y) \quad \text{by } (t_2) \text{ and } (t_3).$$

Repeated application of this result shows that the η-irreducible form $\mathscr{E}_f$ of $\mathscr{E}$ is correctly typed by τ. If $\mathscr{E}_f$ contained a γ-cycle, $X_1 \gamma_{n_1} X_2 \ldots \gamma_{n_k} X_1$, it would follow that $\tau(X_1) \succ \tau(X_2) \succ \ldots \succ \tau(X_1)$, which is impossible, since '$\succ$' is a partial ordering.

Sufficient. We shall first assign types to the letters of $\mathscr{E}_f$ (a non-singular set, by Theorem 3, Corollary 2). Let null type be assigned to the isolated letters, and arbitrary r-fold types to the remaining lowest-level letters. Let it be assumed inductively that types have been assigned to all the letters of some lower section†, S, including all the lowest-level letters. If not all letters of $\mathscr{E}_f$ are in S there is at least one, X, not in S, whose γ_i-descendents are all in S, for every i. The addition of X to S gives a new lower section. By the condition (A 3), X satisfies $X \gamma_i Y_i$, for $i = 1, 2, \ldots, r$ and some letters Y_i. Since $\mathscr{E}_f$ is η-irreducible there is, by (η_2), only one Y_i for each i, and by the inductive hypothesis a type $\tau(Y_i)$ has already been assigned to Y_i. We assign to X the type $(\tau(Y_1)\tau(Y_2)\ldots\tau(Y_r))$. In this way every non-isolated letter of $\mathscr{E}_f$ receives a non-null type, and the conditions (t_1) to (t_4) are evidently all satisfied. We now assign to each letter, X, in $\mathscr{E}$ the type of X_f in $\mathscr{E}_f$. The conditions (t_1) and (t_2) are clearly satisfied. (t_3): if $X \gamma_i Y$ in $\mathscr{E}$, $X_f \gamma_i Y_f$ in $\mathscr{E}_f$, and therefore $\tau_i(X) = \tau(Y)$. (t_4): if X is isolated in $\mathscr{E}$, X_f is isolated in $\mathscr{E}_f$; for the originals of X_f all belong to $\{X\}$, and therefore satisfy no γ_i-relation in $\mathscr{E}$ (Theorem 3). Therefore $\tau(X) = \tau(X_f) = 0$.

Corollary 1. *In a correct typing of a non-singular set level letters have the same type.*

Corollary 2. *In a correct typing of a non-singular set the types of the non-isolated lowest level-classes can be chosen arbitrarily, and the other types are then uniquely determined.*

† The terminology is chosen as if 'Γ' were 'greater than' or 'higher than'. A *lowest-level* letter, X, is one such that $X \Gamma Y$ for no Y (the words 'minimal' and 'maximal' being reserved for the relation $\succ$). A *lower section* of the letters of $\mathscr{E}$ is a subset of them, S, such that if $X \in S$ and $X \Gamma Y$ then $Y \in S$.

COROLLARY 3. *A correct typing of $\mathscr{E}$ determines a correct typing of $\mathscr{E}_f$, and vice versa.*

THEOREM 5. *If τ^1 and τ^2 are correct typings of any two sets, $\mathscr{E}_1$ and $\mathscr{E}_2$, and $\tau^1(\mathbf{X}) = \tau^2(\mathbf{X})$ for all common letters, the combined typing is correct in $\mathscr{E}_1 \cup \mathscr{E}_2$.*†

Let the combined typing be denoted by τ. Conditions (t_1) and (t_2) are clearly satisfied. (t_3): if $\mathbf{X}\gamma_i\mathbf{Y}$ in $\mathscr{E}_1 \cup \mathscr{E}_2$ then $\mathbf{X}\gamma_i\mathbf{Y}$ in $\mathscr{E}_j$, for $j = 1$ or 2, and

$$\tau_i(\mathbf{X}) = \tau_i^j(\mathbf{X}) = \tau^j(\mathbf{Y}) = \tau(\mathbf{Y}).$$

(t_4): an isolated letter, $\mathbf{X}$, in $\mathscr{E}_1 \cup \mathscr{E}_2$ is isolated in both $\mathscr{E}_1$ and $\mathscr{E}_2$, and therefore $\tau(\mathbf{X}) = 0$.

THEOREM 6. *If in a correct typing, τ, of a non-singular set $\mathscr{E}$, all non-isolated lowest level-classes receive different primitive types, a sufficient condition that two non-isolated letters, $\mathbf{X}$ and $\mathbf{Y}$, be level is that $\tau(\mathbf{X}) = \tau(\mathbf{Y})$.*

Let $\tau(\mathbf{X}) = \tau(\mathbf{Y}) = \boldsymbol{\alpha} \neq 0$. We proceed by induction on the *height* of $\boldsymbol{\alpha}$, i.e. the maximum length of a series $\boldsymbol{\alpha} \succ \boldsymbol{\beta}^1 \succ \boldsymbol{\beta}^2 \succ \ldots \succ \boldsymbol{\beta}^q$. If $\boldsymbol{\alpha}$ is primitive, the assertion is true by hypothesis. Suppose, then, that $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1 \ldots \boldsymbol{\alpha}_r)$. Then $\mathbf{X}$ and $\mathbf{Y}$ are not lowest-level letters, and therefore, by (A 3), $\mathbf{X}\gamma_i\mathbf{X}_i$, $\mathbf{Y}\gamma_i\mathbf{Y}_i$, for $i = 1, 2, \ldots, r$; and by (t_3)

$$\tau(\mathbf{X}_i) = \boldsymbol{\alpha}_i = \tau(\mathbf{Y}_i).$$

Hence, by an inductive hypothesis, $\mathbf{X}_i$ is level with $\mathbf{Y}_i$. It follows that in $\mathscr{E}_f$, an η-irreducible form of $\mathscr{E}$, $(\mathbf{X}_i)_f$ is $(\mathbf{Y}_i)_f$; and hence, since $\mathbf{X}_f\gamma_i(\mathbf{X}_i)_f$ and $\mathbf{Y}_f\gamma_i(\mathbf{Y}_i)_f$, it follows that $\mathbf{X}_f\eta\mathbf{Y}_f$; i.e. since $\mathscr{E}_f$ is η-irreducible, $\mathbf{X}_f$ is $\mathbf{Y}_f$. Therefore $\mathbf{X}$ and $\mathbf{Y}$ are level.

9. *Restricted alphabets*. A feature of Church's Theory of Types [Church (4)], which corresponds well with the meaning attributed to types in logical formulae, is the requirement that certain symbols shall have types of a prescribed kind whenever they appear. In the following paragraphs it will be shown how, from a given (Φ, γ)-system, we may derive, first, a '(Φ, p)-system' in which stratification is defined in a modified sense, and secondly a '(Φ, t)-system', in which the modified typings are described; and it will be shown that there is a complete correspondence between the two kinds of system.

In the rest of the paper it is assumed that, in the (Φ, γ)-systems considered, *there are no isolating equations.*

We suppose the letters occurring in equations divided into a number of alphabets, one of which consists of the italic capitals, U, V, ..., so far used, now called the X-*alphabet*. The other alphabets, called A-alphabets, will‡ consist of the letters A_1, A_2, ..., B_1, B_2, ..., etc., the letter denoting the alphabet, and the suffixes the different members of it. Each alphabet contains either an infinity of letters, or just one, which is then called an *invariant*. In applications to formalisms the A-alphabets include all the minimal formulae, so that our sets of equations now include the 'prepared sets' of paragraph 3. This name will still be used for soluble sets of equations in which the minimal letters (relative to $\prec$) belong to A-alphabets, and the others to X-alphabets. An *alphabetical* change of letters is one in which each letter is changed, if at all, into another of the same alphabet.

† $\mathscr{E}_1 \cup \mathscr{E}_2$ is the set of all equations in the two sets $\mathscr{E}_1$ and $\mathscr{E}_2$.

‡ In the general theory. For particular examples the A-alphabets may consist of any symbols.

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To derive a *(Φ, p)-system* from a given (Φ, γ)-system with X- and A-alphabets, certain invariants are first declared to be *basic*; and a set of equations is *b-stratified* if it is stratified, and distinct basic letters belong to distinct lowest level-classes. Secondly, with certain A-alphabets there are associated classes of sets, $\mathscr{P}$, of equations, each class consisting of all the alphabetical isomorphs of one of its members. The following conditions are to be satisfied:

(P 1) a set of equations, $\mathscr{P}$, contains occurrences of just one letter, **A**, of the associated A-alphabet, and apart from this only basic invariants and X-letters;
(P 2) for basic invariants the sets $\mathscr{P}$ are null;
(P 3) every set $\mathscr{P}$ is b-stratified.

In the circumstances of (P 1) the set $\mathscr{P}$ is called a *pedigree* of **A**, and denoted by $\mathscr{P}(\mathbf{A})$. This completes the description of (Φ, p)-systems.

A number of pedigrees, $\mathscr{P}(\mathbf{A})$, $\mathscr{P}(\mathbf{B})$, ..., are *adjusted* relative to each other and to another set $\mathscr{E}_0$ if no letter except (possibly) invariants and **A**, **B**, ... occurs in more than one set. For a given $\mathscr{E}_0$, **A**, **B**, ... it is always possible to choose adjusted forms of the pedigrees. A *closure*, $[\mathscr{E}]$, of a set $\mathscr{E}$ is formed by adding to $\mathscr{E}$ pedigrees of all A-letters occurring in it, adjusted for $\mathscr{E}$ and for each other. $[\mathscr{E}]$ *is its own closure*, since the new letters have no pedigrees.

A set of equations, $\mathscr{E}$, in a (Φ, p)-system, P, is *a-stratified* if a closure of $\mathscr{E}$ is b-stratified. In a formalism derived from P a formula is a-stratified if its set of defining equations is a-stratified.

Example 5. In Church's formalism (C)(4) the symbols $\Pi_{o(o\alpha)}$, $\Pi_{o(o\beta)}$, ..., with various suffixes, must be regarded as different minimal formulae, and to differentiate between them suffixes of a purely distinguishing kind will be placed on the left, thus, ${}_1\Pi_{o(o\alpha)}$, ${}_2\Pi_{o(o\beta)}$, ..., etc. In the untyped theory ${}_1\Pi$, ${}_2\Pi$, ... form one infinite alphabet. The specification of (C) as a (Φ, p)-system, (C, p), is (using $\sim$ and $\vee$ for Church's N and A):

Invariants:	$\sim, \vee, B$,
A-alphabet 1 ('variables'):	$a, b, c, \ldots$,
A-alphabet 2:	${}_1\Pi, {}_2\Pi, \ldots$,
A-alphabet 3:	${}_1\iota, {}_2\iota, \ldots$.

The Φ-equations and γ_i-relations are those already given (Example 3). The only basic invariant is B. The pedigrees are:

$\sim$:	$B = (\sim B)$,
$\vee$:	$X = (\vee B), \quad B = (XB)$,
alphabet 1:	none,
alphabet 2:	$B = ({}_1\Pi X), \quad B = (XY)$,
alphabet 3:	$Y = ({}_1\iota X), \quad B = (XY)$.

The letter B has no meaning; its function is solely to regulate stratification by representing the 'propositional' stratum.

The *well-formed formulae* of the system are the solutions of a-stratified, prepared, defining sets of equations, subject to the condition that in an equation $X = (\lambda AY)$ the letter A must belong to the 'variables' alphabet, $a, b, c, \ldots$.†

THEOREM 7. *Of two alphabetically isomorphic sets of equations, both are a-stratified or neither.*

An alphabetical isomorphism between $\mathscr{E}_1$ and $\mathscr{E}_2$ can be extended to one between $[\mathscr{E}_1]$ and $[\mathscr{E}_2]$ by means of those given to exist between the pedigrees, in $[\mathscr{E}_1]$ and $[\mathscr{E}_2]$, of A-letters of $\mathscr{E}_1$ and their correlates in $\mathscr{E}_2$. The adjustment of the pedigrees ensures that this is indeed an isomorphism between $[\mathscr{E}_1]$ and $[\mathscr{E}_2]$. Hence if $[\mathscr{E}_1]$ is stratified so is $[\mathscr{E}_2]$, and since the isomorphism is an alphabetical one all invariants correspond to themselves. Hence if $[\mathscr{E}_1]$ is b-stratified $[\mathscr{E}_2]$ is also.

COROLLARY. *The property of being a-stratified is independent of the choice of the pedigrees in* $[\mathscr{E}]$.

10. *Restricted types.* The letters $\xi_1, \xi_2, \ldots$ are now regarded as *variable* types, and types in general as *functions*, types that do not contain any ξ_i are *constants*. (Any isomorphic change among the ξ_i's is considered to give the same function with different variables.) A *value* of a type-function is obtained by substituting arbitrary constant types for the variables.

A (Φ, t)-*system* is derived from a (Φ, γ)-system with A- and X-alphabets, and with r γ_i-relations, by first associating fixed primitive constant types (all different), called the *basic* types, with certain invariants; and secondly associating with each A-alphabet, $\{A\}$, an r-fold type, $\phi(A)$, composed entirely of basic types and variables. The name *type-function* is now confined to such types. An *a-correct* typing of a set of equations in a (Φ, t)-system is one that satisfies the conditions (t_1) to (t_4), and also

(t_5) every A-letter has as type a value of the type function assigned to its alphabet.

Example 6. The (Φ, t)-system, (C, t), derived from (C) has Φ-equations, alphabets and γ_i-relations as in Example 5, but instead of specifying basic invariants and pedigrees we assign types as follows, o being the only basic type:‡

B_o, $\sim_{(oo)}$, $\vee_{((oo)o)}$,
alphabet 1: type ξ_1, alphabet 2: type $(o(o\xi_1))$, alphabet 3: type $(\xi_1(o\xi_1))$.

