TYPE-THEORY vs. SET-THEORY

By

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ABSTRACT

The main purpose of this thesis is to investigate the relation between two well-known logical systems. It was my intention to make precise the idea and prove the equivalence of the simple theory of types and Zermelo set-theory. Instead of this I have succeeded in proving a strong theorem from which it follows that the two systems are not equivalent under <u>any</u> reasonable definition of "equivalent."

The relation is then considered between extensions of both systems. A natural series of stronger and stronger logical systems is presented. The problem of truth-definitions is raised and completely solved for all these systems.

Chapter 1 contains a clear statement of the problem and a summary of results. Chapter 2 contains the results concerning the two basic systems, while in Chapter 3 the series of systems is constructed and the previous results are extended to all these systems.

1. The problem.

"When I use a word," Humpty Dumpty said,
"it means just what I choose it to mean."[1]

There are two fundamentally different ways of avoiding the vicious-circle paradoxes. One method leads to the theory of types, the other one to set-theory. There are many variants of both systems. We shall study a typical system of each kind: (1) T is a system usually described as a singulary theory of types of type ω. It is a simple (not ramified) theory of types having only one-place predicates in it, but of all finite types. This system is as strong as the Russell-Whitehead system. (2) Z is a system of set theory based on Zermelo's; the main ideas of its formalization are due to Skolem.[2]

The logical system T.[3]

Primitive symbols: $x_n, y_n, \ldots$ (variables[4]) $($, $)$, $[$, $]$, $\sim$, $\supset$, $\forall$, ι $(n = 0, 1, \ldots)$

w.f.f. and terms of type n:[5] (Definition by recursion.)

1. If a_n is a variable with subscript n, then (a_n) is a term of type n.
2. If a_n is a variable with subscript n and $\mathcal{A}$ is a w.f.f., then $(\iota a_n \mathcal{A})$ is a term of type n.
3. If $\mathcal{A}$, $\mathcal{B}$ are w.f.f. and a_n is a variable, then $[\sim\mathcal{A}]$, $[\mathcal{A}\supset\mathcal{B}]$, $[\forall a_n \mathcal{A}]$ are w.f.f.

4. If A_{n_1}, B_{n_2} are terms of type n_1, n_2, and $n_2 < n_1$, then $[A_{n_1} B_{n_2}]$ is a w.f.f.
5. The sets of w.f.f. and of terms of type n are the smallest sets having all four of the above properties.

Convention:

a_n, b_n, ... are used to stand for variables with subscript n.

A_n, B_n, ... are used to stand for terms of type n.

$\mathcal{A}$, $\mathcal{B}$, ... are used to stand for w.f.f.

We introduce all the usual abbreviations. In particular we introduce the abbreviations:

$a_{n_1} = b_{n_2}$ to stand for $[C_{n_3} a_{n_1}] \equiv_{C_{n_3}} [C_{n_3} b_{n_2}]$ [6]

where $n_3 = \max(n_1, n_2) + 1$

$S^{(a_{m_1})\cdots(a_{m_k})}_{A_{m_1}\cdots A_{m_k}} \mathcal{A}|$ to stand for the result of replacing all free occurrences of the $(a_{n_i})(i=1,\ldots,k)$ by A_{m_i}, simultaneously for all i, in $\mathcal{A}$.

$n > 0$: 0_n to stand for $\iota x_n . \forall y_{n-1} . \sim [x_n y_{n-1}]$

$\{a_n, b_n\}$ to stand for $\iota C_{n+1} . [C_{n+1} d_n] \equiv_{d_n} . d_n = a_n \vee d_n = b_n$

$\langle a_n, b_n \rangle$ to stand for $\{\{a_n, a_n\}, \{a_n, b_n\}\}$.

Axiom schemata:

(1) $\mathcal{A}$ where $\mathcal{A}$ is a substitution instance of a tautology.

(2) $\mathcal{A} \supset_{a_n} \mathcal{B} \supset . \mathcal{A} \supset . \forall a_n \mathcal{B}$ a_n is not free in $\mathcal{A}$

(3) $[\forall a_n \mathcal{A}] \supset [\mathop{S}\limits^{\cdot}{}^{(a_{n_1})}_{B_{n_2}} \mathcal{A} |]$ $n_2 \leq n$, and no free variable of B_{n_2} is bound in $\mathcal{A}$.

(4) $[(b_{n+1})(a_n) \equiv_{a_n} (c_{n+1})(a_n)] \supset [b_{n+1} = c_{n+1}]$

(5*)[7] $[\exists a_n \mathcal{A}] \supset [\mathop{S}\limits^{\cdot}{}^{(a_n)}_{(\iota a_n \mathcal{A})} \mathcal{A} |]$ no variable is both free and bound in $\mathcal{A}$

(6) $[\forall a_n . \sim \mathcal{A}] \supset [(\iota a_n \mathcal{A}) = 0_n]$ $n > 0$.

(7) $[\mathcal{A} \equiv_{a_n} \mathcal{B}] \supset [(\iota a_n \mathcal{A}) = (\iota a_n \mathcal{B})]$

(8) $\exists b_{n+1} \forall a_n . [(b_{n+1})(a_n)] \equiv \mathcal{A}$ b_{n+1} not free in $\mathcal{A}$

(9) $\exists a_3 \exists a_0 . \forall b_0 [\sim [(a_3)(\langle b_0, a_0 \rangle)]] + \forall a_1 [\exists c_0$
$[(a_1)(d_0) \supset_{d_0} . (a_3)(\langle e_0, d_0 \rangle) \supset_{e_0} (a_3)(\langle e_0, c_0 \rangle)] \supset$
$\exists f_0 [[(a_3)(\langle g_0, f_0 \rangle) \equiv_{g_0} (a_1)(g_0)]]$

Rules of inference:

[I] From $\mathcal{A}$ and $[\mathcal{A} \supset \mathcal{B}]$ infer $\mathcal{B}$.
[II] From $\mathcal{A}$ infer $[\forall a_n \mathcal{A}]$.

The logical system Z.[8]

Primitive symbols: $x_n, y_n, \ldots,$ (variables) $\in, (,), [,], \sim, \supset, \forall, \iota$ $(n = 0, 1, \ldots)$

w.f.f. and terms: (Definition by recursion.)

1. If a is a variable, then (a) is a term.
2. If a is a variable and $\mathcal{A}$ a w.f.f., then $(\iota a \mathcal{A})$ is a term.
3. If $\mathcal{A}$, $\mathcal{B}$ are w.f.f. and a is a variable, then $[\sim \mathcal{A}]$, $[\mathcal{A} \supset \mathcal{B}]$, $[\forall a \mathcal{A}]$ are w.f.f.
4. If A, B are terms, then $[B \in A]$ is a w.f.f.
5. The set of w.f.f. and the set of terms are the smallest sets having all four of the above properties.

Convention:

a, b, ... used to stand for variables.
A, B, ... used to stand for terms.
$\mathcal{A}$, $\mathcal{B}$, ... used to stand for w.f.f.

We introduce all the usual abbreviations; in particular:

a = b stands for $[a \in c] \equiv_c [b \in c]$

$\mathop{S}\limits^{\cdot}{}^{(a_1) \cdots (a_K)}_{A_1 \cdots A_K} \mathcal{A} |$ stands for the result of replacing all free occurrences of (a_i) $(i = 1, \ldots, k)$ by A_i, simultaneously for all i, in $\mathcal{A}$.

0 to stand for $\iota x_0 . \forall y_0 . \sim [y_0 \in x_0]$

$\mathcal{P}(a)$ to stand for $\iota b .\, c \in b \equiv_c .\, c \subseteq a$

Axiom schemata:[10]

(1) $\mathcal{A}$ where $\mathcal{A}$ is a substitution instance of a tautology.

(2) $\mathcal{A} \supset_a \mathcal{B} \supset . \mathcal{A} \supset . \forall a \mathcal{B}$ a not free in $\mathcal{A}$

(3) $[\forall a \mathcal{A}] \supset [\mathrm{S}^{(a)}_{\mathcal{B}} \mathcal{A}|]$ no free variable of $\mathcal{B}$ bound in $\mathcal{A}$

(4) $[a \in b \equiv_a a \in c] \supset [b = c]$

(5*)[11] $[\exists a \mathcal{A}] \supset [\mathrm{S}^{(a)}_{(\iota a \mathcal{A})} \mathcal{A}|]$ no variable is free and bound in $\mathcal{A}$

(6) $[\forall a . \sim \mathcal{A}] \supset [(\iota a \mathcal{A}) = 0]$

(7) $[\mathcal{A} \equiv_a \mathcal{B}] \supset [(\iota a \mathcal{A}) = (\iota a \mathcal{B})]$

(8_1) $\forall a \exists b .\, c \in b \equiv_c .\, c \in a \,\&\, \mathcal{A}$ b not free in

(8_2) $\forall a \forall b \exists c .\, d \in c \equiv_d .\, d = a \vee d = b$

(8_3) $\forall a \exists b .\, c \in b \equiv_c \exists d .\, c \in d \,\&\, d \in a$

(8_4) $\forall a \exists b .\, c \in b \equiv_c .\, c \subseteq a$

(9) $\exists a .\, 0 \in a \,\&.\, b \in a \supset_b . [d \in c \supset_d . d \subseteq b] \supset_c c \in a$

Rules of inference.