The relation between (C, p) and (C, t) will appear in the following paragraphs.

11. In any (Φ, t)-system a *blank* typing of a set $\mathscr{E}$ is a typing, satisfying (t_1) to (t_4), in which the basic types are properly assigned, and all other lowest level-classes have types ξ_i, all different.

A (Φ, p)-system, P, and a (Φ, t)-system, T, derived from the same (Φ, γ)-system *correspond* if, first, the basic invariants in P are identical with the invariants that receive basic types in T; and secondly, each pedigree $\mathscr{P}(A)$ admits a blank typing, τ, such that $\tau(A)$ is $\phi(A)$, the function assigned to A in T. The relations between corresponding (Φ, p)- and (Φ, t)-systems are expressed in the following three theorems.

† Since such meaningless conbinations as $(\Pi\sim)$ are (for formal convenience) admitted in (C), it seems unnecessary to forbid B to be a minimal letter in the defining equations.

‡ The other special type, ι, used in Church (4), is needed only in connexion with the axioms and rules of procedure.

Theorem 8. *If a* (Φ, p)*-system,* P, *and a* (Φ, t)*-system,* T, *correspond, a necessary and sufficient condition that a set* $\mathscr{E}$ *be a-stratified in* P *is that it admit an a-typing in* T.

Theorem 9. *Given any* (Φ, p)*-system, a set of type-functions can be found, the association of which with the* A*-alphabets makes the underlying* (Φ, γ)*-system into a corresponding* (Φ, t)*-system.*

For the third theorem we need an additional postulate:

(A 4) there exists a set of equations $\mathscr{E}_0$ which implies $\mathbf{X}\gamma_i\mathbf{X}_i$, for $i = 1, 2, \ldots, r$, and no other γ_i-relation.

Theorem 10. *Given any* (Φ, t)*-system satisfying* (A 4), *a set of pedigrees can be found which make the underlying* (Φ, γ)*-system into a corresponding* (Φ, p)*-system.*

Proof of Theorem 8. *Necessary.* If $[\mathscr{E}]$ is b-stratified, by Corollary 2 of Theorem 4 it admits a correct typing, τ, in which the basic invariants have their basic types, and other lowest level-classes have constant types. This determines a typing of each pedigree, $\mathscr{P}(\mathbf{A})$, contained in $[\mathscr{E}]$, differing from a blank typing only in that each ξ_i is replaced by a constant type $\boldsymbol{\alpha}_i$. Hence $\tau(\mathbf{A})$ is a value of $\phi(\mathbf{A})$, and the typing τ of $\mathscr{E}$ is a-correct.

Sufficient. Let τ be an a-correct typing of $\mathscr{E}$, and $\mathbf{A}$ any A-letter in $\mathscr{E}$. Let $\boldsymbol{\phi}$ be the function $\phi(\mathbf{A})$ with a definitely chosen set of variables, and let σ be that blank typing of $\mathscr{P}(\mathbf{A})$ which gives $\mathbf{A}$ the type $\boldsymbol{\phi}$. By the definition of an a-correct typing, $\tau(\mathbf{A})$ is obtained from $\boldsymbol{\phi}$ by substituting a constant type, $\boldsymbol{\alpha}_i$, for each ξ_i occurring in it. Let $\tau_{\mathbf{A}}$ be the correct typing of $\mathscr{P}(\mathbf{A})$ obtained from the typing σ by assigning the types $\boldsymbol{\alpha}_i$ instead of ξ_i to the lowest-level classes. Then (Theorem 4, Corollary 2) $\tau_{\mathbf{A}}(\mathbf{A}) = \tau(\mathbf{A})$. Let the pedigrees of all other A-letters in $\mathscr{E}$ be similarly typed. Since, in the typings $\tau, \tau_{\mathbf{A}}, \tau_{\mathbf{B}}, \ldots$, of $\mathscr{E}$ and adjusted pedigrees $\mathscr{P}(\mathbf{A}), \mathscr{P}(\mathbf{B}), \ldots$, all common letters receive the same types, the combination of $\tau, \tau_{\mathbf{A}}, \tau_{\mathbf{B}}, \ldots$ is, by Theorem 5, a correct typing of $[\mathscr{E}]$; and hence, by Theorem 4, $[\mathscr{E}]$ is stratified. Since the basic invariants receive primitive types, $[\mathscr{E}]$ is b-stratified, i.e. $\mathscr{E}$ is a-stratified.

Proof of Theorem 9. Let primitive constant types, all different, be assigned to the basic invariants as basic types, and take $\phi(\mathbf{A})$ to be the type that $\mathbf{A}$ receives in a blank typing of the b-stratified set $\mathscr{P}(\mathbf{A})$. It follows immediately from the definition that the (Φ, t)-system so formed corresponds to the original (Φ, p)-system.

Proof of Theorem 10. Let the basic types be $\beta_1, \beta_2, \ldots, \beta_q$, and the invariants to which they are assigned be $B^1, B^2, \ldots, B^q$. These we declare to be the basic invariants of the (Φ, p)-system. (This fixes the meaning of 'b-stratification'.)

A rule will first be given for associating a set of equations, $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$, (possibly empty) with any X-letter, $\mathbf{X}$, and any r-fold type, $\boldsymbol{\phi}$ (constant or function), where r is the number of γ_i-relations in the system. For any $\mathbf{X}$ we take $\mathscr{E}(\mathbf{X}, \boldsymbol{\alpha})$ to be empty if $\boldsymbol{\alpha}$ is primitive. Let $\boldsymbol{\phi}_0$ be any r-fold non-primitive type and $\mathbf{X}_0$ any X-letter. We make the inductive hypothesis that, for types $\boldsymbol{\phi}$ of height† less than that of $\boldsymbol{\phi}_0$ and for any $\mathbf{X}$, $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is already defined as a b-stratified set of equations, such that (i) $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is null if, and only if, $\boldsymbol{\phi}$ is primitive; further, if $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is not null, (ii) a suitable blank

† Cf. the proof of Theorem 6.

typing of $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ gives $\mathbf{X}$ the type $\boldsymbol{\phi}$, (iii) each lowest level-class contains a single letter and (iv) all letters occurring are X-letters or basic invariants. (These are all verified for the primitive types.) Let $\boldsymbol{\phi}_0$ be $(\boldsymbol{\phi}_1 \boldsymbol{\phi}_2 \ldots \boldsymbol{\phi}_r)$. Let X-letters, $\mathbf{Y}_h$, all different, be chosen, one for each of the variables ξ_h in $\boldsymbol{\phi}_0$, and let $\mathbf{X}_i$ be r X-letters different from the $\mathbf{Y}_h$ and each other. For each i, if $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ is not null, let σ^i be a blank typing of it such that $\sigma^i(\mathbf{X}_i)$ is $\boldsymbol{\phi}_i$. Let the X-letters of $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ other than $\mathbf{X}_i$ be so adjusted that the type ξ_h is borne by $\mathbf{Y}_h$ only, but apart from this no two of the sets have a common X-letter, and none of them contains $\mathbf{X}_0$. Finally let $\mathscr{E}_0(\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_r)$ be a set of equations which imply $\mathbf{X}_0 \gamma_i \mathbf{X}_i$, for $i = 1, 2, \ldots, r$, and no other γ_i-relations (A 4). From the positional character of the γ_i-relations, any letter other than $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_r$ in $\mathscr{E}_0$ can be altered to $\mathbf{X}_0$ without disturbing the relations $\mathbf{X}_0 \gamma_i \mathbf{X}_i$, or introducing any new ones: this we suppose already done. The set $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$ is defined to be the union of the adjusted sets $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ $(i = 1, 2, \ldots, r)$ and the set $\mathscr{E}_0(\mathbf{X}_0, \mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_r)$, where $\mathbf{H}_i$ is $\mathbf{Y}_h$ if $\boldsymbol{\phi}_i$ is ξ_h, B^j if $\boldsymbol{\phi}_i$ is β_j, and $\mathbf{X}_i$ if it is neither. (Note that if $\mathbf{H}_i$ is B^j, $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ is null, and $\mathbf{X}_i$ does not occur in $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$.)

The inductive hypothesis is true of $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$. *Parts* (i) and (iv) are evidently true. *Part* (ii). The required typing, σ^0, is the combination of the σ^i's with the types $\boldsymbol{\phi}_0$ for $\mathbf{X}_0$ and $\boldsymbol{\phi}_i$ for $\mathbf{H}_i$ $(i = 1, 2, \ldots, r)$. Since the only relations in $\mathscr{E}_0$ are $\mathbf{X}_0 \gamma_i \mathbf{H}_i$, for $i = 1, 2, \ldots, r$, $\mathscr{E}_0$ is correctly typed; and since letters common to any two sets have the same type in both, the combination of all the types is a correct typing of $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$ (i.e. satisfies t_1 to t_4). Since $\mathbf{H}_i$ is an invariant if, and only if, $\boldsymbol{\phi}_i$ is β_j, and since in σ^i basic types are correctly assigned, the same is true of σ^0. The typing is a 'blank' one, since two X-letters carry the same type ξ_h only if they are identical. *Part* (iii). A lowest level-class bears the type ξ_h or β_j, and contains the single letter $\mathbf{Y}_h$ or B^j respectively.

The inductive definition of $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is therefore complete. We assign as pedigrees to every A-letter, $\mathbf{A}$, the set $\mathscr{E}(\mathbf{A}, \boldsymbol{\phi})$ and its alphabetical isomorphs, where $\boldsymbol{\phi}$ is $\phi(\mathbf{A})$. The conditions (P 1) and (P 2) are evidently satisfied, and it has just been proved that (P 3) holds. A (Φ, p)-system has therefore been defined, which clearly corresponds to the original (Φ, t)-system.

Example 7. If this process is applied to (C, t), (Example 6), taking $\mathscr{E}_0(\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2)$ to be $\mathbf{X}_1 = (\mathbf{X}_0 \mathbf{X}_2)$, the pedigrees found for $\sim$, $\vee$, and alphabets 1 and 2 are precisely those given in Example 5.

12. *Some generalizations.* (These are concerned with the '(Φ, γ)'-theory, without special alphabets or invariants.)

I. The single set of relations γ_i may be replaced by a finite or infinite number of sets of relations

$$\gamma_{ji} \quad (j = 1, 2, 3, \ldots;\ i = 1, 2, \ldots, r_j).$$

The assumption (A 3) must be replaced by

(A* 3) if the relation $\mathbf{X}_m \gamma_{ji} \mathbf{X}_n$ follows from $\mathbf{X}_0 = \Phi \mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_k$, $\mathbf{X}_m \gamma_{jh} \mathbf{X}_{n_h}$ also follows, for the same j and $h = 1, 2, \ldots, r_j$;

and the following new assumption is required:

(A 5) if $r_j = r_k$ then $j = k$.

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(This apparently rather arbitrary assumption merely expresses the fact that only if $r_j \neq r_k$ need the γ_{ji} and γ_{kh} be distinguished.) In the definition of stratification the new condition must be added, that the same **X** shall not satisfy relations $\mathbf{X}\gamma_{ji}\mathbf{Y}$ in $\mathscr{E}$ and $\mathbf{X}\gamma_{kh}\mathbf{Z}$ in $\mathscr{E}$, for $j \neq k$. In the definition of a correct typing conditions (t_2) and (t_4) are replaced by the new condition

(t^*) if $\mathbf{X}\gamma_{ji}\mathbf{Y}$, $\tau(\mathbf{X})$ has r_j factors, and $\tau_i(\mathbf{X})$ is $\tau(\mathbf{Y})$.

All the main theorems survive these changes, with little modification of the proofs. From (A* 3) and the new stratification condition it follows that in a stratified $\mathscr{E}_f$ a letter which is not at the lowest level has a single set of descendents $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{r_j}$, with $\mathbf{X}\gamma_{ji}\mathbf{Y}_i$, and hence in Theorem 4 we may take $\tau(\mathbf{X})$ to be $(\tau(\mathbf{Y}_1)\tau(\mathbf{Y}_2)\ldots\tau(\mathbf{Y}_{r_j}))$. (The condition (A 5) is used in the proof of Theorem 6.)

Example 8. In *Principia Mathematica* the stratification underlying the 'simple' theory of types (as modified by Chwistek, Ramsey, Carnap and others †) is applied only to the variables. It is not affected by the distribution of the logical constants and quantifiers, but depends only on the functional inter-relation of the letters. The relevant Φ-equations therefore have the form

$$X = f(x_1, x_2, \ldots, x_j),$$

and this gives $f\gamma_{ji}x_i$, for $i = 1, 2, \ldots, j$ $(r_j = j)$, the only γ-relations.

Another formalism in which the γ-relations fall into groups is considered in the next example.

II. A number of further possibilities are illustrated by the system developed by Quine in his *System of Logistic*. Although this system has been superseded by others in Quine's own writings, it is of interest to see how the use of pedigree equations simplifies the specification of complicated systems.‡

The Φ-equations are of the forms

$$\mathbf{X} = [\mathbf{Y}], \quad \mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}, \quad \mathbf{X} = (\mathbf{Y}, \mathbf{Z}).$$

There are three γ-relations, falling into a group of one, γ_{11} $(r_1 = 1)$, and a group of two, γ_{21} and γ_{22} $(r_2 = 2)$. Each of the equations $\mathbf{X} = [\mathbf{Y}]$ and $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$ gives $\mathbf{X}\gamma_{11}\mathbf{Y}$, and $\mathbf{X} = (\mathbf{Y}, \mathbf{Z})$ gives $\mathbf{X}\gamma_{21}\mathbf{Y}$ and $\mathbf{X}\gamma_{22}\mathbf{Z}$. In addition, a pair of equations $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$ and $\mathbf{Z} = (\mathbf{U}, \mathbf{V})$ in $\mathscr{E}$ together give $\mathbf{U}\gamma_{11}\mathbf{V}$ in $\mathscr{E}$.