[I] From $\mathcal{A}$ and $[\mathcal{A} \supset \mathcal{B}]$ infer $\mathcal{B}$.

[II] From $\mathcal{A}$ infer $[\forall a \mathcal{A}]$.

It has been generally believed that these two systems are equivalent. This concept is ambiguous, but the people who stated that these systems are equivalent left it ambiguous. We shall prove that these systems are not equivalent, and we will assume the burden of proving this for <u>any</u> reasonable meaning of "equivalent."

One possible meaning of equivalence was made precise by me in a previous paper.[12] This equivalence makes the concept of "equally good for formalizing mathematical systems" precise. It is a consequence of Corollary VI that T and Z are not equivalent inthe above sense; namely, there is a set of integers expressible (definable) in Z but not in T.

But the concept of equivalence often takes another form. It is assumed that there is a method of "translating" w.f.f. of one system into the other system in such a way that meaning is preserved and that theorems go into theorems and non-theorems into non-theorems. This is expressed by statements of the form: "Anything that can be proven in T can be proven in Z, and vice-versa." We will show that the first part of the statement is true, but that the "vice-versa" is false. To make this precise, we introduce a method of translation from T into Z which preserves meaning (the only possible translation according to the intended interpretations).

The translation from T to Z.

In Z we can define sets $\psi(n)$ recursively, by well known methods,[13] so that

$$\psi(0) = 0$$
$$\psi(n+1) = \mathcal{P}(\psi(n))$$

Let $\psi(\omega)$ stand for $\iota x_0 .\ y_0 \in x_0 \equiv_{y_0} \exists z_0 .\ z_0 \in \omega \ \&\ y_0 \in \psi(z_0)$

It follows from (9) that $\psi(\omega)$ has all the intended properties. (9) is actually equivalent to

(9[1]) $\psi(\omega) \neq 0$.

We define $\psi(\omega+n)$ recursively so that

$$\psi(\omega+0) = \psi(\omega)$$
$$\psi(\omega+n+1) = \mathcal{P}(\psi(\omega+n))$$

Let $\overline{a_n}$ stand for $[a_n \in \psi(\omega+n)]$.

For every term A (w.f.f. $\mathcal{A}$) of T we define a corresponding term A* (w.f.f. $\mathcal{A}$*) of Z as follows (by recursion):

1. $(a_n)^*$ is (a_n)
2. $(\iota a_n \mathcal{A})^*$ is $(\iota a_n .\ \overline{a_n}\ \&\ \mathcal{A}^*)$

3. $[\sim \mathcal{A}]^*$ is $[\sim \mathcal{A}^*]$

   $[\mathcal{A} \supset \mathcal{B}]^*$ is $[\mathcal{A}^* \supset \mathcal{B}^*]$

   $[\forall a_n \mathcal{A}]^*$ is $[\forall a_n .\ \overline{a_n} \supset \mathcal{A}^*]$
4. $[A_{n_1} B_{n_2}]^*$ is $[B^*_{n_2} \in A^*_{n_1}]$

For every w.f.f. $\mathcal{A}$ of T we define a w.f.f. $\mathcal{A}'$ of Z:

If $\mathcal{A}$ has the free variables $a_{n_1}, \ldots, a_{n_k}$, then $\mathcal{A}'$ is $[[\overline{a_{n_1}}\ \&\ \cdots\ \&\ \overline{a_{n_k}}] \supset_{a_{n_1} \ldots a_{n_k}} \mathcal{A}^*]$.
If, in particular, $\mathcal{A}$ has no free variables, then $\mathcal{A}'$ is $\mathcal{A}^*$.

$\mathcal{A}'$ is the "translation" of $\mathcal{A}$.

It will be proven (corollary II) that if $\mathcal{A}$ is a theorem of T, then $\mathcal{A}'$ is a theorem of Z. But there is a w.f.f. of T, C_T, which is not a theorem of T (assuming the consistency of T), but whose translation, C'_T, is a theorem of Z. This should be sufficient to establish that the two systems are not equivalent, but that Z is in a definite sense "stronger" than T.

But someone might object that we considered only one method of translation--even if it is the natural one. So we shall carry out a more general consideration. If there were a translation from one system to the other carrying theorem into theorem and non-theorem into non-theorem, then we could prove that one system is consistent if and only if the other one is; we shall call this equiconsistency. Equiconsistency seems like a minimum requirement for equivalence. By equiconsistency one usually means a proof in an elementary system (a system just strong enough to serve as a syntax language) that the consistency of either system implies the consistency of the other. Not only is this impossible, but we shall prove the stronger result that equiconsistency cannot even be proven in the strong system T, unless T and Z are both inconsistent (Corollary V). We may sum this up by saying that T and Z are not equivalent in any sense, unless they are both inconsistent--which is hardly what was meant by the people who believed these systems to be equivalent.

On the contrary, it is shown that Z is stronger than T both in the sense of our being able to prove more theorems in Z, and of being able to define more mathematical entities (e.g., sets of integers) in Z. On the other hand, since the consistency of Z implies the consistency of T (corollary 4), but not vice-versa, we may say that T is a "safer" system. It is now interesting to see which of the theorems which have been proved in Z can also be proved in T.

The main "tool" used in these proofs is a truth-definition for T given within Z. The fact that this is possible already shows that Z is stronger than T.[14]

For the sake of completeness, these results are extended to a series of systems. We arrive at a natural series of transfinite type- (and set-) theories, each one of which (from a certain system on) is sufficiently stronger than all previous systems to allow a truth-definition for all the previous systems. These results give us an insight into the relation between the extensions of T and Z. They also give us an elegant systematic method of introducing stronger and stronger logical systems.

2. T and Z.

"And if you take 1 from 365, what remains?"
"364, of course."
Humpty Dumpty looked doubtful.
"I'd rather see that done on paper," he said.[15]

We now proceed to give a truth-definition for T in Z. Truth is defined by means of a concept of satisfaction. Therefore, our first task is a formal definition of satisfaction.

By Gödel's method we assign an integer to each w.f.f. of T. W_m will stand for the m^{th} w.f.f. of T. Thus each w.f.f. of T is represented in Z by an integer. But while W_m is represented by m, the proposition W_m is expressed in Z by W'_m. These few remarks will make the meaning of the theorems to be proved clear. We enumerate the variables. Let v_k stand for the k^{th} variable.

The definition of satisfaction, and later of truth, will be given in the system Z. We shall define a set Tr, such that $[m \in Tr]$ expresses in Z that W_m is true. We shall define a relation of satisfaction on level k first. It is a relation between a k-tuple x, and an integer m. x satisfies m if W_m has no other variables than $v_1, \ldots, v_k$, if x is a k-tuple so constructed that its i^{th} member $(i = 1, \ldots, k)$ is of the set corresponding to the type of v_i, and if putting the

i^{th} member of x for every free occurrence of v_i makes W^*_m true. This corresponds to our intuitive notion of satisfaction, except for the complications caused by the parameter k. But without this additional parameter the definition is not possible.

A relation in Z is expressed by a set of ordered pairs. Satisfaction on level k will be expressed by the set of ordered pairs S(k). I.e., "x satisfies m" is expressed by "$\langle x, m \rangle \in S(k)$." S(k) must be defined recursively. There are seven recursion conditions corresponding to the different ways of forming w.f.f. S(k) is defined as the smallest set satisfying all these conditions. This is done by the usual method: we define S(k) as the set such that y belongs to it if and only if y belongs to every set satisfying the seven recursion conditions.

The formal definition is much too lengthy to be given, unless we introduce abbreviations. On the following pages we give a list of abbreviations which will be used throughout this chapter. We make use of the well-known metatheorem that every primitive recursive function and relation is calculable in the system Z.[16] Thus we feel free to introduce abbreviations for the w.f.f. expressing such functions or relations without actually writing down these w.f.f.

Abbreviations.