Certain *equations* have pedigrees which must be added to give their 'closures' when considering stratification:

$$\mathbf{X} = [\mathbf{Y}] \text{ has the pedigree } \mathbf{Z} = \hat{\mathbf{X}}\mathbf{U},\ \mathbf{U} = (\mathbf{Y}, \mathbf{V}),$$
$$\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z} \text{ has the pedigree } \mathbf{Z} = (\mathbf{U}, \mathbf{V}).$$

The formulae admitted into Quine's calculus are the solutions of prepared defining sets of equations (the minimal formulae being the letters of a single infinite italic alphabet), such that (1) the closure is stratified, (2) in all equations $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$, **Y** is a minimal letter.

† See, e.g. Carnap (2), pp. 84 ff.
‡ The briefest specification hitherto given is in Church's review (3).

A correct typing in our sense is identical with Quine's if $(\boldsymbol{\alpha})$ and $(\boldsymbol{\alpha\beta})$ are replaced by $\boldsymbol{\alpha}!$ and $\boldsymbol{\alpha}\uparrow\boldsymbol{\beta}$ respectively, and only one primitive type, λ, is used. For if $\mathbf{X} = [\mathbf{Y}]$ it follows from the pedigree equations that $\mathbf{Y}\,\gamma_{11}\,\mathbf{V}$, and hence that $\tau(\mathbf{Y})$ is of the form $(\boldsymbol{\alpha})$. If $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$, let $\mathbf{V}$, in the pedigree equation, have type $\boldsymbol{\alpha}$. Then $\tau(\mathbf{U})$ is $(\boldsymbol{\alpha})$, and $\tau(\mathbf{Z})$ is $((\boldsymbol{\alpha})\,\boldsymbol{\alpha})$, which is of the prescribed form for a 'propositional' formula.

The system differs from those considered in this paper, first by the presence of pedigrees of equations, secondly in that γ_{11} is not positional. It is, however, preserved under homomorphic changes, and all the main theorems remain true. Note, in particular that Theorem 5 holds. For if $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$ is in $\mathscr{E}_1$ and $\mathbf{Z} = (\mathbf{U}, \mathbf{V})$ is in $\mathscr{E}_2$, the first shows that $\tau^2(\mathbf{Z}) = \tau^1(\mathbf{Z}) = ((\boldsymbol{\alpha})\,\boldsymbol{\alpha})$, and therefore from the second $\tau^2(\mathbf{U}) = (\boldsymbol{\alpha})$, $\tau^2(\mathbf{V}) = \boldsymbol{\alpha}$, as required by the relation $\mathbf{U}\,\gamma_{11}\,\mathbf{V}$, which holds in the combined set $\mathscr{E}_1 \cup \mathscr{E}_2$.

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STRATIFIED SYSTEMS OF LOGIC

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The suffixes used in logic to indicate differences of type may be regarded either as belonging to the formalism itself, or as being part of the machinery for deciding which rows of symbols (without suffixes) are to be admitted as significant. The two different attitudes do not necessarily lead to different formalisms, but when types are regarded as only one way of regulating the calculus it is natural to consider other possible ways, in particular the direct characterization of the significant formulae. Direct criteria for stratification were given by Quine, in his 'New Foundations for Mathematical Logic'(7). In the corresponding typed form of this theory ordinary integers are adequate as type-suffixes, and the direct description is correspondingly simple, but in other theories, including that recently proposed by Church(4), a partially ordered set of types must be used. In the present paper criteria, equivalent to the existence of a correct typing, are given for a general class of formalisms, which includes Church's system, several systems proposed by Quine, and (with some slight modifications, given in the last paragraph) *Principia Mathematica*. (The discussion has been given this general form rather with a view to clarity than to comprehensiveness.)

The effect of stratification on the rules of procedure is not discussed in this paper, except in so far as all formulae occurring are required to be stratified; † and the question of possible relaxations of the stratification conditions is therefore also not considered. The object is rather, by showing how existing type-systems could be axiomatically treated, to provide a convenient machinery for such generalizations.

1. Stratification can be defined for any kind of 'scheme' in which a finite number of identifiable places are filled by letters (e.g. formulae, matrices, sets of equations), provided that for every scheme, $\mathscr{S}$, of the system the incidence of two relations between letters, '$\mathbf{X}\,\eta\,\mathbf{Y}$ in $\mathscr{S}$' and '$\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$', is determined in such a way that

(A 1) η is symmetrical,

(A 2) η and γ are preserved under any homomorphic change of letters.‡

(The conditions are satisfied, e.g., if the schemes are rows of letters, and '$\mathbf{X}\,\eta\,\mathbf{Y}$ in $\mathscr{S}$' means 'an X occurs next to a Y in $\mathscr{S}$', and '$\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$' means 'an X followed by a Y occurs in $\mathscr{S}$'; but (A 2) is not satisfied if '$\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$' means 'every X in $\mathscr{S}$ is followed by a Y'.)

† On this point cf. *P.M.* vol. 3, p. 75 and *256·66; and Quine(10), p. 136.

‡ 'Letter' always means 'kind of letter', not a particular occurrence.

A change of letters is *homomorphic* if places in $\mathscr{S}$ which contained the same letter before the change continue to do so after it; it is *isomorphic* if it and its inverse are homomorphic. A relation $\mathbf{X}\,\mathrm{R}\,\mathbf{Y}$ is 'preserved under homomorphic changes of letters' if $\mathbf{X}\,\mathrm{R}\,\mathbf{Y}$ in $\mathscr{S}$ implies $\mathbf{X}'\,\mathrm{R}\,\mathbf{Y}'$ in $\mathscr{S}'$, where $\mathbf{X}'$ and $\mathbf{Y}'$ replace X and Y in any homomorphic change that turns $\mathscr{S}$ into $\mathscr{S}'$

An equivalence relation, 'level in $\mathscr{S}$', is derived from η as follows. If $\mathbf{X}\,\eta\,\mathbf{Y}$ in $\mathscr{S}$, and $\mathbf{X}$ and $\mathbf{Y}$ are different letters, it is an *η-reduction* to replace $\mathbf{Y}$ everywhere by $\mathbf{X}$. This may give rise to some new η-relations among the remaining letters, but since the number of different letters is diminished, repeated η-reductions lead finally to an η-irreducible scheme, $\mathscr{S}_f$. '$\mathbf{X}$ is level with $\mathbf{Y}$ in $\mathscr{S}$' shall mean that $\mathbf{X}$ and $\mathbf{Y}$ are replaced by the same letter in $\mathscr{S}_f$. This equivalence relation divides the letters of $\mathscr{S}$ into *level-classes*; that containing $\mathbf{X}$ is denoted by $\{\mathbf{X}\}$.

There may be some freedom of choice in the order of η-reductions, but *the level-classes are independent of the order of reduction.* For if not it must happen that, in the series of schemes $\mathscr{S}$, $\mathscr{S}_1$, $\mathscr{S}_2$, ..., leading by η-reduction to an irreducible scheme $\mathscr{S}_f$, a scheme $\mathscr{S}_n$ is changed to $\mathscr{S}_{n+1}$ by a substitution of $\mathbf{U}$ for $\mathbf{V}$ that puts $\mathbf{U}$ into two places, p and q, filled by different letters in $\mathscr{S}_f'$, another irreducible form of $\mathscr{S}$. It may be assumed that $(\mathscr{S}_n, \mathscr{S}_{n+1})$ is the first such η-reduction in the series. But then $\mathscr{S}_f'$ is obtained from $\mathscr{S}_n$ by a homomorphic change of letters; and since $\mathbf{U}\,\eta\,\mathbf{V}$ in $\mathscr{S}_n$, $\mathbf{U}'\,\eta\,\mathbf{V}'$ in $\mathscr{S}_f'$, where $\mathbf{U}'$ and $\mathbf{V}'$ occupy the places p and q, and hence by hypothesis are different. Therefore $\mathscr{S}_f'$ is not η-irreducible, contrary to hypothesis.

It follows from this result that all η-irreducible forms of $\mathscr{S}$ are alphabetically isomorphic.

A relation Γ among level-classes is next determined by the rules

(Γ_1) if $\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{S}$, $\{\mathbf{X}\}\,\Gamma\,\{\mathbf{Y}\}$ in $\mathscr{S}$;

(Γ_2) Γ is transitive.

The relation '$\{\mathbf{X}\}\,\Gamma\,\{\mathbf{Y}\}$ in $\mathscr{S}$' holds only if it is deducible from (Γ_1) and (Γ_2); and '$\mathbf{X}\,\Gamma\,\mathbf{Y}$ in $\mathscr{S}$' has the same meaning. Clearly a necessary and sufficient condition that $\mathbf{X}\,\Gamma\,\mathbf{Y}$ in $\mathscr{S}$ is that either $\mathbf{X}_f\,\gamma\,\mathbf{Y}_f$, or $\mathbf{X}_f\,\gamma\,\mathbf{X}_1\,\gamma\,\mathbf{X}_2\,\gamma \ldots \gamma\,\mathbf{X}_k\,\gamma\,\mathbf{Y}_f$ in $\mathscr{S}_f$ for some $\mathbf{X}_i$'s, where $\mathbf{X}_f$ and $\mathbf{Y}_f$ replace $\mathbf{X}$ and $\mathbf{Y}$ in $\mathscr{S}_f$.

Finally, the scheme $\mathscr{S}$ is *stratified* if $\mathbf{X}\,\Gamma\,\mathbf{X}$ holds for no $\mathbf{X}$, i.e. if Γ is a partial ordering of the letters in $\mathscr{S}$.†

2. The definitions of the preceding paragraph will be applied to logical formalisms, not directly, but through their 'defining equations', an expression which will now be explained.

In most symbolisms that are used in mathematical logic, the formulae are built up step by step from certain minimal formulae. A single step consists in placing a number of formulae already constructed in a row, say $\mathfrak{X}_1\,\mathfrak{X}_2 \ldots \mathfrak{X}_k$, and indicating, by adding an 'operational symbol' and appropriate brackets, which 'function' of them is wanted. (In some formalisms mere juxtaposition, or inclusion of one formula in a pair of brackets, is a method of construction.) If $\mathfrak{X}$ is the new formula, $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_k$ will be called its *factors*; and factors, factors of factors, and so on, are the *segments* of $\mathfrak{X}$. (A formula is not a segment of itself.) As an example, in Church's untyped 'λ-K-calculus' [Church (5)], if $\mathfrak{A}$ and $\mathfrak{B}$ denote formulae and $\mathfrak{x}$ a variable, we can construct the new formulae $\mathfrak{X} = (\mathfrak{A}\mathfrak{B})$ and $\mathfrak{Y} = (\lambda\mathfrak{x}\mathfrak{A})$. To give a complete specification of $\mathfrak{X}$ it would be necessary to give similar equations for the factors $\mathfrak{A}$ and $\mathfrak{B}$, say $\mathfrak{A} = (\lambda\mathfrak{y}\mathfrak{C})$, $\mathfrak{B} = (\mathfrak{C}\mathfrak{C})$, and so on, the process terminating when all the factors on the right-hand

† In the main application to be made, 'level' will correspond roughly to 'having the same type', 'Γ' to 'having higher type'.

side are minimal formulae. Any formula can be specified in this way by giving the defining equations leading back step by step to the minimal formulae of which it is composed. For example, if $\mathfrak{X}$ is the formula $(a(a)\,(b(ab)))$ of the λ-calculus, $\mathfrak{X}$ is seen to be determined by the equations

$$\mathfrak{X} = (a\mathfrak{Y}\mathfrak{Z}), \quad \mathfrak{Y} = (a), \quad \mathfrak{Z} = (b\mathfrak{U}), \quad \mathfrak{U} = (ab),$$

in all of which only single letters appear as 'factors' inside the brackets on the right-hand side.

It is such *sets of equations*, $\mathscr{E}$, that play the part of the schemes $\mathscr{S}$ in paragraph 1. The letters occurring in equations are therefore object letters, not syntactical names, in our discussion. The italic capitals U, V, W, X, Y, Z, with or without suffixes, will accordingly be used from now on for the letters in equations, German letters being reserved for the names of the formulae which, in certain instances, are the solutions of the equations. Clarendon capitals, **U**, **V**, ... are used as syntactical names for single letters in equations.

For the purposes of the general discussion it may be supposed that the 'operational' symbols are $\Phi, \Phi_1, \Phi_2, \ldots$, and are placed at the beginning of a formula, mere juxtaposition and bracketing being excluded (though in discussing particular formalisms the symbols in ordinary use will be retained). Our theory is therefore concerned with sets of formal equations such as $X = \Phi_1 YZ$, $Y = \Phi_2 Y$, called 'Φ-equations'. For the present the Φ-equations are restricted only by the condition that each Φ_i shall always be followed by the same number of letters, and a *Φ-system* is set up by specifying this number for a finite number of Φ_i's. No condition of 'solubility' is imposed on the sets of equations. (N.B. The symbols Φ_i themselves are not 'letters' in our theory.)

3. A preliminary application of the concepts of paragraph 1 leads immediately to a formal theory of 'equality' and 'solubility' for such sets of equations. Let '$\mathbf{X}\,e\,\mathbf{Y}$ in $\mathscr{E}$' mean that the set of equations $\mathscr{E}$ contains a pair of equations $\mathbf{X} = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$ and $\mathbf{Y} = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$, with identical right-hand sides. Then *equality* in $\mathscr{E}$ ('$\mathbf{X} = \mathbf{Y}$ in $\mathscr{E}$') is, by definition, the level-relation derived as in paragraph 1, on taking η to be e. The relation e has the required properties of symmetry and of being preserved under a homomorphic change of letters, and letters that are 'equal' according to this definition are in fact those which can be proved equal by means of the equations $\mathscr{E}$.†

We now take the relation '$\mathbf{X}\,\gamma\,\mathbf{Y}$ in $\mathscr{E}$' of paragraph 1 to be '$\mathscr{E}$ contains an equation $\mathbf{X} = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$, where $\mathbf{Y}$ is one of the letters $\mathbf{X}_i$'. This also is a relation preserved in homomorphic changes. The Γ-relation derived from it, in combination with e as 'η', will be denoted by $\succ$, ('$\mathbf{X}\succ\mathbf{Y}$' will also be written '$\mathbf{Y}\prec\mathbf{X}$'). An e-irreducible set of equations is *soluble* if (1) it is stratified relative to $\succ$, and (2) no letter stands on the left of more than one equation; and any set $\mathscr{E}$ is soluble if its e-reduced form is soluble. There must in this case be *minimal* letters for the partial ordering $\succ$,‡ and the system can in fact be 'solved' in terms of them for the remaining letters, by repeatedly substituting the right-hand sides of equations, enclosed in brackets, for their left-hand sides. If there is a single maximal letter, $\mathbf{X}$, $\mathscr{E}$ is called a *defining set* for $\mathbf{X}$.

A *formalism* is derived from a Φ-system by, first, specifying certain 'minimal formulae'

† '$\mathbf{X}=\mathbf{Y}$' is never used between equation-letters to mean '$\mathbf{X}$ and $\mathbf{Y}$ are the same letter'.
‡ Letters, $\mathbf{X}$, such that $\mathbf{X}\succ\mathbf{Y}$ in $\mathscr{E}$ for no $\mathbf{Y}$.

(which may be any symbols whatever), the substitution of which for the minimal letters in the solutions of defining sets gives the *formulae* of the system, and secondly, stating (possibly) other conditions to be satisfied by a formula, or its defining equations, in order that it may be a significant, or *well-formed*, formula of the system. If the minimal formulae are substituted for the minimal letters in the equations themselves of a soluble set, we obtain a *prepared set*, whose solutions are formulae of the system. The 'values' obtained for letters other than **X**, when a prepared defining set for **X** is solved, are the segments, as above defined, of the value of **X** itself; and the relation '$\prec$', carried over from letters **X** to their values, becomes 'is a segment of'.

4. The stratification proper of logical formulae is, of course, entirely distinct from the 'solubility' just defined. It depends on relations γ and η which must be specified separately for each Φ-system, and are subject to no restriction, in the first place, beyond those imposed in paragraph 1.

Example 1. In the system developed in Quine's 'New foundations for mathematical logic'(7), there are three kinds of equation,

$$\mathbf{X} = (\mathbf{Y} \in \mathbf{Z}), \quad \mathbf{X} = (\mathbf{U} \downarrow \mathbf{V}), \quad \mathbf{X} = (\mathbf{P})\,\mathbf{Q}.$$

For the stricter, 'classical', stratification proposed in (7) '$\mathbf{X}\gamma\mathbf{Y}$ in $\mathscr{E}$' must be defined to mean '$\mathscr{E}$ contains an equation $\mathbf{Z} = (\mathbf{Y} \in \mathbf{X})$'; and '$\mathbf{X}\eta\mathbf{Y}$' to mean '$\mathbf{X}\,e\,\mathbf{Y}$, or $\mathbf{X}\gamma\mathbf{U}$ and $\mathbf{Y}\gamma\mathbf{U}$ for some **U**, or $\mathbf{V}\gamma\mathbf{X}$ and $\mathbf{V}\gamma\mathbf{Y}$ for some **V**'.

The minimal formulae are small italic letters, and for a 'significant' formula, in addition to the condition of stratification, the equations must be such that those letters, and only those, are minimal that occur on the right of equations $\mathbf{X} = (\mathbf{Y} \in \mathbf{Z})$. Hence the test for stratification, applied directly to formulae occurring in the theory, is as follows. If *either* $\mathbf{x} \in \mathbf{u}$ and $\mathbf{y} \in \mathbf{u}$, *or* $\mathbf{v} \in \mathbf{x}$ and $\mathbf{v} \in \mathbf{y}$, occur in the formula, replace **y** throughout by **x**, and continue this process as long as possible (making also any possible e-reductions). If in the final result there is no cycle $\mathbf{x}_1 \in \mathbf{x}_2 \in \ldots \in \mathbf{x}_k \in \mathbf{x}_1$, the original formula is stratified.†

5. In order to bring stratification into correlation with types some restriction on the relations η and γ is necessary. A form of theory sufficiently general to cover most extant formalisms, including that of Church(4), will now be described. Some generalizations, enabling *Principia Mathematica* and some other systems to be brought within the scope of the theory, are briefly described at the end of the paper.

The assumptions made in the previous paragraph, about the existence and properties of the relations γ and η, are now replaced by the assumption that any equation $\mathbf{X}_0 = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$ determines a number of *positional relations*, $\mathbf{X}_m\gamma_i\mathbf{X}_n$, i.e. relations that hold if, and only if, certain places in the equation are filled by $\mathbf{X}_m$ and $\mathbf{X}_n$. There are to be a finite number, r, of the relations γ_i, and they are to satisfy

(A 3) if the relation $\mathbf{X}_m\gamma_i\mathbf{X}_n$ follows from $\mathbf{X}_0 = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$, then $\mathbf{X}_m\gamma_h\mathbf{X}_{n_h}$ also follows, for $h = 1, 2, \ldots, r$ and some $\mathbf{X}_{n_h}$ $(1 \leqslant n_h \leqslant k)$.

† A simpler test given by Quine in (7), p. 78, is not quite correct. See Bernays's review of the paper (1).

A (Φ, γ)-*system* is set up by specifying, for a number of operators Φ_j, how many variables each takes, and what γ_i-relations follow from any equation $\mathbf{X} = \Phi_j \mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_k$; provided that the relations are positional and satisfy (A 3).

The relations η and γ are now defined in terms of the γ_i: '$\mathbf{X}\gamma\mathbf{Y}$ in $\mathscr{E}$' means that $\mathbf{X}\gamma_i\mathbf{Y}$ in $\mathscr{E}$ for some i, and '$\mathbf{X}\eta\mathbf{Y}$ in $\mathscr{E}$' that one of the following holds:

(η_1) $\mathbf{X} e \mathbf{Y}$;

(η_2) $\mathbf{U}\gamma_i\mathbf{X}$ and $\mathbf{U}\gamma_i\mathbf{Y}$ in $\mathscr{E}$ for some $\mathbf{U}$ and i;

(η_3) $\mathbf{X}\gamma_i\mathbf{U}_i$ and $\mathbf{Y}\gamma_i\mathbf{U}_i$ in $\mathscr{E}$ for $i = 1, 2, \ldots, r$, and some $\mathbf{U}_i$.

Condition (A 1) is clearly satisfied, and from the positional character of the γ_i it follows that η and γ satisfy (A 2). Hence the definitions and results of the previous paragraph may be carried over to any (Φ, γ)-system, and to any formalism derived from it. (Note that η is *not* a positional relation.)

Example 2. Returning to the formalism of Example 1, we see that there is only one γ_i-relation, γ, and that η is defined in accordance with (η_1), (η_2) and (η_3).

Example 3. In Church's λ-K-calculus [Church(5)] there are two kinds of defining equations, $\mathbf{Z} = (\mathbf{XY})$ and $\mathbf{Z} = (\lambda\mathbf{XY})$. The stratification introduced in Church(4) involves two γ_i-relations: the equation $\mathbf{Z} = (\mathbf{XY})$ gives $\mathbf{X}\gamma_1\mathbf{Z}$ and $\mathbf{X}\gamma_2\mathbf{Y}$, and $\mathbf{Z} = (\lambda\mathbf{XY})$ gives $\mathbf{Z}\gamma_1\mathbf{Y}$ and $\mathbf{Z}\gamma_2\mathbf{X}$. As an illustration consider first $(f(fx))$. Its equations are $X = (fx)$, $Y = (fX)$. By (η_2), $x\eta X$, whence, putting x for X, the two equations give $x = Y$. Putting x for Y we obtain two copies of the equation $x = (fx)$, an irreducible set with no γ-cycle: the original formula is stratified. Secondly, consider $(x(f(fx)))$. The defining equations are

$$X = (fx), \quad Y = (fX), \quad Z = (xY),$$

in which x, X, Y are again level. The η-reduced form of the last equation is $Z = (xx)$, giving $x\gamma_2 x$. The formula is therefore not stratified.

6. *Types* are the formulae of a system in which the minimal formulae are small Greek letters, and the only principle of construction is to enclose a row of given formulae in a pair of round brackets. Types in general will be denoted by heavy small Greek letters. A type is *primitive* if it consists of a single letter, not in brackets, and, in accordance with our previous notations, if $\boldsymbol{\alpha}$ is $(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 \ldots \boldsymbol{\alpha}_k)$, the $\boldsymbol{\alpha}_i$ are the *factors* of $\boldsymbol{\alpha}$.

A formula is *typed* by attaching types in a random way to it and its segments. 'Correct' typing is defined only if the formalism is derived from a (Φ, γ)-system, and should then mean that segments of a given formula are so matched with types that the relations $\succ$ for types and Γ for segments correspond. Since $\succ$ is a partial ordering of the types, this will ensure that Γ is a partial ordering of the segments, and hence that the formula is stratified. The object of using types, however, is to avoid the process of determining the level-classes. The criteria should therefore, while aiming at the above correlation, be expressed directly in terms of the relations γ_i.

We consider first the attaching of types to the letters, or rather to the places, in Φ-equations. It is convenient to regard a place to which no type is attached as having null type.

An r-*fold type* is defined inductively for positive integral r to be either a primitive type, or a type with r factors, each of which is an r-fold type.

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The following are the conditions to be satisfied for a *correct typing*:

(t_1) Each letter has the same type (or none) at each of its occurrences. When this condition is satisfied, we denote the type of **X** by $\tau(\mathbf{X})$, and write $\tau(\mathbf{X}) = 0$ if the type is null. If $\tau(\mathbf{X})$ is neither null nor primitive, $\tau_i(\mathbf{X})$ denotes its ith factor.

(t_2) $\tau(\mathbf{X})$ is either null or an r-fold type (where r is the number of γ_i-relations).

(t_3) If $\mathbf{X}\,\gamma_i\,\mathbf{Y}$, $\tau(\mathbf{Y})$ is $\tau_i(\mathbf{X})$.

(t_4) A letter satisfying no γ_i-relation has null type.

Condition (t_3) implies that, if $\mathbf{X}\,\gamma_i\,\mathbf{Y}$, neither letter has null type.

A *formula*, in a formalism derived from a (Φ, γ)-system, is correctly typed by assigning to its segments the types given to the corresponding letters in a correct typing (if such exists) of a prepared set of defining equations.

Example 4. In the formalism of Examples 1 and 2 a single primitive type suffices, and the other types are obtained by enclosing it in any number of pairs of brackets. We may therefore use the positive integers instead. The letters with non-null types are those that appear on the right of equations $\mathbf{X} = (\mathbf{Y} \in \mathbf{Z})$, and by ($t_3$) the type of **Z** is 1 higher than that of **Y**.

In the typed λ-K-calculus (without logical constants, Example 3) our rules state that if $\mathbf{Z} = (\mathbf{XY})$, and **Z** and **Y** have types $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, **X** has the type $(\boldsymbol{\alpha\beta})$; if $\mathbf{Z} = (\lambda\mathbf{XY})$, and **X** and **Y** have the types $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, **Z** has the type $(\boldsymbol{\beta\alpha})$. These are the rules given by Church (4), save that there may now be any number of different primitive types.

7. Difficulties may arise in connexion with letters of the kind occurring in (t_4). Consider, as an illustration, the (Φ, γ)-system, $\boldsymbol{\Phi}$, with three kinds of equation:

$$\mathbf{X} = \Phi_1\mathbf{Y}, \quad \mathbf{X} = \Phi_2\mathbf{UV}, \quad \mathbf{W} = \Phi_3\mathbf{XYZ},$$

and a single relation γ_1. Let the first equation imply no γ_1-relation, the second $\mathbf{X}\,\gamma_1\,\mathbf{U}$ and $\mathbf{X}\,\gamma_1\,\mathbf{V}$, the third $\mathbf{Y}\,\gamma_1\,\mathbf{Z}$. In this system consider the equations

$$Y = \Phi_1 U, \quad Z = \Phi_1 V, \quad X = \Phi_2 UV, \quad W = \Phi_3 XYZ \tag{1}$$

(a defining set). The following typing satisfies (t_1) to (t_4):

$$Y_{(\beta)} = \Phi_1 U_\alpha, \quad Z_\beta = \Phi_1 V_\alpha, \quad X_{(\alpha)} = \Phi_2 U_\alpha V_\alpha, \quad W = \Phi_3 X_{(\alpha)} Y_{(\beta)} Z_\beta,$$

but the set of equations is not stratified. For the relations $X\,\gamma_1\,U$ and $X\,\gamma_1\,V$ give $U\,\eta\,V$. On putting U for V the first two equations give $Y = Z$, and this, combined with $Y\,\gamma_1\,Z$ (from the last equation), gives a γ-cycle. Moreover, in the solution

$$W = \Phi_3((\Phi_2 UV)(\Phi_1 U)(\Phi_1 V))$$

of the equations, the types, (β) and β, of the last two factors depend not merely on their own structure, but on that of the first factor, a situation which seems to be at variance with the idea of a logical type.