Terms:[17]

T_l	stands in Z for	the type of v_l.
$\Sigma(k)$	" " "	the set of k-tuples whose l^{th} member is an element of $\Psi(\omega + T_l)$[18]
K_m	" " "	the highest k such that v_k occurs in W_m.
$M(l, k, x)$	" " "	the l^{th} member of the k-tuple x.
$D(k, x, l, t)$	" " "	the element of $\Sigma(k)$ which we get by replacing the l^{th} member of x by t.
$Neg(m)$	" " "	the no. of $[\sim W_m]$
$Eq(l_1, l_2)$	" " "	the no. of $[(v_{l_1}) = (v_{l_2}$

W.f.f.:[19]

$\epsilon_1(m, l_1, l_2)$	expresses in Z that	W_m is $[(v_{l_1})(v_{l_2})]$
$\epsilon_2(m, l_1, m_1, l_2)$	" " " "	W_m is $[(\iota v_{l_1} W_{m_1})(v_{l_2})]$
$\epsilon_3(m, l_1, m_1, l_2)$	" " " "	W_m is $[(v_{l_1})(\iota v_{l_2} W_m$
$\epsilon_4(m, l_1, m_1, l_2, m_2)$	" " " "	W_m is $[(\iota v_{l_1} W_{m_1})(\iota v_{l_2}$
$\sim(m, m_1)$	" " " "	W_m is $[\sim W_{m_1}]$
$\supset(m, m_1, m_2)$	" " " "	W_m is $[W_{m_1} \supset W_{m_2}]$
$\forall(m, l_1, m_1)$	" " " "	W_m is $[\forall v_{l_1} W_{m_1}]$

$B_T(k, m)$	expresses in T that	k is the no. of a proof of W_m in T
$Bew_T(m)$	" " "	$\vdash_T W_m$
$Bew_Z(m)$	" " " "	$\vdash_Z W_m'$
C_T	" " " "	T is consistent
C_Z	" " " "	Z is consistent
$Fr(l, m)$	expresses in Z that	v_l is free in W_m
$Bd(l, m)$	" " " "	v_l is bound in W_m
$S(m, l_1, l_2, m_1)$	" " " "	W_m is $S \frac{(v_{l_1})}{(v_{l_2})} W_{m_1}\vert$
$S(m, l_1, l_2, m_1, m_2)$	" " " "	W_m is $S \frac{(v_{l_1})}{(\iota v_{l_2} W_{m_1})} W_{m_2}\vert$
$Ac(m, l_1, l_2, m_1)$	" " " "	W_m is gotten from W_{m_1} by changing the bound variable v_{l_1} to v_{l_2}, and that v_{l_1} is not free in W_{m_1}, and that v_{l_2} does not occur in W_{m_1}.

$Rec_1^k z$ stands for $[[\exists l_1 \exists l_2 . \epsilon_1(m_1, l_1, l_2) \& . M(l_2, k, x_1) \in M(l_1, k, x_1)] \supset J_1 \in z]$

$Rec_2^k z$ " " $[[\exists l_1 \exists l_2 \exists m_2 . \epsilon_2(m_1, l_1, m_2, l_2) \& . M(l_2, k, x_1) \in (\iota t[\langle D(k, x_1, l_1, t), m_2\rangle \in z])] \supset J_1 \in z]$

$\mathrm{Rec}^k_3 z$ stands for $[[\exists \ell_1 \exists \ell_2 \exists m_2 . \epsilon_3(m_1, \ell_1, m_2, \ell_2)$
$\&. (\iota t[\langle D(k, x_1, \ell_2, t), m_2\rangle \in z]) \in$
$M(\ell_1, k, x_1)] \supset \mathfrak{z}_1 \in z]$

$\mathrm{Rec}^k_4 z$ " " $[[\exists \ell_1 \exists \ell_2 \exists m_2 \exists m_3 . \epsilon_4(m_1, \ell_1, m_2,$
$\ell_2, m_3) \&. (\iota t[\langle D(k, x_1, \ell_2, t), m_3 \in z])$
$(\iota w[\langle D(k, x_1, \ell_1, w), m_2\rangle \in z])] \supset \mathfrak{z}_1 \in$

$\mathrm{Rec}^k_5 z$ " " $[[\exists m_2 . \sim(m_1, m_2) \& \sim[\langle x_1, m_2\rangle \in z]] \supset \mathfrak{z}_1 \in$

$\mathrm{Rec}^k_6 z$ " " $[[\exists m_2 \exists m_3 . \supset(m_1, m_2, m_3) \&. [\langle x_1, m_2\rangle$
$\in z] \supset [\langle x_1, m_3\rangle \in z]] \supset \mathfrak{z}_1 \in z]$

$\mathrm{Rec}^k_7 z$ " " $[[\exists \ell_1 \exists m_2 . \forall(m_1, \ell_1, m_2) \&. t \in$
$\Psi(\omega + T\ell_1) \supset_t . \langle D(k, x_1, \ell_1, t), m_2\rangle \in z] \supset \mathfrak{z}_1 \in$

$\mathrm{Rec}^k z$ " " $[[\mathfrak{z}_1 = \langle x_1, m_1\rangle \& x_1 \in \Sigma(k) \& K_{m_1} \leq k$
$\supset_{\mathfrak{z}_1 x_1 m_1} . \mathrm{Rec}^k_1 z \& \mathrm{Rec}^k_2 z \& \cdots \& \mathrm{Rec}^k_7$

$S(k)$ " " $(\iota v: \mathfrak{z} \in v \equiv_{\mathfrak{z}} . \mathfrak{z} \in (\Sigma(k) \times \omega) \&.$
$\mathrm{Rec}^k z \supset_z \mathfrak{z} \in z)$

By the subset and description axioms:[20]

$$\vdash \mathfrak{z} \in S(k) \equiv_{\mathfrak{z}} . \mathfrak{z} \in (\Sigma(k) \times \omega) \&. \mathrm{Rec}^k z \supset_z \mathfrak{z} \in z$$

By the usual methods of recursive definition, we get seven theorems corresponding to the seven recursion conditions:

Rec 1: $\vdash [k \in \omega \& \epsilon_1(m, \ell_1, \ell_2)] \supset . \langle x, m\rangle \in S(k) \equiv_x .$
$K_m \leq k \&. x \in \Sigma(k) \& [M(\ell_2, k, x) \in M(\ell_1, k, x)]$

Rec 2: $\vdash [k \in \omega \& \epsilon_2(m, \ell_1, m_1, \ell_2)] \supset . \langle x, m\rangle \in S(k) \equiv_x .$
$K_m \leq k \&. x \in \Sigma(k) \& [M(\ell_2, k, x) \in (\iota t [$
$\langle D(k, x, \ell_1, t), m_1\rangle \in S(k)])]$

Rec 3: $\vdash [k \in \omega \& \epsilon_3(m, \ell_1, m_1, \ell_2)] \supset . \langle x, m\rangle \in S(k) \equiv_x .$
$K_m \leq k \&. x \in \Sigma(k) \& [(\iota t[\langle D(k, x, \ell_2, t),$
$m_1\rangle \in S(k)]) \in M(\ell_1, k, x)]$

Rec 4: $\vdash [k \in \omega \& \epsilon_4(m, \ell_1, m_1, \ell_2, m_2)] \supset . \langle x, m\rangle \in S(k) \equiv_x .$
$K_m \leq k \&. x \in \Sigma(k) \& [(\iota t[\langle D(k, x, \ell_2, t), m_2\rangle$
$\in S(k)]) \in (\iota w[\langle D(k, x, \ell_1, t), m_1\rangle \in S(k)])]$

Rec 5: $\vdash [k \in \omega \& \sim(m, m_1)] \supset . \langle x, m\rangle \in S(k) \equiv_x .$
$K_m \leq k \&. x \in \Sigma(k) \& \sim [\langle x, m_1\rangle \in S(k)]$

Rec 6: $\vdash [k \in \omega \& \supset(m, m_1, m_2)] \supset . \langle x, m\rangle \in S(k) \equiv_x .$
$K_m \leq k \&. x \in \Sigma(k) \& [\langle x, m_1\rangle \in S(k) \supset . \langle x, m_2\rangle \in S(k$

Rec 7: $\vdash [k \in \omega \& \forall(m, \ell_1, m_1)] \supset . \langle x, m_1\rangle \in S(k) \equiv_x .$
$K_m \leq k \&. x \in \Sigma(k) \& [t \in \Psi(\omega + T\ell_1) \supset_t .$
$\langle D(k, x, \ell_1, t), m_1\rangle \in S(k)]$

Metatheorems about Z will be proved in English, but in such a way that they are formalizable in any system adequate for Arithmetic (e.g., T). They will be numbered by Roman numerals.

Lemma I. For every pair of integers k, m,

$$\vdash \langle X, m\rangle \in S(k) \equiv_X . X \in \Sigma(k) \,\&.\, K_m \leq k \,\&. \underset{\cdot}{S}\, {}^{(v_1)}_{M(1,k,X)} \cdots {}^{(v_k)}_{M(k,k,X)} W^*_m \Big|$$ [21]

Proof: By the length of a w.f.f. we understand the number of occurrences of $\sim$, $\supset$, $\forall$, and ι. Proof is by induction on the length of W_m.

Length = 0. W_m is $[(v_{1_1})(v_{1_2})]$.

$\vdash \in_1(m, \ell_1, \ell_2)$

W^*_m is $(v_{1_2}) \in (v_{1_1})$

Lemma follows immediately from Rec. 1.

Assume for length $\leq$ s.

Length = s + 1. There are six cases.

Case 1., W_m is $\sim W_{m_1}$.