Both these difficulties could be overcome by requiring, in place of (t_4), that any letter occurring on the left of an isolating equation in $\mathscr{E}$ shall have null type; where $\mathbf{X}_0 = \Phi\mathbf{X}_1\mathbf{X}_2 \ldots \mathbf{X}_k$ is an *isolating equation* if it implies no γ_i-relation involving $\mathbf{X}_0$. The letters Y and Z are then required to be untyped in (1), and no correct typing is possible. But apart from being rather unnatural, this stronger condition would merely displace the difficulty, for with the new rule the last three equations of (1), which form a stratified set, would be untypeable.

It seems that this is a point where the ideas of stratification and typing diverge. The system Φ admits a satisfactory definition of stratification, but not a satisfactory system of types, if all sets of equations are admitted. We shall therefore introduce a restriction which excludes such sets as (1).

8. A letter occurring in a set of equations, but satisfying no γ_i-relation, is said to be *isolated*. A set of equations is *non-singular* if all letters that appear on the left of isolating equations are isolated.

THEOREM 1. *The equation* $X = \Phi X_1 X_2 \dots X_k$ *determines the type of* X *absolutely in non-singular sets, in terms of the types of* $X_1, X_2, \dots, X_k$.

The meaning is that if the equation $Y = \Phi Y_1 Y_2 \dots Y_k$ occurs in the same or another correctly typed non-singular set, and if† $\tau(X_i) = \tau(Y_i)$, then $\tau(X) = \tau(Y)$.

If the equation is an isolating one, both X and Y are isolated, and by (t_4) have null type. If not, either, for some i and h, $X_h \gamma_i X$ follows from the equation, and then $Y_h \gamma_i Y$, giving $\tau(X) = \tau_i(X_h) = \tau_i(Y_h) = \tau(Y)$; or, by (A 3), $X \gamma_i X_{n_i}$, for $i = 1, 2, \dots, r$ and some n_i's; and then $Y \gamma_i Y_{n_i}$, and

$$\begin{aligned}\tau(X) &= (\tau(X_{n_1})\,\tau(X_{n_2}) \dots \tau(X_{n_r})) \\ &= (\tau(Y_{n_1})\,\tau(Y_{n_2}) \dots \tau(Y_{n_r})) = \tau(Y).\end{aligned}$$

THEOREM 2. *If* $\tau(X) = \tau(Y)$ *in a correct typing of any set* $\mathscr{E}$, *and if* $\mathscr{E}'$ *is the result of substituting* X *everywhere for* Y, *the same typing*‡ *of* $\mathscr{E}'$ *is correct.*

Condition (t_1) is satisfied since $\tau(X) = \tau(Y)$, (t_2) and (t_3) by their positional character. As regards (t_4): if a letter, Z, other than X is isolated in $\mathscr{E}'$ it is isolated in $\mathscr{E}$, and $\tau(Z) = 0$; and if X is isolated in $\mathscr{E}'$ it is *a fortiori* isolated in $\mathscr{E}$.

THEOREM 3. *A level-class in a non-singular set either consists entirely of isolated letters, or else contains none.*

The theorem is certainly true of an η-irreducible set, since each level-class then contains only one letter. Hence if it is false for a set $\mathscr{E}$, the paired letters in one of the η-reductions leading from $\mathscr{E}$ to $\mathscr{E}_f$ must be an isolated letter, X_0, and a non-isolated letter, Y_0, with $X_0 \eta Y_0$. This relation η cannot follow from (η_2) or (η_3), since X_0 satisfies no γ_i-relation. It therefore follows from (η_1), say

$$X_0 = \Phi X_1 X_2 \dots X_k \quad \text{and} \quad Y_0 = \Phi X_1 X_2 \dots X_k.$$

Since Y_0 is a non-isolated letter these are not isolating equations. Hence the first of them implies a γ_i-relation involving X_0, contrary to the assumption that X_0 is isolated.

In view of this result we may speak of 'isolated' and 'non-isolated level-classes' in non-singular sets of equations. All the members of an isolated level-class in $\mathscr{E}$ are equal in $\mathscr{E}$.

† The use of '=' between types to mean 'is identical with' can hardly be confused with the formal '=' between equation-letters.

‡ I.e. the same distribution of types among the *places* in $\mathscr{E}'$.

COROLLARY 1. *In a non-singular set the isolated level-classes are all lowest level-classes.*†

COROLLARY 2. *If $\mathscr{E}$ is non-singular, $\mathscr{E}_f$ is non-singular.* Let $\mathbf{X}_0 = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$ be an isolating equation in $\mathscr{E}_f$. This corresponds to at least one equation $\mathbf{Y}_0 = \Phi\mathbf{Y}_1\mathbf{Y}_2\ldots\mathbf{Y}_k$ in $\mathscr{E}$, where $\mathbf{X}_i$ is $(\mathbf{Y}_i)_f$. Since the Y-equation is also isolating $\mathbf{Y}_0$ is isolated in $\mathscr{E}$, and therefore all members of $\{\mathbf{Y}_0\}$ are isolated. But if $\mathbf{X}_0$ satisfied a γ_i-relation in $\mathscr{E}_f$, some original of $\mathbf{X}_0$ in $\mathscr{E}$, i.e. some member of $\{\mathbf{Y}_0\}$, would satisfy a γ_i-relation in $\mathscr{E}$. Therefore $\mathbf{X}_0$ is isolated in $\mathscr{E}_f$.

THEOREM 4. *A necessary and sufficient condition that a non-singular set of equations admit a correct typing is that it be stratified.*

Necessary. Let the typing be τ. It will be shown that if an η-reduction turns $\mathscr{E}$ into $\mathscr{E}'$, the same typing of $\mathscr{E}'$ is correct. In view of Theorem 2 it is sufficient to show that if $\mathbf{X}\,\eta\,\mathbf{Y}$ in $\mathscr{E}$, $\tau(\mathbf{X}) = \tau(\mathbf{Y})$:

(i) if $\mathbf{X}\,\eta\,\mathbf{Y}$ by (η_1), say $\mathbf{X} = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$ and $\mathbf{Y} = \Phi\mathbf{X}_1\mathbf{X}_2\ldots\mathbf{X}_k$, $\tau(\mathbf{X}) = \tau(\mathbf{Y})$ by Theorem 1;

(ii) if $\mathbf{X}\,\eta\,\mathbf{Y}$ by (η_2), say $\mathbf{U}\,\gamma_i\,\mathbf{X}$ and $\mathbf{U}\,\gamma_i\,\mathbf{Y}$, $\tau(\mathbf{X}) = \tau_i(\mathbf{U}) = \tau(\mathbf{Y})$ by (t_3);

(iii) if $\mathbf{X}\,\eta\,\mathbf{Y}$ by (η_3), say $\mathbf{X}\,\gamma_i\,\mathbf{U}_i$ and $\mathbf{Y}\,\gamma_i\,\mathbf{U}_i$ for $i = 1, 2, \ldots, r$,

$$\tau(\mathbf{X}) = (\tau(\mathbf{U}_1)\,\tau(\mathbf{U}_2)\ldots\tau(\mathbf{U}_r)) = \tau(\mathbf{Y}) \quad \text{by } (t_2) \text{ and } (t_3).$$

Repeated application of this result shows that the η-irreducible form $\mathscr{E}_f$ of $\mathscr{E}$ is correctly typed by τ. If $\mathscr{E}_f$ contained a γ-cycle, $\mathbf{X}_1\gamma_{n_1}\mathbf{X}_2\ldots\gamma_{n_k}\mathbf{X}_1$, it would follow that $\tau(\mathbf{X}_1) \succ \tau(\mathbf{X}_2) \succ \ldots \succ \tau(\mathbf{X}_1)$, which is impossible, since '$\succ$' is a partial ordering.

Sufficient. We shall first assign types to the letters of $\mathscr{E}_f$ (a non-singular set, by Theorem 3, Corollary 2). Let null type be assigned to the isolated letters, and arbitrary r-fold types to the remaining lowest-level letters. Let it be assumed inductively that types have been assigned to all the letters of some lower section†, S, including all the lowest-level letters. If not all letters of $\mathscr{E}_f$ are in S there is at least one, X, not in S, whose γ_i-descendents are all in S, for every i. The addition of X to S gives a new lower section. By the condition (A 3), X satisfies $\mathbf{X}\,\gamma_i\,\mathbf{Y}_i$, for $i = 1, 2, \ldots, r$ and some letters $\mathbf{Y}_i$. Since $\mathscr{E}_f$ is η-irreducible there is, by (η_2), only one $\mathbf{Y}_i$ for each i, and by the inductive hypothesis a type $\tau(\mathbf{Y}_i)$ has already been assigned to $\mathbf{Y}_i$. We assign to X the type $(\tau(\mathbf{Y}_1)\,\tau(\mathbf{Y}_2)\ldots\tau(\mathbf{Y}_r))$. In this way every non-isolated letter of $\mathscr{E}_f$ receives a non-null type, and the conditions (t_1) to (t_4) are evidently all satisfied. We now assign to each letter, X, in $\mathscr{E}$ the type of $\mathbf{X}_f$ in $\mathscr{E}_f$. The conditions (t_1) and (t_2) are clearly satisfied. (t_3): if $\mathbf{X}\,\gamma_i\,\mathbf{Y}$ in $\mathscr{E}$, $\mathbf{X}_f\,\gamma_i\,\mathbf{Y}_f$ in $\mathscr{E}_f$, and therefore $\tau_i(\mathbf{X}) = \tau(\mathbf{Y})$. (t_4): if X is isolated in $\mathscr{E}$, $\mathbf{X}_f$ is isolated in $\mathscr{E}_f$; for the originals of $\mathbf{X}_f$ all belong to $\{\mathbf{X}\}$, and therefore satisfy no γ_i-relation in $\mathscr{E}$ (Theorem 3). Therefore $\tau(\mathbf{X}) = \tau(\mathbf{X}_f) = 0$.

COROLLARY 1. *In a correct typing of a non-singular set level letters have the same type.*

COROLLARY 2. *In a correct typing of a non-singular set the types of the non-isolated lowest level-classes can be chosen arbitrarily, and the other types are then uniquely determined.*

† The terminology is chosen as if 'Γ' were 'greater than' or 'higher than'. A *lowest-level* letter, X, is one such that $\mathbf{X}\,\Gamma\,\mathbf{Y}$ for no Y (the words 'minimal' and 'maximal' being reserved for the relation $\succ$). A *lower section* of the letters of $\mathscr{E}$ is a subset of them, S, such that if $\mathbf{X} \in \mathrm{S}$ and $\mathbf{X}\,\Gamma\,\mathbf{Y}$ then $\mathbf{Y} \in \mathrm{S}$.

COROLLARY 3. *A correct typing of $\mathscr{E}$ determines a correct typing of $\mathscr{E}_f$, and vice versa.*

THEOREM 5. *If τ^1 and τ^2 are correct typings of any two sets, $\mathscr{E}_1$ and $\mathscr{E}_2$, and $\tau^1(\mathbf{X}) = \tau^2(\mathbf{X})$ for all common letters, the combined typing is correct in $\mathscr{E}_1 \cup \mathscr{E}_2$.*†

Let the combined typing be denoted by τ. Conditions (t_1) and (t_2) are clearly satisfied. (t_3): if $\mathbf{X}\gamma_i\mathbf{Y}$ in $\mathscr{E}_1 \cup \mathscr{E}_2$ then $\mathbf{X}\gamma_i\mathbf{Y}$ in $\mathscr{E}_j$, for $j = 1$ or 2, and

$$\tau_i(\mathbf{X}) = \tau_i^j(\mathbf{X}) = \tau^j(\mathbf{Y}) = \tau(\mathbf{Y}).$$

(t_4): an isolated letter, $\mathbf{X}$, in $\mathscr{E}_1 \cup \mathscr{E}_2$ is isolated in both $\mathscr{E}_1$ and $\mathscr{E}_2$, and therefore $\tau(\mathbf{X}) = 0$.

THEOREM 6. *If in a correct typing, τ, of a non-singular set $\mathscr{E}$, all non-isolated lowest level-classes receive different primitive types, a sufficient condition that two non-isolated letters, $\mathbf{X}$ and $\mathbf{Y}$, be level is that $\tau(\mathbf{X}) = \tau(\mathbf{Y})$.*

Let $\tau(\mathbf{X}) = \tau(\mathbf{Y}) = \boldsymbol{\alpha} \neq 0$. We proceed by induction on the *height* of $\boldsymbol{\alpha}$, i.e. the maximum length of a series $\boldsymbol{\alpha} \succ \boldsymbol{\beta}^1 \succ \boldsymbol{\beta}^2 \succ \ldots \succ \boldsymbol{\beta}^q$. If $\boldsymbol{\alpha}$ is primitive, the assertion is true by hypothesis. Suppose, then, that $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1 \ldots \boldsymbol{\alpha}_r)$. Then $\mathbf{X}$ and $\mathbf{Y}$ are not lowest-level letters, and therefore, by (A 3), $\mathbf{X}\gamma_i\mathbf{X}_i$, $\mathbf{Y}\gamma_i\mathbf{Y}_i$, for $i = 1, 2, \ldots, r$; and by (t_3)

$$\tau(\mathbf{X}_i) = \boldsymbol{\alpha}_i = \tau(\mathbf{Y}_i).$$

Hence, by an inductive hypothesis, $\mathbf{X}_i$ is level with $\mathbf{Y}_i$. It follows that in $\mathscr{E}_f$, an η-irreducible form of $\mathscr{E}$, $(\mathbf{X}_i)_f$ is $(\mathbf{Y}_i)_f$; and hence, since $\mathbf{X}_f\gamma_i(\mathbf{X}_i)_f$ and $\mathbf{Y}_f\gamma_i(\mathbf{Y}_i)_f$, it follows that $\mathbf{X}_f\eta\mathbf{Y}_f$; i.e. since $\mathscr{E}_f$ is η-irreducible, $\mathbf{X}_f$ is $\mathbf{Y}_f$. Therefore $\mathbf{X}$ and $\mathbf{Y}$ are level.