$\vdash \sim (m, m_1)$

W_{m_1} has length s, hence by assumption:

$$\vdash \langle X, m_1\rangle \in S(k) \equiv_X . X \in \Sigma(k) \,\&\, K_{m_1} \leq k \,\&. \underset{\cdot}{S}\, {}^{(v_1)}_{M(1,k,X)} \cdots W^*_{m_1} \Big|$$

$$\vdash \langle X, m\rangle \in S(k) \equiv_X . X \in \Sigma(k) \,\&.\, K_m \leq k \,\&.\, \sim[\langle X, m_1\rangle \in S(k)]$$ (By Rec. 5)

$$\vdash K_m \leq k \supset K_{m_1} \leq k$$

$$\vdash \langle X, m\rangle \in S(k) \equiv_X . X \in \Sigma(k) \,\&.\, K_m \leq k \,\&. \sim[\underset{\cdot}{S}\, {}^{(v_1)}_{M(1,k,X)} \cdots W^*_{m_1} |]$$

But $\sim[\underset{\cdot}{S}\, {}^{(v_1)}_{M(1,k,X)} \cdots W^*_{m_1} |]$ is the same as $\underset{\cdot}{S}\, {}^{(v_1)}_{M(1,k,X)} \cdots [\sim W^*_{m_1}] |$ and $[\sim W^*_{m_1}]$ is W^*_m.

Hence lemma.

Case 2., W_m is $[W_{m_1} \supset W_{m_2}]$

Proof exactly analogous, only it uses Rec. 6 in place of Rec. 5.

Case 3., W_m is $[\forall v_{l_1} W_{m_1}]$

$\vdash \forall(m, l_1, m_1)$

W_{m_1} has length s, hence

$$\vdash \langle x, m_1\rangle \in S(k) \equiv_x . x\in\Sigma(k) \supset . K_m \le k \supset . \; \mathsf{S}^{(v_1)\cdots}_{M(1,k,x)\cdots} W^*_{m_1}|$$

$$\vdash \langle x, m\rangle \in S(k) \equiv_x . x\in\Sigma(k) \supset . K_m \le k \supset .$$

$$t\in\Psi(\omega+T_{l_1}) \supset_t . \langle D(k,x,l_1,t), m_1\rangle \in S(k)$$ (By Rec. 7)

$$x\in\Sigma(k),\ K_m\le k,\ t\in\Psi(\omega+T_{l_1}) \vdash K_{m_1}\le k$$

" $\vdash D(k,x,l_1,t)\in\Sigma(k)$

" $\vdash \langle D(k,x,l_1,t), m_1\rangle \in S(k)\equiv . \; \mathsf{S}^{(v_1)\cdots}_{M(1,k,D)\cdots} W^*_{m_1}|$ [22]

" $\vdash \langle D(k,x,l_1,t), m_1\rangle \in S(k)\equiv . \; \mathsf{S}^{(v_1)\cdots(v_{l_1})\cdots(v_k)}_{M(1,k,x)\cdots(t)\cdots M(k,k,x)} W^*_{m_1}|$

$$\vdash \langle x, m\rangle \in S(k) \equiv_x . x\in\Sigma(k) \supset . K_m \le k \supset . \; t\in\Psi(\omega+T_{l_1}) \supset_t . \; \mathsf{S}^{(v_1)\cdots(v_{l_1})\cdots(v_k)}_{M(1,k,x)\cdots(t)\cdots M(k,k,x)} W^*_{m_1}|$$

$$\vdash \langle x, m\rangle \in S(k) \equiv_x . x\in\Sigma(k) \supset . K_m \le k \supset . \; v_{l_1}\in\Psi(\omega+T_{l_1}) \supset_{v_{l_1}} . \; \mathsf{S}^{(v_1)\cdots(v_{l_1})\cdots(v_k)}_{M(1,k,x)\cdots(v_{l_1})\cdots M(k,k,x)} W^*_{m_1}|$$

$$\vdash \langle x, m\rangle \in S(k) \equiv_x . x\in\Sigma(k) \supset . K_m \le k \supset . \; \mathsf{S}^{(v_1)\cdots}_{M(1,k,x)\cdots} W^*_{m}|$$ (Since v_{l_1} is not free in W^*_m.)

Hence lemma.

The last three cases are so similar that we prove only a typical one.

Case 4., W_m is $[(\iota v_{l_1} W_{m_1})(v_{l_2})]$

Proof similar to case 5. Using Rec. 2.

Case 5., W_m is $[(v_{l_1})(\iota v_{l_2} W_{m_1})]$

$\vdash \epsilon_3(m, l_1, m_1, l_2)$

W_{m_1} has length s, hence

$$\vdash \langle x, m_1\rangle \in S(k) \equiv_x . x\in\Sigma(k) \supset . K_{m_1} \le k \supset . \; \mathsf{S}^{(v_1)\cdots}_{M(1,k,x)\cdots} W^*_{m_1}|$$

$\vdash \langle x, m\rangle \in S(k) \equiv_x . x\in\Sigma(k)$ (By Rec. 3.) $\supset . K_m \le k \supset . (\iota t[\langle D(k,x,l_2,t), m_1\rangle \in S(k)]) \in M(l_1,k,x)$

$X \in \Sigma(k),\ K_m \leqslant k \vdash K_{m_1} \leqslant k$

" $\vdash D(k, X, \ell_2, t) \in \Sigma(k) \equiv_t t \in \Psi(\omega + T_{\ell_2})$

" $\vdash \langle D, m_1 \rangle \in S(k) \equiv .\ t \in \Psi(\omega + T_{\ell_2}) \wedge .$
$\underset{\cdot}{S}^{(v_1)}_{M(1,k,X)} \cdots {}^{(v_{\ell_2})}_{(t)} \cdots {}^{(v_k)}_{M(k,k,X)} W^{*}_{m_1} |$

" $\vdash \langle D, m_1 \rangle \in S(k) \equiv \underset{\cdot}{S}^{(v_1)}_{M(1,k,X)} \cdots {}^{(v_{\ell_2})}_{(t)} \cdots$
${}^{(v_k)}_{M(k,k,X)} [v_{\ell_2} \in \Psi(\omega + T_{\ell_2}) \wedge W^{*}_{m_1}] |$

" $\vdash (\iota t [\langle D, m_1 \rangle \in S(k)]) = (\iota t [\underset{\cdot}{S}^{(v_1)}_{M(1,k,X)} \cdots$
${}^{(v_{\ell_2})}_{(t)} \cdots {}^{(v_k)}_{M(k,k,X)} [v_{\ell_2} \in \Psi(\omega + T_{\ell_2}) \wedge W^{*}_{m_1}] |$
(By ext. of descr. axiom)

" $\vdash (\iota t [\langle D, m_1 \rangle \in S(k)]) = (\iota v_{\ell_2} [\underset{\cdot}{S}^{(v_1)}_{M(1,k,X)}$
$\cdots {}^{(v_{\ell_2})}_{(v_{\ell_2})} \cdots {}^{(v_k)}_{M(k,k,X)} [v_{\ell_2} \in \Psi(\omega + T_{\ell_2}) \wedge W^{*}_{m_1}] |$

" $\vdash (\iota t [\langle D, m_1 \rangle \in S(k)]) = \underset{\cdot}{S}^{(v_1)}_{M(1,k,X)} \cdots$
${}^{(v_k)}_{M(k,k,X)} (\iota v_{\ell_2} W^{*}_{m_1}) |$ (Since v_{1_2} not free in last term.)

$\vdash \langle X, m \rangle \in S(k) \equiv_X .\ X \in \Sigma(k) \wedge .\ K_m \leqslant k \wedge .\ \underset{\cdot}{S} \because\because (\iota v_{\ell_2} W_{m_1})^{*}$
$\in M(\ell_1, k, X)$

$\vdash \langle X, m \rangle \in S(k) \equiv_X .\ X \in \Sigma(k) \wedge .\ K_m \leqslant k \wedge .$
$\underset{\cdot}{S} \because\because [(\iota v_{\ell_2} W_{m_1})^{*} \in (v_{\ell_1})^{*}] |$

Hence lemma.

Case 6., W_m is $[(\iota v_{1_1} W_{m_1})(\iota v_{1_2} W_{m_2})]$

Proof similar to case 5. Using Rec. 4.

Hence lemma follows by induction.

Q. E. D.

T_r is $(\iota z.\ z \in z \equiv_z .\ z \in \omega \wedge .\ X \in \Sigma(K_z) \supset_X .$
$\langle X, z \rangle \in S(K_z))$[23]

By subset and description axioms:

$\vdash z \in T_r \equiv_z .\ z \in \omega \wedge .\ X \in \Sigma(K_z) \supset_X .\ \langle X, z \rangle \in S(K_z)$

<u>Theorem I.</u> For every integer m, $\vdash m \in T_r \equiv W'_m$.

Proof: $\vdash K_m \leqslant K_m$ (m is now a fixed integer)

$\vdash \langle X, m \rangle \in S(K_m) \equiv_X .\ X \in \Sigma(K_m) \wedge .$

$\underset{\cdot}{S}^{(v_1)}_{M(1,K_m,X)} \cdots {}^{(v_{K_m})}_{M(K_m,K_m,X)} W^{*}_{m} |$ (lemma I.)

Let $v_{1_1}, \ldots, v_{1_k}$ be the free variables of W_m.