9. *Restricted alphabets.* A feature of Church's Theory of Types [Church (4)], which corresponds well with the meaning attributed to types in logical formulae, is the requirement that certain symbols shall have types of a prescribed kind whenever they appear. In the following paragraphs it will be shown how, from a given (Φ, γ)-system, we may derive, first, a '(Φ, p)-system' in which stratification is defined in a modified sense, and secondly a '(Φ, t)-system', in which the modified typings are described; and it will be shown that there is a complete correspondence between the two kinds of system.

In the rest of the paper it is assumed that, in the (Φ, γ)-systems considered, *there are no isolating equations.*

We suppose the letters occurring in equations divided into a number of alphabets, one of which consists of the italic capitals, U, V, ..., so far used, now called the X-*alphabet*. The other alphabets, called A-alphabets, will‡ consist of the letters A_1, A_2, ..., B_1, B_2, ..., etc., the letter denoting the alphabet, and the suffixes the different members of it. Each alphabet contains either an infinity of letters, or just one, which is then called an *invariant*. In applications to formalisms the A-alphabets include all the minimal formulae, so that our sets of equations now include the 'prepared sets' of paragraph 3. This name will still be used for soluble sets of equations in which the minimal letters (relative to $\prec$) belong to A-alphabets, and the others to X-alphabets. An *alphabetical* change of letters is one in which each letter is changed, if at all, into another of the same alphabet.

† $\mathscr{E}_1 \cup \mathscr{E}_2$ is the set of all equations in the two sets $\mathscr{E}_1$ and $\mathscr{E}_2$.

‡ In the general theory. For particular examples the A-alphabets may consist of any symbols.

To derive a *(Φ, p)-system* from a given (Φ, γ)-system with X- and A-alphabets, certain invariants are first declared to be *basic*; and a set of equations is *b-stratified* if it is stratified, and distinct basic letters belong to distinct lowest level-classes. Secondly, with certain A-alphabets there are associated classes of sets, $\mathscr{P}$, of equations, each class consisting of all the alphabetical isomorphs of one of its members. The following conditions are to be satisfied:

(P 1) a set of equations, $\mathscr{P}$, contains occurrences of just one letter, **A**, of the associated A-alphabet, and apart from this only basic invariants and X-letters;

(P 2) for basic invariants the sets $\mathscr{P}$ are null;

(P 3) every set $\mathscr{P}$ is b-stratified.

In the circumstances of (P 1) the set $\mathscr{P}$ is called a *pedigree* of **A**, and denoted by $\mathscr{P}(\mathbf{A})$. This completes the description of (Φ, p)-systems.

A number of pedigrees, $\mathscr{P}(\mathbf{A})$, $\mathscr{P}(\mathbf{B})$, ..., are *adjusted* relative to each other and to another set $\mathscr{E}_0$ if no letter except (possibly) invariants and **A**, **B**, ... occurs in more than one set. For a given $\mathscr{E}_0$, **A**, **B**, ... it is always possible to choose adjusted forms of the pedigrees. A *closure*, $[\mathscr{E}]$, of a set $\mathscr{E}$ is formed by adding to $\mathscr{E}$ pedigrees of all A-letters occurring in it, adjusted for $\mathscr{E}$ and for each other. $[\mathscr{E}]$ *is its own closure*, since the new letters have no pedigrees.

A set of equations, $\mathscr{E}$, in a (Φ, p)-system, P, is *a-stratified* if a closure of $\mathscr{E}$ is b-stratified. In a formalism derived from P a formula is a-stratified if its set of defining equations is a-stratified.

Example 5. In Church's formalism (C)(4) the symbols $\Pi_{o(o\alpha)}$, $\Pi_{o(o\beta)}$, ..., with various suffixes, must be regarded as different minimal formulae, and to differentiate between them suffixes of a purely distinguishing kind will be placed on the left, thus, ${}_1\Pi_{o(o\alpha)}$, ${}_2\Pi_{o(o\beta)}$, ..., etc. In the untyped theory ${}_1\Pi$, ${}_2\Pi$, ... form one infinite alphabet. The specification of (C) as a (Φ, p)-system, (C, p), is (using $\sim$ and v for Church's N and A):

Invariants:	$\sim$, v, B,
A-alphabet 1 ('variables'):	$a, b, c, \ldots,$
A-alphabet 2:	${}_1\Pi$, ${}_2\Pi$, ...,
A-alphabet 3:	${}_1\iota$, ${}_2\iota$, ...

The Φ-equations and γ_i-relations are those already given (Example 3). The only basic invariant is B. The pedigrees are:

$\sim$:	$B = (\sim B)$,
v:	$X = (\mathsf{v} B), \quad B = (XB)$,
alphabet 1:	none,
alphabet 2:	$B = ({}_1\Pi X), \quad B = (XY)$,
alphabet 3:	$Y = ({}_1\iota X), \quad B = (XY)$.

The letter B has no meaning; its function is solely to regulate stratification by representing the 'propositional' stratum.

The *well-formed formulae* of the system are the solutions of a-stratified, prepared, defining sets of equations, subject to the condition that in an equation $X = (\lambda AY)$ the letter A must belong to the 'variables' alphabet, $a, b, c, \ldots$.†

THEOREM 7. *Of two alphabetically isomorphic sets of equations, both are a-stratified or neither.*

An alphabetical isomorphism between $\mathscr{E}_1$ and $\mathscr{E}_2$ can be extended to one between $[\mathscr{E}_1]$ and $[\mathscr{E}_2]$ by means of those given to exist between the pedigrees, in $[\mathscr{E}_1]$ and $[\mathscr{E}_2]$, of A-letters of $\mathscr{E}_1$ and their correlates in $\mathscr{E}_2$. The adjustment of the pedigrees ensures that this is indeed an isomorphism between $[\mathscr{E}_1]$ and $[\mathscr{E}_2]$. Hence if $[\mathscr{E}_1]$ is stratified so is $[\mathscr{E}_2]$, and since the isomorphism is an alphabetical one all invariants correspond to themselves. Hence if $[\mathscr{E}_1]$ is b-stratified $[\mathscr{E}_2]$ is also.

COROLLARY. *The property of being a-stratified is independent of the choice of the pedigrees in* $[\mathscr{E}]$.

10. *Restricted types.* The letters $\xi_1, \xi_2, \ldots$ are now regarded as *variable* types, and types in general as *functions*, types that do not contain any ξ_i are *constants*. (Any isomorphic change among the ξ_i's is considered to give the same function with different variables.) A *value* of a type-function is obtained by substituting arbitrary constant types for the variables.

A (Φ, t)-*system* is derived from a (Φ, γ)-system with A- and X-alphabets, and with r γ_i-relations, by first associating fixed primitive constant types (all different), called the *basic* types, with certain invariants; and secondly associating with each A-alphabet, $\{A\}$, an r-fold type, $\phi(A)$, composed entirely of basic types and variables. The name *type-function* is now confined to such types. An *a-correct* typing of a set of equations in a (Φ, t)-system is one that satisfies the conditions (t_1) to (t_4), and also

(t_5) every A-letter has as type a value of the type function assigned to its alphabet.

Example 6. The (Φ, t)-system, (C, t), derived from (C) has Φ-equations, alphabets and γ_i-relations as in Example 5, but instead of specifying basic invariants and pedigrees we assign types as follows, o being the only basic type:‡

B_o, $\sim_{(oo)}$, $\vee_{((oo)o)}$,

alphabet 1: type ξ_1, alphabet 2: type $(o(o\xi_1))$, alphabet 3: type $(\xi_1(o\xi_1))$.

The relation between (C, p) and (C, t) will appear in the following paragraphs.

11. In any (Φ, t)-system a *blank* typing of a set $\mathscr{E}$ is a typing, satisfying (t_1) to (t_4), in which the basic types are properly assigned, and all other lowest level-classes have types ξ_i, all different.

A (Φ, p)-system, P, and a (Φ, t)-system, T, derived from the same (Φ, γ)-system *correspond* if, first, the basic invariants in P are identical with the invariants that receive basic types in T; and secondly, each pedigree $\mathscr{P}(A)$ admits a blank typing, τ, such that $\tau(A)$ is $\phi(A)$, the function assigned to A in T. The relations between corresponding (Φ, p)- and (Φ, t)-systems are expressed in the following three theorems.

† Since such meaningless conbinations as $(\Pi\sim)$ are (for formal convenience) admitted in (C), it seems unnecessary to forbid B to be a minimal letter in the defining equations.

‡ The other special type, ι, used in Church (4), is needed only in connexion with the axioms and rules of procedure.

THEOREM 8. *If a* (Φ, p)*-system*, P, *and a* (Φ, t)*-system*, T, *correspond, a necessary and sufficient condition that a set* $\mathscr{E}$ *be a-stratified in* P *is that it admit an a-typing in* T.

THEOREM 9. *Given any* (Φ, p)*-system, a set of type-functions can be found, the association of which with the* A*-alphabets makes the underlying* (Φ, γ)*-system into a corresponding* (Φ, t)*-system.*

For the third theorem we need an additional postulate:

(A 4) there exists a set of equations $\mathscr{E}_0$ which implies $\mathbf{X}\gamma_i\mathbf{X}_i$, for $i = 1, 2, \ldots, r$, and no other γ_i-relation.

THEOREM 10. *Given any* (Φ, t)*-system satisfying* (A 4), *a set of pedigrees can be found which make the underlying* (Φ, γ)*-system into a corresponding* (Φ, p)*-system.*

Proof of Theorem 8. *Necessary.* If [$\mathscr{E}$] is b-stratified, by Corollary 2 of Theorem 4 it admits a correct typing, τ, in which the basic invariants have their basic types, and other lowest level-classes have constant types. This determines a typing of each pedigree, $\mathscr{P}(\mathbf{A})$, contained in [$\mathscr{E}$], differing from a blank typing only in that each ξ_i is replaced by a constant type $\boldsymbol{\alpha}_i$. Hence $\tau(\mathbf{A})$ is a value of $\phi(\mathbf{A})$, and the typing τ of $\mathscr{E}$ is a-correct.

Sufficient. Let τ be an a-correct typing of $\mathscr{E}$, and $\mathbf{A}$ any A-letter in $\mathscr{E}$. Let $\boldsymbol{\phi}$ be the function $\phi(\mathbf{A})$ with a definitely chosen set of variables, and let σ be that blank typing of $\mathscr{P}(\mathbf{A})$ which gives $\mathbf{A}$ the type $\boldsymbol{\phi}$. By the definition of an a-correct typing, $\tau(\mathbf{A})$ is obtained from $\boldsymbol{\phi}$ by substituting a constant type, $\boldsymbol{\alpha}_i$, for each ξ_i occurring in it. Let $\tau_\mathbf{A}$ be the correct typing of $\mathscr{P}(\mathbf{A})$ obtained from the typing σ by assigning the types $\boldsymbol{\alpha}_i$ instead of ξ_i to the lowest-level classes. Then (Theorem 4, Corollary 2) $\tau_\mathbf{A}(\mathbf{A}) = \tau(\mathbf{A})$. Let the pedigrees of all other A-letters in $\mathscr{E}$ be similarly typed. Since, in the typings τ, $\tau_\mathbf{A}$, $\tau_\mathbf{B}$, ..., of $\mathscr{E}$ and adjusted pedigrees $\mathscr{P}(\mathbf{A})$, $\mathscr{P}(\mathbf{B})$, ..., all common letters receive the same types, the combination of τ, $\tau_\mathbf{A}$, $\tau_\mathbf{B}$, ... is, by Theorem 5, a correct typing of [$\mathscr{E}$]; and hence, by Theorem 4, [$\mathscr{E}$] is stratified. Since the basic invariants receive primitive types, [$\mathscr{E}$] is b-stratified, i.e. $\mathscr{E}$ is a-stratified.

Proof of Theorem 9. Let primitive constant types, all different, be assigned to the basic invariants as basic types, and take $\phi(\mathbf{A})$ to be the type that $\mathbf{A}$ receives in a blank typing of the b-stratified set $\mathscr{P}(\mathbf{A})$. It follows immediately from the definition that the (Φ, t)-system so formed corresponds to the original (Φ, p)-system.

Proof of Theorem 10. Let the basic types be $\beta_1, \beta_2, \ldots, \beta_q$, and the invariants to which they are assigned be $B^1, B^2, \ldots, B^q$. These we declare to be the basic invariants of the (Φ, p)-system. (This fixes the meaning of 'b-stratification'.)

A rule will first be given for associating a set of equations, $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$, (possibly empty) with any X-letter, $\mathbf{X}$, and any r-fold type, $\boldsymbol{\phi}$ (constant or function), where r is the number of γ_i-relations in the system. For any $\mathbf{X}$ we take $\mathscr{E}(\mathbf{X}, \boldsymbol{\alpha})$ to be empty if $\boldsymbol{\alpha}$ is primitive. Let $\boldsymbol{\phi}_0$ be any r-fold non-primitive type and $\mathbf{X}_0$ any X-letter. We make the inductive hypothesis that, for types $\boldsymbol{\phi}$ of height † less than that of $\boldsymbol{\phi}_0$ and for any $\mathbf{X}$, $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is already defined as a b-stratified set of equations, such that (i) $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is null if, and only if, $\boldsymbol{\phi}$ is primitive; further, if $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is not null, (ii) a suitable blank

† Cf. the proof of Theorem 6.

typing of $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ gives $\mathbf{X}$ the type $\boldsymbol{\phi}$, (iii) each lowest level-class contains a single letter and (iv) all letters occurring are X-letters or basic invariants. (These are all verified for the primitive types.) Let $\boldsymbol{\phi}_0$ be $(\boldsymbol{\phi}_1 \boldsymbol{\phi}_2 \ldots \boldsymbol{\phi}_r)$. Let X-letters, $\mathbf{Y}_h$, all different, be chosen, one for each of the variables ξ_h in $\boldsymbol{\phi}_0$, and let $\mathbf{X}_i$ be r X-letters different from the $\mathbf{Y}_h$ and each other. For each i, if $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ is not null, let σ^i be a blank typing of it such that $\sigma^i\,(\mathbf{X}_i)$ is $\boldsymbol{\phi}_i$. Let the X-letters of $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ other than $\mathbf{X}_i$ be so adjusted that the type ξ_h is borne by $\mathbf{Y}_h$ only, but apart from this no two of the sets have a common X-letter, and none of them contains $\mathbf{X}_0$. Finally let $\mathscr{E}_0(\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_r)$ be a set of equations which imply $\mathbf{X}_0 \gamma_i \mathbf{X}_i$, for $i = 1, 2, \ldots, r$, and no other γ_i-relations (A 4). From the positional character of the γ_i-relations, any letter other than $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_r$ in $\mathscr{E}_0$ can be altered to $\mathbf{X}_0$ without disturbing the relations $\mathbf{X}_0 \gamma_i \mathbf{X}_i$, or introducing any new ones: this we suppose already done. The set $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$ is defined to be the union of the adjusted sets $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ $(i = 1, 2, \ldots, r)$ and the set $\mathscr{E}_0(\mathbf{X}_0, \mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_r)$, where $\mathbf{H}_i$ is $\mathbf{Y}_h$ if $\boldsymbol{\phi}_i$ is ξ_h, B^j if $\boldsymbol{\phi}_i$ is β_j, and $\mathbf{X}_i$ if it is neither. (Note that if $\mathbf{H}_i$ is B^j, $\mathscr{E}(\mathbf{X}_i, \boldsymbol{\phi}_i)$ is null, and $\mathbf{X}_i$ does not occur in $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$.)

The inductive hypothesis is true of $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$. *Parts* (i) and (iv) are evidently true. *Part* (ii). The required typing, σ^0, is the combination of the σ^i's with the types $\boldsymbol{\phi}_0$ for $\mathbf{X}_0$ and $\boldsymbol{\phi}_i$ for $\mathbf{H}_i$ $(i = 1, 2, \ldots, r)$. Since the only relations in $\mathscr{E}_0$ are $\mathbf{X}_0 \gamma_i \mathbf{H}_i$, for $i = 1, 2, \ldots, r$, $\mathscr{E}_0$ is correctly typed; and since letters common to any two sets have the same type in both, the combination of all the types is a correct typing of $\mathscr{E}(\mathbf{X}_0, \boldsymbol{\phi}_0)$ (i.e. satisfies t_1 to t_4). Since $\mathbf{H}_i$ is an invariant if, and only if, $\boldsymbol{\phi}_i$ is β_j, and since in σ^i basic types are correctly assigned, the same is true of σ^0. The typing is a 'blank' one, since two X-letters carry the same type ξ_h only if they are identical. *Part* (iii). A lowest level-class bears the type ξ_h or β_j, and contains the single letter $\mathbf{Y}_h$ or B^j respectively.

The inductive definition of $\mathscr{E}(\mathbf{X}, \boldsymbol{\phi})$ is therefore complete. We assign as pedigrees to every A-letter, $\mathbf{A}$, the set $\mathscr{E}(\mathbf{A}, \boldsymbol{\phi})$ and its alphabetical isomorphs, where $\boldsymbol{\phi}$ is $\phi(\mathbf{A})$. The conditions (P 1) and (P 2) are evidently satisfied, and it has just been proved that (P 3) holds. A (Φ, p)-system has therefore been defined, which clearly corresponds to the original (Φ, t)-system.

Example 7. If this process is applied to (C, t), (Example 6), taking $\mathscr{E}_0(\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2)$ to be $\mathbf{X}_1 = (\mathbf{X}_0 \mathbf{X}_2)$, the pedigrees found for $\sim$, $\vee$, and alphabets 1 and 2 are precisely those given in Example 5.

12. *Some generalizations.* (These are concerned with the '(Φ, γ)'-theory, without special alphabets or invariants.)

I. The single set of relations γ_i may be replaced by a finite or infinite number of sets of relations

$$\gamma_{ji} \quad (j = 1, 2, 3, \ldots; \; i = 1, 2, \ldots, r_j).$$

The assumption (A 3) must be replaced by

(A* 3) if the relation $\mathbf{X}_m \gamma_{ji} \mathbf{X}_n$ follows from $\mathbf{X}_0 = \Phi \mathbf{X}_1 \mathbf{X}_2 \ldots \mathbf{X}_k$, $\mathbf{X}_m \gamma_{jh} \mathbf{X}_{n_h}$ also follows, for the same j and $h = 1, 2, \ldots, r_j$;

and the following new assumption is required:

(A 5) if $r_j = r_k$ then $j = k$.

(This apparently rather arbitrary assumption merely expresses the fact that only if $r_j \neq r_k$ need the γ_{ji} and γ_{kh} be distinguished.) In the definition of stratification the new condition must be added, that the same $\mathbf{X}$ shall not satisfy relations $\mathbf{X}\gamma_{ji}\mathbf{Y}$ in $\mathscr{E}$ and $\mathbf{X}\gamma_{kh}\mathbf{Z}$ in $\mathscr{E}$, for $j \neq k$. In the definition of a correct typing conditions (t_2) and (t_4) are replaced by the new condition

(t^*) if $\mathbf{X}\gamma_{ji}\mathbf{Y}$, $\tau(\mathbf{X})$ has r_j factors, and $\tau_i(\mathbf{X})$ is $\tau(\mathbf{Y})$.

All the main theorems survive these changes, with little modification of the proofs. From (A* 3) and the new stratification condition it follows that in a stratified $\mathscr{E}_f$ a letter which is not at the lowest level has a single set of descendents $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{r_j}$, with $\mathbf{X}\gamma_{ji}\mathbf{Y}_i$, and hence in Theorem 4 we may take $\tau(\mathbf{X})$ to be $(\tau(\mathbf{Y}_1)\tau(\mathbf{Y}_2)\ldots\tau(\mathbf{Y}_{r_j}))$. (The condition (A 5) is used in the proof of Theorem 6.)

Example 8. In *Principia Mathematica* the stratification underlying the 'simple' theory of types (as modified by Chwistek, Ramsey, Carnap and others†) is applied only to the variables. It is not affected by the distribution of the logical constants and quantifiers, but depends only on the functional inter-relation of the letters. The relevant Φ-equations therefore have the form

$$X = f(x_1, x_2, \ldots, x_j),$$

and this gives $f\gamma_{ji}x_i$, for $i = 1, 2, \ldots, j$ $(r_j = j)$, the only γ-relations.

Another formalism in which the γ-relations fall into groups is considered in the next example.

II. A number of further possibilities are illustrated by the system developed by Quine in his *System of Logistic*. Although this system has been superseded by others in Quine's own writings, it is of interest to see how the use of pedigree equations simplifies the specification of complicated systems.‡

The Φ-equations are of the forms

$$\mathbf{X} = [\mathbf{Y}], \quad \mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}, \quad \mathbf{X} = (\mathbf{Y}, \mathbf{Z}).$$

There are three γ-relations, falling into a group of one, γ_{11} $(r_1 = 1)$, and a group of two, γ_{21} and γ_{22} $(r_2 = 2)$. Each of the equations $\mathbf{X} = [\mathbf{Y}]$ and $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$ gives $\mathbf{X}\gamma_{11}\mathbf{Y}$, and $\mathbf{X} = (\mathbf{Y}, \mathbf{Z})$ gives $\mathbf{X}\gamma_{21}\mathbf{Y}$ and $\mathbf{X}\gamma_{22}\mathbf{Z}$. In addition, a pair of equations $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$ and $\mathbf{Z} = (\mathbf{U}, \mathbf{V})$ in $\mathscr{E}$ together give $\mathbf{U}\gamma_{11}\mathbf{V}$ in $\mathscr{E}$.

Certain *equations* have pedigrees which must be added to give their 'closures' when considering stratification:

$$\mathbf{X} = [\mathbf{Y}] \text{ has the pedigree } \mathbf{Z} = \hat{\mathbf{X}}\mathbf{U},\ \mathbf{U} = (\mathbf{Y}, \mathbf{V}),$$

$$\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z} \text{ has the pedigree } \mathbf{Z} = (\mathbf{U}, \mathbf{V}).$$

The formulae admitted into Quine's calculus are the solutions of prepared defining sets of equations (the minimal formulae being the letters of a single infinite italic alphabet), such that (1) the closure is stratified, (2) in all equations $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$, $\mathbf{Y}$ is a minimal letter.

† See, e.g. Carnap (2), pp. 84 ff.

‡ The briefest specification hitherto given is in Church's review (3).

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A correct typing in our sense is identical with Quine's if $(\boldsymbol{\alpha})$ and $(\boldsymbol{\alpha\beta})$ are replaced by $\boldsymbol{\alpha}!$ and $\boldsymbol{\alpha}\uparrow\boldsymbol{\beta}$ respectively, and only one primitive type, λ, is used. For if $\mathbf{X} = [\mathbf{Y}]$ it follows from the pedigree equations that $\mathbf{Y}\,\gamma_{11}\,\mathbf{V}$, and hence that $\tau(\mathbf{Y})$ is of the form $(\boldsymbol{\alpha})$. If $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$, let $\mathbf{V}$, in the pedigree equation, have type $\boldsymbol{\alpha}$. Then $\tau(\mathbf{U})$ is $(\boldsymbol{\alpha})$, and $\tau(\mathbf{Z})$ is $((\boldsymbol{\alpha})\,\boldsymbol{\alpha})$, which is of the prescribed form for a 'propositional' formula.

The system differs from those considered in this paper, first by the presence of pedigrees of equations, secondly in that γ_{11} is not positional. It is, however, preserved under homomorphic changes, and all the main theorems remain true. Note, in particular that Theorem 5 holds. For if $\mathbf{X} = \hat{\mathbf{Y}}\mathbf{Z}$ is in $\mathscr{E}_1$ and $\mathbf{Z} = (\mathbf{U}, \mathbf{V})$ is in $\mathscr{E}_2$, the first shows that $\tau^2(\mathbf{Z}) = \tau^1(\mathbf{Z}) = ((\boldsymbol{\alpha})\,\boldsymbol{\alpha})$, and therefore from the second $\tau^2(\mathbf{U}) = (\boldsymbol{\alpha})$, $\tau^2(\mathbf{V}) = \boldsymbol{\alpha}$, as required by the relation $\mathbf{U}\,\gamma_{11}\,\mathbf{V}$, which holds in the combined set $\mathscr{E}_1 \cup \mathscr{E}_2$.

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The central problem is this: Given a time parameter t and a condition $\phi(t)$ concerning the times of excitation of the afferent neurons of a net, find a method of constructing the net so that a specified efferent neuron will fire (be in a state of excitation) at time t if and only if the condition $\phi(t)$ is satisfied. If a sufficient time interval is allowed between the firing of the afferent neurons and the firing of the efferent neuron, the required network can always be constructed without difficulty, at least if $\phi(t)$ does not involve quantifiers. This is true because the problem is easily solved for conditions $[\phi(t) \,\&\, \psi(t)]$, $[\phi(t) \,\&\, \sim\psi(t)]$, and $[\phi(t) \vee \psi(t)]$ if it can be solved for $\phi(t)$ and $\psi(t)$. Conversely the net may be already given and we may seek a condition on the afferent neurons necessary and sufficient for firing of some specified neuron of the net at a time t. This converse problem can be easily solved, the authors show, if the net does not involve neural pathways that return upon themselves. McCulloch and Pitts, however, go further and deal with networks involving such reentrant pathways and also with conditions involving quantifiers, but proper evaluation of this part of their theory is practically impossible because of numerous errors. FREDERIC B. FITCH

H. D. LANDAHL, W. S. MCCULLOCH, and WALTER PITTS. *A statistical consequence of the logical calculus of nervous nets.* Ibid., pp. 135–137.

As the title implies, this is a study of the statistical relations among the frequencies of neuron impulses and is based on the results of the paper reviewed immediately above.

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M. H. A. NEWMAN. *Stratified systems of logic.* ***Proceedings of the Cambridge Philosophical Society,*** vol. 39 (1943), pp. 69–83.

This paper is a contribution to what Carnap (IV 82) would call "general syntax," and is related to papers on "Semiotik" by students of Scholz (Hermes, IV 87, and Schröter, VIII 77, IX 20).