Let $\mathcal{A}(x, v_1, \ldots, v_{K_m})$ express that x is the K_m-tuple whose i^{th} member is v_2; x is a variable not yet used.

$$m \in Tr,\ [\overline{v_1} \cdot \ldots \cdot \overline{v_{K_m}}],\ A(x, v_1, \ldots, v_{K_m}) \vdash x \in \Sigma(K_m)$$

$$'' \qquad \vdash \langle x, m \rangle \in S(K_m)$$

$$'' \qquad \vdash \underset{M(1,K_m,x)}{\overset{(v_1)}{S}} \cdots \underset{M(K_m,K_m,x)}{\overset{(v_{K_m})}{\cdots}} W^*_m|$$

$$'' \qquad \vdash W^*_m$$

$$m \in Tr,\ [\overline{v_1} \cdot \ldots \cdot \overline{v_{K_m}}] \vdash [\exists x\, A(x, v_1, \ldots, v_{K_m})] \supset W^*_m$$

$$'' \qquad \vdash W^*_m$$

$$m \in Tr \quad \vdash \quad [\overline{v_1} \cdot \ldots \cdot \overline{v_{K_m}}] \supset_{v_1 \ldots v_{K_m}} W^*_m$$

$$'' \qquad \vdash \exists v_1 \cdots \exists v_{K_m} [\overline{v_1} \cdot \ldots \cdot \overline{v_{K_m}}]$$

$$'' \qquad \vdash [\overline{v_{e_1}} \cdot \ldots \cdot \overline{v_{e_K}}] \supset_{v_{e_1} \ldots v_{e_K}} W^*_m$$

(Since only these are free in W^*_m.)

$$\vdash m \in Tr \supset W'_m$$

$$W'_m,\ x \in \Sigma(K_m) \vdash M(\ell_i, K_m, x) \in \Psi(\omega + Te_i) \qquad 1 \le i \le$$

$$'' \qquad \vdash \underset{M(\ell_1,K_m,x)}{\overset{(v_{e_1})}{S}} \cdots \underset{M(\ell_K,K_m,x)}{\overset{(v_{e_K})}{\cdots}} W^*_m|$$

$$'' \qquad \vdash \underset{M(1,K_m,x)}{\overset{(v_1)}{S}} \cdots \underset{M(K_m,K_m,x)}{\overset{(v_{K_m})}{\cdots}} W^*_m$$

(Since others are not free in W^*_m.)

$$W'_m,\ x \in \Sigma(K_m) \vdash \langle x, m \rangle \in S(K_m) \qquad \text{(See above)}$$

$$W'_m \vdash m \in Tr$$

$$\therefore \vdash m \in Tr \equiv W'_m$$

Q. E. D.

We shall prove only one typical one of the corollaries of Thm. I. (For others see Tarski.)[24]

Corollary I. For every m, if W_m has no free variables,

$$\vdash [m \in Tr \ \vee\ Neg(m) \in Tr]$$

Proof: For fixed m, such that W_m has no free vars.

$$\vdash W'_m \ \vee\ [\sim W'_m]$$

Since W_m has no free variables,

$[\sim W'_m]$ is $W'_{Neg(m)}$

$$\vdash [W'_m \ \vee\ W'_{Neg(m)}]$$

$$\vdash m \in Tr \equiv W'_m \qquad \text{(Thm. I.)}$$

$$\vdash Neg(m) \in Tr \equiv W'_{Neg(m)} \qquad \text{(Thm. I.)}$$

$$\therefore \vdash [m \in Tr \ \vee\ Neg(m) \in Tr]$$

Q. E. D.

Before proving the next theorem, it is convenient to put down a few lemmas. The first three are lemmas about Z which follow directly from the axioms.

Lemma II. For every w.f.f. $\mathcal{A}$, if a not free in $\mathcal{A}$, and $k_1 > k_2$, $\vdash k_1 \in \omega \,\&\, k_2 \in \omega \supset . \exists a .$

$$a \in \Psi(\omega + k_1) \,\&. \, b \in \Psi(\omega + k_2) \supset_b . \, b \in a \equiv \mathcal{A}$$

Proof: $\exists a . \, b \in a \equiv_b . \, b \in \Psi(\omega + k_2) \,\&\, \mathcal{A}$ (Subset axiom)

Since this a is a subset of $\Psi(\omega + k_2)$, if $k_1 \in \omega$, and $k_2 \in k_1$, then $a \in \Psi(\omega + k_1)$.

Lemma follows immediately.

Q. E. D.

Lemma III. For every term A, w.f.f. $\mathcal{A}$, if a not free in either one, and no variable free and bound in $\mathcal{A}$, $\vdash c \in (\iota a [b \in a \equiv_b . \, b \in A$

$$\&\, \mathcal{A}]) \equiv_c . \, c \in A \,\&\, \dot{S}^{(a)}_{(c)} \mathcal{A}| .$$

Proof: $\exists a . \, b \in a \equiv_b . \, b \in A \,\&\, \mathcal{A}$ (Subset axiom)

Lemma follows by description axiom.

Q. E. D.

Lemma IV. $(\iota a \mathcal{A}) \neq 0 \supset . \, \dot{S}^{(a)}_{(\iota a \mathcal{A})} \mathcal{A}|$ if no variable free and bound in $\mathcal{A}$.

Immediate by description axiom.

Q. E. D.

The following lemmas are formal theorems about S(k), Tr.

Lemma 5. $\vdash k \in \omega \supset . \langle X, m \rangle \in S(k) \supset . \, X \neq 0 .$

Proof: $k \in \omega \quad \vdash \sim . \, 0 \in \Sigma(k)$

$\quad\text{"}\quad \vdash \langle X, m \rangle \in S(k) \supset . \, X \in \Sigma(k)$

$\quad\text{"}\quad \vdash \langle X, m \rangle \in S(k) \supset . \, X \neq 0$

Q. E. D.

The remaining lemmas express obvious facts about S(k), Tr. However, their proofs require (formal) inductions on the length of W_m, which would take up too much space. The reader will find no difficulty in supplying the proofs, if he wishes to try it.

Lemma 6. $\vdash k \in \omega \supset . \sim \mathfrak{Fr}(l, m) \supset . \, t \in \Psi(\omega + T_l) \supset .$

$$\langle X, m \rangle \in S(k) \equiv_X . \langle D(k, X, l, t), m \rangle \in S(k).$$

Lemma 7. $\vdash \dot{S}(m, l_1, l_2, m_1) \supset . \, K_m \leq k \,\&\, K_{m_1} \leq k \supset .$

$$\sim \mathfrak{Bd}(l_2, m_1) \supset . \langle X, m \rangle \in S(k) \equiv_X .$$

$$\langle D(k, X, l_1, M(l_2, k, X)), m_1 \rangle \in S(k).$$

Lemma 8. $\vdash \underset{\cdot}{S}(m, l_1, l_2, m_1, m_2) \supset . K_m \leq k \cdot . K_{m_2} \leq k \supset . [\mathfrak{Fr}(l, m_1) \supset_l \sim \mathfrak{Bd}(l, m_2)] \supset . \langle X, m \rangle \in S(k) \equiv_X . \langle D(k, X, l_1, (\iota t[\langle D(k, X, l_2, t), m_1 \rangle \in S(k)])), m_2 \rangle \in S(k)$

Lemma 9. $\vdash m_1 \in Tr \supset . Ac(m, l_1, l_2, m_1) \supset m \in Tr$

Lemma 10. $\vdash k \in \omega \supset . K_m \leq k \supset . m \in Tr \equiv . X \in \Sigma(k) \supset_X . \langle X, m \rangle \in S(k)$

Lemma 11. $\vdash l \in \omega \supset . k \in \omega \supset . (\iota t[\langle D(k, x, l, t), m \rangle \in S(k)]) \in \Psi(\omega + T_l)$.

(If there is such a t, by definition of $\Sigma(k)$, description axiom, and Lemma 5; if not then it is obvious since $0 \in \Psi(\omega + T_l)$.)

We now proceed to prove the second theorem. This is a formal theorem of Z, expressing that every theorem of T is true. Since the proof is very long, it is convenient to use the following trick: In several places we shall use the English language as a substitute for Z. It will always be done in such a manner that it is clear what its formal analogue is. We shall say "if $\mathcal{A}$ then $\mathcal{B}$ " in place of "$[\mathcal{A} \supset \mathcal{B}]$," "A is an element of B" in place of "$A \in B$," etc. This is,

of course, only a method for introducing abbreviations in a systematic method without having to state explicitly what the abbreviations stand for. This method has often been used to great advantage in the literature.[25] It not only shortens the proof considerably, but it enables the reader to concentrate on the essentials of the proof rather than on the symbolism.

Theorem 2. $\vdash \mathfrak{Bew}'_T(m) \supset_m m \in Tr$.

Proof: We shall show first that if W_m is an axiom of T, then $m \in Tr$.

Next we shall show that if $m_1 \in Tr$, $(m_2 \in Tr)$ and W_m follows from W_{m_1} (and W_{m_2}) by a rule of T, then $m \in Tr$.

Hence every step W_m in a proof of T must be such that $m \in Tr$ (by induction). Hence the theorem will follow.