The author introduces, as a method in logical syntax, the use of "defining equations" for the formulas of a logical formalism (object language).

E.g., in the system of Quine's *New foundations* (II 86), the formula $(x)(y)((x\epsilon y)|(y\epsilon x))$ has the defining equations: $X = (x)Y$, $Y = (y)Z$, $Z = (U|V)$, $U = (x\epsilon y)$, $V = (y\epsilon x)$. Here the small italic letters are the letters ("minimal formulas") of Quine's system itself, and the capital italic letters are new letters introduced for the purpose. The defining equations constitute in obvious fashion an nalysis or description of the formula; and by a process of repeated substitution the equations can be "solved" for the capital letters in terms of the small letters, the "value" obtained for X being the formula itself, and the value for each of the other capital letters being one of the "segments" of the formula. —Using bold capital letters as syntactical variables (for italic small and capital letters), we have for this system that defining equations must have one of the three forms: $\mathbf{X}_1 = (\mathbf{Y}\epsilon\mathbf{Z})$, $\mathbf{X}_2 = (\mathbf{U}|\mathbf{V})$, $\mathbf{X}_3 = (\mathbf{P})\mathbf{Q}$. If a set of such equations can be solved uniquely for the italic capital letters, as above, and if this solution reveals a single "maximal letter" **X**, the set of equations is said to be a "defining set" for **X**, or a defining set of the formula obtained as a value for **X**. In order that this latter formula be "significant," or well-formed, the condition must be imposed on the defining equations severally that the letters **Y**, **Z**, **P** and those only are small italic letters (are minimal).

As another example, for the λ-K-calculus of the reviewer's ***Calculi of λ-conversion*** (VI 171), defining equations have one of the two forms, $\mathbf{X}_1 = (\mathbf{YZ})$ and $\mathbf{X}_2 = (\lambda\mathbf{PQ})$, where, for significance, the condition must be imposed that **P** be minimal. In particular the formula $(\lambda a((a(ac))(b(ab))))$ has the defining equations: $X = (\lambda aY)$, $Y = (ZU)$, $Z = (aV)$, $U = (bW)$, $V = (ac)$, $W = (ab)$.

In the two preceding examples we have particular cases of what the author calls a "Φ-system."

The author explicitly says that his italic capitals are not syntactical. But it would seem to the reviewer desirable to make the relatively minor change of construing these italic capitals as syntactical variables, namely as variables for formulas of the object language. The small italic letters in the examples above then belong to the object language, but are also used autonymously in the syntax language. And the bold capitals are metasyntactical variables, i.e., they are variables for syntactical letters.

On this basis it would be possible to make a slight reformulation and generalization of the author's definition of a Φ-system, as follows. We begin with a set of operations Φ_1, Φ_2, $\cdots$ upon an initially undefined set of objects. These objects are eventually to be identified as formulas of the object language and may therefore from the beginning be called "formulas"; italic capitals will be used as variables for them. If each Φ_i operates upon a specified number k_i of formulas, to yield a formula, and if the Φ_i's are independent in the sense that $\Phi_i X_1 X_2 \cdots X_{k_i} = \Phi_j Y_1 Y_2 \cdots Y_{k_j}$ always implies $i = j$, $X_1 = Y_1$, $X_2 = Y_2$, $\cdots$, $X_{k_i} = Y_{k_j}$, then the Φ_i's together constitute a Φ-system. If in a set of "Φ-equations," i.e., a set of equations $\mathbf{U} = \Phi_i \mathbf{U}_1 \mathbf{U}_2 \cdots \mathbf{U}_{k_i}$, $\mathbf{V} = \Phi_j \mathbf{V}_1 \mathbf{V}_2 \cdots \mathbf{V}_{k_j}$, $\cdots$, $\mathbf{Z} = \Phi_u \mathbf{Z}_1 \mathbf{Z}_2 \cdots \mathbf{Z}_{k_u}$, it is possible to distinguish some of the variables (italic capitals) as minimal in such a way that the equations can be solved uniquely for the other letters in terms of the minimal letters, and if this solution reveals a single maximal letter **X**, the set of equations is called a defining set for **X**. A "formalism" is derived from a Φ-system by specifying certain minimal formulas, the substitution of which for the minimal letters in the solutions of defining sets gives the formulas of the formalism, and also (possibly) stating other conditions to be satisfied by the defining equations in order that a formula be significant. If the minimal formulas are substituted for the minimal letters in a defining set of equations, we obtain a "prepared set." A prepared set of equations may also be spoken of as a defining set, and in the two particular examples given above it is actually prepared sets which were used.

A (Φ, γ)-system is obtained from a Φ-system by introducing a set of r binary relations γ_1, γ_2, $\cdots$, γ_r, and specifying for certain Φ_j's that from the presence of any equation $\mathbf{X}_0 = \Phi_j \mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_{k_j}$ certain γ_i-relations follow between the letters $\mathbf{X}_0, \mathbf{X}_1, \cdots, \mathbf{X}_{k_j}$; provided, first, that the relations γ_i are "positional" in the sense that, for a given j, $\mathbf{X}_m \gamma_i \mathbf{X}_n$ follows from the presence of the equation $\mathbf{X}_0 = \Phi_j \mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_{k_j}$ if and only if certain places in the equation are filled by $\mathbf{X}_m$ and $\mathbf{X}_n$, and secondly, that if $\mathbf{X}_m \gamma_i \mathbf{X}_n$ follows from the presence of $\mathbf{X}_0 = \Phi_j \mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_{k_j}$ then for every γ_h $(1 \leqq h \leqq r)$ $\mathbf{X}_m \gamma_h \mathbf{X}_{n_h}$ follows for some $\mathbf{X}_{n_h}$ $(1 \leqq n_h \leqq k_j)$. Given a set of Φ-equations, $\mathbf{X}\gamma\mathbf{Y}$ is used to mean that $\mathbf{X}\gamma_i\mathbf{Y}$ for some γ_i, and $\mathbf{X}\eta\mathbf{Y}$ is used to mean that either (1) there are two equations $\mathbf{X} = \Phi_i \mathbf{U}_1 \mathbf{U}_2 \cdots \mathbf{U}_{k_i}$, $\mathbf{Y} = \Phi_i \mathbf{U}_1 \mathbf{U}_2 \cdots \mathbf{U}_{k_i}$ with identical right-hand sides, or (2) for some **U** and some γ_i, $\mathbf{U}\gamma_i\mathbf{X}$ and $\mathbf{U}\gamma_i\mathbf{Y}$, or (3) for every γ_i there is some $\mathbf{U}_i$ such that $\mathbf{X}\gamma_i\mathbf{U}_i$ and $\mathbf{Y}\gamma_i\mathbf{U}_i$. If $\mathbf{X}\eta\mathbf{Y}$ in a given set of Φ-equations, it is an η-reduction to replace **Y** everywhere by **X**; this may give rise to new η-relations among the remaining letters, but repetitions of the process must lead finally to an η-irreducible set of equations. **X** is said to be "level" with **Y** in a given set of Φ-equations if, upon reducing the set to an η-irreducible set as just described, **X** and **Y** are replaced by the same letter. The class of letters level with **X** is denoted by $\{\mathbf{X}\}$, and is called a "level-class." A relation Γ among level-classes is introduced by the rules that if $\mathbf{X}\gamma\mathbf{Y}$ then $\{\mathbf{X}\}\Gamma\{\mathbf{Y}\}$, and that Γ is transitive—i.e., the relation Γ holds between two level-classes when and only when its doing so follows from these two rules. A set of Φ-equations, or the formula determined by a defining set of equations, is "stratified" if $\{\mathbf{X}\}\Gamma\{\mathbf{X}\}$ holds for no **X**.

For example, in the system of Quine's *New foundations*, $r = 1$, and $\mathbf{Z}\gamma_1\mathbf{Y}$ follows from the presence of an equation $\mathbf{X}_1 = (\mathbf{Y}\epsilon\mathbf{Z})$. In the λ-$K$-calculus, $r = 2$, and for $\mathbf{X}_1 = (\mathbf{YZ})$ we have $\mathbf{Y}\gamma_1\mathbf{X}_1$ and $\mathbf{Y}\gamma_2\mathbf{Z}$, while for $\mathbf{X}_2 = (\lambda\mathbf{PQ})$ we have $\mathbf{X}_2\gamma_1\mathbf{Q}$ and $\mathbf{X}_2\gamma_2\mathbf{P}$. The author applies his general theory to determination of stratification of formulas of these two formalisms—that the matter is not trivial even in the relatively simple case of Quine's system is seen from an error in this regard which was made in Quine's original paper and corrected in Bernays's review (II 86).

"Types" are the formulas of a system in which the minimal formulas are small Greek letters, and the only principle of construction is to enclose a finite row of given formulas in a pair of parentheses. Bold small Greek letters are used as syntactical variables for this system, i.e., as variables for types. A type is "primitive" if it consists of a single letter not in parentheses, and if $\boldsymbol{\alpha}$ is $(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\cdots\boldsymbol{\alpha}_k)$ then $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\cdots$, $\boldsymbol{\alpha}_k$ are the "factors" of $\boldsymbol{\alpha}$. A "k-fold type" is defined inductively to be either a primitive type or a type with k factors each of which is k-fold. A "correct typing" τ of a set of Φ-equations satisfies the four conditions: (1) each letter is assigned the same type at each of its occurrences, or else is assigned no type at any of its occurrences (the type assigned to **X** is denoted by $\tau(\mathbf{X})$ and if no type is assigned to $\mathbf{X}$ then $\tau(\mathbf{X}) = 0$, moreover the ith factor of $\tau(\mathbf{X})$ is denoted by $\tau_i(\mathbf{X})$); (2) if $\tau(\mathbf{X}) \neq 0$ then $\tau(\mathbf{X})$ is an r-fold type, where r is the number of γ_i-relations; (3) if $\mathbf{X}\gamma_i\mathbf{Y}$ then $\mathbf{X}$ and $\mathbf{Y}$ are assigned types such that $\tau(\mathbf{Y}) = \tau_i(\mathbf{X})$; (4) no type is assigned to a letter satisfying no γ_i-relation. In a formalism derived from a (Φ, γ)-system, a correct typing of a formula is obtained by assigning to its segments the same types assigned to the corresponding letters in a correct typing (if such exists) of a prepared set of defining equations.

As examples, we may apply this scheme of typing to the system of *New foundations*, using a single primitive type λ; or we may apply it to the λ-K-calculus, using any number of primitive types. In the latter case, if two primitive types ι and o are used, the typings obtained are those of the reviewer's *A formulation of the simple theory of types* (V 114).

An equation $\mathbf{X}_0 = \Phi_j\mathbf{X}_1\mathbf{X}_2\cdots\mathbf{X}_{k_j}$ is an "isolating equation" if it yields no γ_i-relation involving $\mathbf{X}_0$. A letter occurring in a set of equations is "isolated" if it satisfies no γ_i-relation. A set of equations is "non-singular" if all letters that appear on the left of isolating equations are isolated. Among other results concerning correct typing, levelness, level-classes, and stratification, it is proved that *a necessary and sufficient condition that a non-singular set of Φ-equations admit a correct typing is that it be stratified.*

The foregoing is an outline of the main points in the first eight sections of the paper, here given at greater length than usual, because of the large amount of new devices and terminology. The four remaining sections are concerned with certain extensions of the method which are necessary in order to deal with more complex formalisms. One of these extensions consists in introducing conventions according to which a given set of Φ-equations is enlarged by adding to it certain sets of equations called "pedigrees," the enlarged set of equations being then considered in defining stratification. This is necessary in particular in connection with formalisms in which certain symbols are required to have types of a prescribed kind (the system of the reviewer's *A formulation of the simple theory of types* is used as an example). Another extension consists in replacing the single set of relations γ_i by a finite or infinite number of sets, each of a finite number of γ_{ji}-relations. For the system of ***Principia mathematica,*** as modified to conform to the simple theory of types, an infinite set of sets of γ_{ji}-relations is used. For Quine's *system of logistic* (458*5*) a set of one and another set of two γ_{ji}-relations are used; it is in this case necessary also to use pedigrees; and one of the γ_{ji}-relations is not positional (but it does have the essential property of being preserved under substitution of letters, and all the main theorems remain true).

The reader's first impression of Newman's paper may be that the machinery introduced is heavy in comparison with the results obtained. The value of the paper is in fact difficult to estimate at present, as this will depend on the extent to which results obtained in the future by Newman's methods justify the weight of machinery. The reviewer would, however, venture a prediction on one point, namely that the kind of analysis of formulas which is involved in the use of defining sets of equations will for many purposes prove a more fruitful approach to "Semiotik" than the analysis of formulas as consisting of "atoms" combined by "concatenation" (as used by Tarski 285*16*, Quine I 116(3), Hermes, Schröter).

Alonzo Church

Nelson Goodman. *On the simplicity of ideas.* ***The journal of symbolic logic,*** vol. 8 (1943), pp. 107–121.

In this paper Dr. Goodman seeks to establish techniques for measuring the complexity of any non-logical set of primitive ideas. The problem is complicated by the fact that the number of distinct primitives in any such set can be reduced by standard procedures to one. Merely counting primitives will accordingly not determine the complexity of the base if every decrease in complexity, and hence every gain in economy, is to be recognized as significant. The complexity of each primitive itself must be taken into account.

This may be done as follows. Let the logical product of a class A and any cardinal number be called *a cardinal subclass of* A, and let the number of A's non-null cardinal subclasses be called *the numerical variegation of* A. The complexity of A is merely its numerical variegation when A is either a class of individuals (identified with their unit classes) or a