We shall make repeated uses of Rec 1-Rec 7. It will always be in cases where $k = K_m$ or at least it is clear that $K_m \leq k$, hence this clause will be omitted. Similarly for clauses like $m \in \omega$.

Case 1., W_m is a substitution instance of a tautology.

Say W_m is $\underset{\cdot}{S}\ {}^{p_1\ \ldots\ p_k}_{W_{m_1} \ldots W_{m_k}}\ \mathfrak{T}|$,

where $\mathfrak{T}$ is a tautology in $p_1, \ldots, p_k$.

$$\langle X, m\rangle \in S(K_m) \equiv_X .\ X \in \Sigma(K_m) \supset .$$
$$\underset{\cdot}{S}\ {}^{p_1 \qquad \ldots \qquad p_k}_{\langle X, m_1\rangle \in S(K_m) \ldots \langle X, m_k\rangle \in S(K_m)}\ \mathfrak{T}| \quad \text{(By Rec 5 and 6 repeated.)}$$

The last clause is a substitution instance of a tautology, hence a theorem.

$$\therefore\ X \in \Sigma(K_m) \supset_X .\ \langle X, m\rangle \in S(K_m)$$

$$\therefore\ m \in Tr$$

Case 2., W_m is $W_{m_1} \supset_{v_{1_1}} W_{m_2} \supset .\ W_{m_1} \supset .\ \forall v_{1_1} W_{m_2}$

and $\sim \mathfrak{Fr}(1_1, m_1)$.

Let W_{m_3} be $W_{m_1} \supset_{v_{2_1}} W_{m_2}$;

W_{m_4} be $\forall v_{2_1} W_{m_2}$;

W_{m_5} be $W_{m_1} \supset W_{m_2}$.

$$\langle X, m\rangle \in S(K_m) \equiv_X .\ X \in \Sigma(K_m) \supset .$$

$$\langle X, m_3\rangle \in S(K_m) \supset .\ \langle X, m_1\rangle \in S(K_m) \supset .\ \langle X, m_4\rangle \in S(K_m) \quad \text{(By Rec 6 repeated)}$$

Suppose $X \in \Sigma(K_m)$, $\langle X, m_3\rangle \in S(K_m)$, $\langle X, m_1\rangle \in S(K_m)$,

then $t \in \Psi(\omega + T_{e_1}) \supset_t .$

$\langle D(K_m, X, l_1, t), m_5\rangle \in S(K_m)$ (Rec 7)[26]

" $t \in \Psi(\omega + T_{e_1}) \supset_t .\ \langle D(K_m, X, l_1, t), m_1\rangle \in S(K_m)$

$\supset .\ \langle D(K_m, X, l_1, t), m_2\rangle \in S(K_m)$ (Rec 6)

" $t \in \Psi(\omega + T_{e_1}) \supset_t .\ \langle X, m_1\rangle \in S(K_m) \supset .$

$\langle D(K_m, X, l_1, t), m_2\rangle \in S(K_m)$ (Lemma 6)

" $t \in \Psi(\omega + T_{e_1}) \supset_t .\ \langle D(K_m, X, l_1, t), m_2\rangle \in S(K_m)$

" $\langle X, m_4\rangle \in S(K_m)$ (Rec 7)

Hence $X \in \Sigma(K_m)$ implies that $\langle X, m_3\rangle \in S(K_m) \supset .$
$\langle X, m_1\rangle \in S(K_m) \supset .\ \langle X, m_4\rangle \in S(K_m)$.

$$\therefore\ X \in \Sigma(K_m) \supset_X .\ \langle X, m\rangle \in S(K_m)$$

$$\therefore\ m \in Tr$$

Case 3., W_m is $[\forall v_{1_1} W_{m_1}] \supset [\underset{\cdot}{S}\ {}^{(v_{1_1})}_{A}\ W_{m_1}|]$

where A may be v_{1_2} or $(\iota v_{1_2} W_{m_2})$, and

$T_{e_2} \leqslant T_{e_1}$, and no free variable of A is bound in W_{m_1}. (W_m is $W_{m_3} \supset W_{m_4}$.)

$$\langle x, m\rangle \in S(K_m) \equiv_x .\ x \in \Sigma(K_m)\ \&.\ \langle x, m_3\rangle \in S(K_m) \supset .\ \langle x, m_4\rangle \in S(K_m)$$ (Rec 6)

If A is v_{1_2}, let $t_2 = M(1_2, K_m, x)$

If A is $(\iota v_{1_2} W_{m_2})$, let $t_2 = (\iota t[\langle D(K_m, x, 1_2, t), m_2\rangle \in S(K_m)])$

Suppose $x \in \Sigma(K_m)$, $\langle x, m_3\rangle \in S(K_m)$,

then $t \in \Psi(\omega + T_{e_1}) \supset_t .\ \langle D(K_m, x, l_1, t), m_1\rangle \in S(K_m)$ (Rec 7)

" $t_2 \in \Psi(\omega + T_{e_2})$ (By definition of $\Sigma(k)$ or by Lemma 11.)

" $t_2 \in \Psi(\omega + T_{e_1})$ (Since $T_{1_2} \leq T_{1_1}$.)

" $\langle D(K_m, x, l_1, t_2), m_1\rangle \in S(K_m)$

" $\langle x, m_4\rangle \in S(K_m)$ (By lemma 7 or 8.)

$x \in \Sigma(K_m) \supset_x .\ \langle x, m_3\rangle \in S(K_m) \supset .\ \langle x, m_4\rangle \in S(K_m)$

$\therefore x \in \Sigma(K_m) \supset_x .\ \langle x, m\rangle \in S(K_m)$

$\therefore m \in Tr$

Case 4., W_m is $[[(v_{1_2})(v_{1_1}) \equiv_{v_{1_1}} (v_{1_3})(v_{1_1})] \supset [(v_{1_2}) = (v_{1_3})]]$

W_m is $W_{m_1} \supset W_{m_2}$. W_{m_1} is $\forall v_{1_1} W_{m_3}$.

where $T_{1_2} = T_{1_3} = T_{1_1} + 1$.

$$\langle x, m\rangle \in S(K_m) \equiv_x .\ x \in \Sigma(K_m)\ \&.\ \langle x, m_1\rangle \in S(K_m) \supset .\ \langle x, m_2\rangle \in S(K_m)$$ (Rec 6)

Suppose $x \in \Sigma(K_m)$, $\langle x, m_1\rangle \in S(K_m)$,

then $t \in \Psi(\omega + T_{e_1}) \supset_t .\ \langle D(K_m, x, l_1, t), m_3\rangle \in S(K_m)$ (Rec 7)

" $t \in \Psi(\omega + T_{e_1}) \supset_t .\ M(l_1, K_m, D) \in M(l_2, K_m, D) \equiv M(l_1, K_m, D) \in M(l_3, K_m, D)$ (Rec 1, 5, 6 repeated)

" $t \in \Psi(\omega + T_{e_1}) \supset_t .\ t \in M(l_2, K_m, x) \equiv t \in M(l_3, K_m, x)$

" $t \in M(l_2, K_m, x) \supset_t t \in \Psi(\omega + T_{e_1})$ (Since $T_{1_2} = T_{1_3} = T_{1_1} + 1$)

" $t \in M(l_3, K_m, x) \supset_t t \in \Psi(\omega + T_{e_1})$

" $t \in M(l_2, K_m, x) \equiv_t .\ t \in M(l_3, K_m, x)$

" $M(l_2, K_m, x) = M(l_3, K_m, x)$ (Ext. Axiom)

It is now convenient to make use of the following easily provable lemma:

Lemma 12. $\vdash k \in \omega\ \&\ K_{Eq(l_1, l_2)} \leq k \supset .\ \langle x, Eq(l_1, l_2)\rangle \in S(k) \equiv_x .\ x \in \Sigma(k)\ \&.\ M(l_1, k, x) = M(l_2, k, x)$

$\therefore \langle x, m_2 \rangle \in S(K_m)$ (By lemma 12.)

$\therefore m \in Tr$ (As in previous cases.)

Case 5., W_m is $[\sim \forall v_{l_1} \sim [W_{m_1} \& W_{m_2}]] \supset W_{m_3}$.

Where W_{m_2} is $\underset{\cdot}{S} \begin{matrix} (v_{e_1}) \\ (v_{e_2}) \end{matrix} W_{m_1} | \supset_{v_{e_2}} . v_{e_2} = v_{e_1}$

W_{m_3} is $\underset{\cdot}{S} \begin{matrix} (v_{e_1}) \\ (\iota v_{e_1} M_1) \end{matrix} W_{m_1} |$

W_{m_4} is $\underset{\cdot}{S} \begin{matrix} (v_{e_1}) \\ (v_{e_2}) \end{matrix} W_{m_1} |$

$T_{l_2} \leq T_{l_1}$, no variable is both free and bound in W_{m_1}, and v_{l_2} does not occur in W_{m_1};

Suppose x satisfies (on level K_m) the first clause of W_m,

then $\sim . t \in \Psi(\omega + T_{e_1}) \supset_t . \sim [\langle D(K_m, x, l_1, t), m_1 \rangle \in S(K_m) \& \langle D, m_2 \rangle \in S(K_m)]$ (Rec 5, 6, 7.)

" $\exists t . \langle D, m_1 \rangle \in S(K_m) \& \langle D, m_2 \rangle \in S(K_m)$

" $\exists t . \langle D, m_1 \rangle \in S(K_m) \& . w \in \Psi(\omega + T_{e_2}) \supset_w . \langle D(K_m, D, l_2, w), m_4 \rangle \in S(K_m) \supset .$

$M(l_2, K_m, D(K_m, D, l_2, w)) = M(l_1, K_m, D(K_m, D, l_2, w))$

(By Rec 6, 7, lemma 12.)

then $\exists t . \langle D, m_1 \rangle \in S(K_m) \& . w \in \Psi(\omega + T_{e_2}) \supset_w . \langle D(K_m, D, l_2, w), m_4 \rangle \in S(K_m) \supset . w = t$

" $\exists t . \langle D, m_1 \rangle \in S(K_m) \& . w \in \Psi(\omega + T_{e_2}) \supset_w . \langle D(K_m, x, l_2, w), m_4 \rangle \in S(K_m) \supset . w = t$ (Lemma 6, since $\| \sim Fr(l_1, m_4)$.)

" $\exists t . \langle D, m_1 \rangle \in S(K_m) \& . \langle D(K_m, x, l_2, w), m_4 \rangle \in S(K_m) \supset_w . w = t$ (Argument as in lemma 11.)

" $\exists t . \langle D, m_1 \rangle \in S(K_m) \& . \langle D(K_m, D(K_m, x, l_2, w), l_1, w), m_1 \rangle \in S(K_m) \supset_w . w = t$ (Lemma 7.)

" $\exists t . \langle D, m_1 \rangle \in S(K_m) \& . \langle D(K_m, x, l_1, w), m_1 \rangle \in S(K_m) \supset_w . w = t$ (Lemma 6.)

let $t_1 = (\iota t [\langle D(K_m, x, l_1, t), m_1 \rangle \in S(K_m)])$

then $\langle D(K_m, x, l_1, t_1), m_1 \rangle \in S(K_m)$ (Descr. axiom.)

" $\langle x, m_3 \rangle \in S(K_m)$ (Lemma 8, since no variable is both free and bound in W_{m_1}.)

$\therefore m \in Tr$ (as before)

Case 5*., Suppose we replace the description axiom in T by the choice axiom. (Then we are allowed to use the choice axiom in Z.)

W_m is $[\sim \forall v_1 \sim W_{m_1}] \supset W_{m_3}$; $\left. \begin{matrix} W_{m_1} \\ W_{m_3} \end{matrix} \right\}$ as before.

If x satisfies the first clause (on level K_m),

then $\exists t. \langle D(K_m, X, \ell_1, t), m_1\rangle \in S(K_m)$ (Just as above.)

let $t_1 = (\iota t[\langle D(K_m, X, \ell_1, t), m_1\rangle \in S(K_m)]$

then $\langle D(K_m, X, \ell_1, t_1), m_1\rangle \in S(K_m)$ (Choice axiom.)

" $\langle X, m_3\rangle \in S(K_m)$ (Lemma 8, since no variable is free and bound in W_{m_1}.)

$\therefore m \in Tr$ (as before.)[27]

Case 6., W_m is $[\forall v_{e_1} \sim W_{m_1}] \supset [(\iota v_{e_2} W_{m_1}) = (\iota v_{e_2} W_{m_2})]$

where W_{m_2} is $\forall v_{e_3}. \sim [(v_{e_2})(v_{e_3})]$

and $T_{e_1} = T_{e_2} = T_{e_3} + 1.$

Suppose x satisfies the first clause (on level K_m),

then $t \in \Psi(\omega + T_{e_1}) \supset_t . \sim [\langle D(K_m, X, \ell_1, t), m_1] \in S(K_m)]$ (Rec 5, 7.)

" $\sim [t \in \Psi(\omega + T_{e_1})]$ implies that $D \equiv 0$, hence that

$\sim [\langle D, m_1\rangle \in S(K_m)$ (Lemma 5.)

" $\forall t. \sim [\langle D, m_1\rangle \in S(K_m)]$

" $(\iota t[\langle D, m_1\rangle \in S(K_m)]) = 0$; call this set t_1.

let $t_2 = (\iota w[\langle D(K_m, X, \ell_2, w), m_2\rangle \in S(K_m)])$

then $\langle D(K_m, X, \ell_2, w), m_2\rangle \in S(K_m)$ if and only if

$z \in \Psi(\omega + T_{e_3}) \supset_z . \sim [z \in w]$ (Rec 5, 7, 1.)

but since $T_{1_2} = T_{1_3} + 1$, this means that $\forall z. \sim [z \in w]$

So this can happen just in case $w = 0.$

Then $t_2 = 0.$ (Descr. axiom.)

" $t_1 = t_2.$

Then we can show that x satisfies (on level K_m) the second clause of W_m, by Rec 4 and a proof like that of Lemma 12.

$\therefore m \in Tr$ (as before.)

Case 7., W_m is $[W_{m_1} \equiv_{v_1} W_{m_2}] \supset [(\iota v_1 W_{m_1}) = (\iota v_1 W_{m_2})]$

Let $t_1 = (\iota t[\langle D(K_m, X, \ell, t), m_1\rangle \in S(K_m)])$

" $t_2 = (\iota t[\langle D(K_m, X, \ell, t), m_2\rangle \in S(K_m)])$

Suppose x satisfies the first clause (on level K_m),

then we have to show that $t_1 = t_2$, then we can proceed as in case 6.

But then we have $t \in \Psi(\omega + T_e) \supset_t . \langle D(K_m, X, \ell, t), m_1\rangle \in S(K_m) \equiv . \langle D, m_2\rangle \in S(K_m)$ (Rec 5, 6, 7.)

then $\langle D, m_1\rangle \in S(K_m) \equiv_t . \langle D, m_2\rangle \in S(K_m)$ (As in lemma 11.)

" $t_1 = t_2$ (Extension. of descr. axiom.)

Case 8., W_m is $\sim \forall v_{1_1} \sim W_{m_2}$.

Where W_{m_2} is $[(v_{\ell_1})(v_{\ell_2}) \equiv_{v_{\ell_2}} W_{m_1}]$,

$\sim Fr(\ell_1, m_1)$, $T_{\ell_1} = T_{\ell_2} + 1$.

$\langle x, m \rangle \in S(K_m)$ if and only if

$x \in \Sigma(K_m) \mathbin{\&} \exists t.\ t \in \Psi(\omega + T_{\ell_1}) \mathbin{\&}.$

$\langle D(K_m, x, \ell_1, t), m_2 \rangle \in S(K_m)$ (By Rec 7 as in case 5.)

or $x \in \Sigma(K_m) \mathbin{\&} \exists t.\ t \in \Psi(\omega + T_{\ell_1}) \mathbin{\&}.$

$w \in \Psi(\omega + T_{\ell_2}) \supset_w .\ w \in t \equiv . \langle D(K_m,$

$D, \ell_2, w), m_1 \rangle \in S(K_m)$ (Rec 1, 5, 6, 7.)

or $x \in \Sigma(K_m) \mathbin{\&} \exists t.\ t \in \Psi(\omega + T_{\ell_1}) \mathbin{\&}.\ w \in \Psi(\omega + T_{\ell_2}) \supset_w$

$w \in t \equiv . \langle D(K_m, x, \ell_2, w), m_1 \rangle \in S(K_m)$ (Lemma 6.)

but $\exists t.\ t \in \Psi(\omega + T_{\ell_1}) \mathbin{\&}.\ w \in \Psi(\omega + T_{\ell_2}) \supset_w .$

$w \in t \equiv . \langle D(K_m, x, \ell_2, w), m_1 \rangle \in S(K_m)$

(Lemma II.)

$\therefore\ x \in \Sigma(K_m) \supset_x . \langle x, m \rangle \in S(K_m)$

$\therefore\ m \in Tr$

Case 9., W_m is an instance of the axiom of infinity.

Let W_{m_0} be one instance. Then W_m is gotten from W_{m_0} by a series of alphabetic changes of bound variables. If we can show that $m_0 \in Tr$, it will follow by several applications of lemma 9 that $m \in Tr$

We make use of Theorem I to prove $m_0 \in Tr \equiv W_{m_0}$. Thus it will be sufficient to prove W'_{m_0}.

Instead of stating W_{m_0} we state W'_{m_0}.

W'_{m_0} is of the form $\exists x_3 .\ \overline{x_3} \mathbin{\&} .\ \exists x_0 \mathcal{A}$,

Where $\mathcal{A}$ is $[\overline{x_0} \mathbin{\&} .\ \overline{y_0} \supset_{y_0} \sim [\langle y_0, x_0 \rangle \in x_3] \mathbin{\&} .\ \overline{x_1} \supset_{x_1} .$

$\exists z_0 [\overline{z_0} \mathbin{\&} .\ \overline{w_0} \supset_{w_0} .\ w_0 \in x_1 \supset .\ \overline{v_0} \supset_{v_0} .$

$\langle v_0, w_0 \rangle \in x_3 \supset . \langle v_0, z_0 \rangle \in x_3] \supset \exists u_0 .$

$\overline{u_0} \mathbin{\&} .\ \overline{t_0} \supset_{t_0} . \langle t_0, u_0 \rangle \in x_3 \equiv .\ t_0 \in x_1]$

Let B be $(\iota x_3 .\ x_2 \in x_3 \equiv_{x_2} .\ x_2 \in \Psi(\omega + 2) \mathbin{\&} .$
$\exists x_0 \exists y_0 .\ x_2 = \langle x_0, y_0 \rangle \mathbin{\&} .\ x_0 \in y_0)$

$x_2 \in B \equiv_{x_2} .\ x_2 \in \Psi(\omega + 2) \mathbin{\&} .\ \exists x_0 \exists y_0 .$

$x_2 = \langle x_0, y_0 \rangle \mathbin{\&} .\ x_0 \in y_0$ (Lemma III.)

$x_0 \in \Psi(\omega) \mathbin{\&} y_0 \in \Psi(\omega) \supset_{x_0 y_0} . \langle x_0, y_0 \rangle \in \Psi(\omega + 2)$

$x_0 \in \Psi(\omega) \mathbin{\&} y_0 \in \Psi(\omega) \supset_{x_0 y_0} . \langle x_0, y_0 \rangle \in B \equiv .\ x_0 \in y_0$

$W'_{m_0} \equiv \exists x_3 \exists x_0 .\ \overline{x_3} \mathbin{\&} \mathcal{A}$

$$\overset{X_3}{\underset{B}{S}}\ \overset{X_0}{\underset{0}{}}[\overline{X_3} \wedge \mathcal{A}]| \equiv . B\in\Psi(\omega+3) \wedge . [0\in\Psi(\omega)$$
$$\wedge . \overline{y_0} \supset_{y_0} \sim[y_0 \in 0]] \wedge . \overline{X_1} \supset_{X_1} . \exists z_0[\overline{z_0} \wedge .$$
$$\overline{w_0} \supset_{w_0} . w_0 \in X_1 \supset . \overline{v_0} \supset_{v_0} . v_0 \in w_0 \supset . v_0 \in z_0] \supset$$
$$\exists u_0[\overline{u_0} \wedge . \overline{t_0} \supset_{t_0} . t_0 \in u_0 \equiv . t_0 \in X_1]$$

$$B\in\Psi(\omega+3); \quad 0\in\Psi(\omega); \quad \forall y_0 . \sim[y_0\in 0].$$

So the first two clauses may be dropped.

If $\overline{X_1}$, then every element of x_1 and every element of an element of x_1 is an element of $\Psi(\omega)$. And if $\overline{u_0}$, then all elements of u_0 are elements of $\Psi(\omega)$.

$$\overset{X_3}{\underset{B}{S}}\ \overset{X_0}{\underset{0}{}}[\overline{X_3} \wedge \mathcal{A}]| \equiv . \overline{X_1} \supset_{X_1} . \exists z_0[\overline{z_0} \wedge . [w_0\in X_1$$
$$\wedge\, v_0 \in w_0] \supset . v_0 \in z_0] \supset \exists u_0[\overline{u_0} \wedge . t_0 \in u_0 \equiv_{t_0} t_0 \in X_1]$$

If $\overline{z_0} \wedge . [w_0 \in X_1 \wedge v_0 \in w_0] \supset_{v_0 w_0} . v_0 \in z_0$,

then $w_0 \in X_1 \supset_{w_0} . w_0 \subseteq z_0$; then $X_1 \subseteq \mathcal{P}(z_0)$.

" $X_1 \in \mathcal{P}(\mathcal{P}(z_0))$

" there is an n, $n \in \omega$, and $z_0 \in \Psi(n)$.

" $X_1 \in \Psi(n+2)$; hence $X_1 \in \Psi(\omega)$.

" $X_1 \in \Psi(\omega) \wedge . t_0 \in X_1 \equiv_{t_0} t_0 \in X_1$

" $\exists u_0 [\overline{u_0} \wedge . t_0 \in u_0 \equiv_{t_0} t_0 \in X_1]$

$\therefore\ \overset{X_3}{\underset{B}{S}}\ \overset{X_0}{\underset{0}{}}[\overline{X_3} \wedge \mathcal{A}]|$; $\therefore W_{m_0}$.

This completes the proof that if W_m is an axiom of T, then $m \in Tr$.

Rule I., $m_1 \in Tr$, $m_2 \in Tr$, and W_{m_2} is $[W_{m_1} \supset W_m]$.

$X \in \Sigma(K_{m_2}) \supset_X . \langle X, m_2\rangle \in S(K_{m_2})$ (since $m_2 \in Tr$)

$X \in \Sigma(K_{m_2}) \supset_X . \langle X, m_2\rangle \in S(K_{m_2}) \supset .$
$\langle X, m\rangle \in S(K_{m_2})$ (Rec 6.)

$K_{m_1} \leq K_{m_2}$

$X \in \Sigma(K_{m_2}) \supset_X . \langle X, m_1\rangle \in S(K_{m_2})$ (since $m_1 \in Tr$ and lemma 10.)

$X \in \Sigma(K_{m_2}) \supset_X . \langle X, m\rangle \in S(K_{m_2})$

$K_m \leq K_{m_2}$

$\therefore m \in Tr$ (Lemma 10.)

Rule II., $m_1 \in Tr$ and W_m is $[\forall v_\ell\, W_{m_1}]$.

$K_{m_1} \leq K_m$

$X \in \Sigma(K_m) \supset_X . \langle X, m_1\rangle \in S(K_m)$ ($m_1 \in Tr$ and lemma 10.)

$\langle X, m\rangle \in S(K_m) \equiv_X . X \in \Sigma(K_m) \wedge .$

$t \in \Psi(\omega + T_\ell) \supset_t . \langle D(K_m, X, \ell, t), m_1\rangle$
$\in S(K_m)$ (Rec 7.)

If $X \in \Sigma(K_m)$, then $t \in \Psi(\omega + T_e) \supset_t$.

$$D(K_m, X, \ell, t) \in \Sigma(K_m),$$

then $t \in \Psi(\omega + T_e) \supset_t$.

$$\langle D, m_1 \rangle \in S(K_m).$$

$\therefore\ X \in \Sigma(K_m) \supset_X . \langle X, m \rangle \in S(K_m)$

$\therefore\ m \in Tr$

This completes the second part of the proof.

Q. E. D.

Corollary II. If $\vdash_T W_m$, then $\vdash_Z W'_m$.

Proof: Suppose $\vdash_T W_m$,
then it has a proof, say of number k;

then $\vdash_Z \mathfrak{B}'_T(k, m)$ ($\mathfrak{B}'_T$ calculable in Z.)

" $\vdash_Z \mathfrak{Bew}'_T(m)$

" $\vdash_Z m \in Tr$ (Theorem 2.)

" $\vdash_Z W'_m$ (Theorem I.)

Q. E. D.

We could also prove the formal theorem of T:

$$\vdash_T \mathfrak{Bew}_T(x) \supset_x \mathfrak{Bew}_Z(x).$$

Corollary 3. $\vdash_Z C'_T$.

Proof: Let W_{m_0} be the w.f.f. of case 9, theorem 2.

W'_{m_0} (Proof given loc. cit.)[28]

$\sim [\sim W'_{m_0}]$

$\sim W'_{Neg(m_0)}$ (Same as previous step.)

$Neg(m_0) \in Tr \equiv W'_{Neg(m_0)}$ (Proof by Th. I.)

$\mathfrak{Bew}'_T(Neg(m_0)) \supset . Neg(m_0) \in Tr$ (Theorem 2.)

$\mathfrak{Bew}'_T(Neg(m_0)) \supset . W'_{Neg(m_0)}$[29]

$\sim \mathfrak{Bew}'_T(Neg(m_0))$

$\sim C'_T \supset . x \in \omega \supset_x \mathfrak{Bew}'_T(x)$ (Well known.)

$\sim C'_T \supset \mathfrak{Bew}'_T(Neg(m_0))$

C'_T

Q. E. D.

Corollary 4. $\vdash_T C_Z \supset C_T$.

Proof: We carry out the formal analogue of corollary II in T, to prove

$\mathfrak{B}ew_T(x) \supset_x \mathfrak{B}ew_Z(x)$.

Write down the proof of W_{m_0} (consisting of one step) in T. Let its number be k. Because $\mathfrak{B}_T$ is calculable in T, we can prove

$\mathfrak{B}_T(k, m_0)$

$\mathfrak{B}ew_T(m_0)$

$\mathfrak{B}ew_Z(m_0)$

$C_Z \supset . \sim \mathfrak{B}ew_Z(Neg(m_0))$

$\sim \mathfrak{B}ew_Z(Neg(m_0)) \supset \sim \mathfrak{B}ew_T(Neg(m_0))$

$\sim C_T \supset \mathfrak{B}ew_T(Neg(m_0))$ (As in cor. 3.)

$\therefore C_Z \supset C_T$.

Q. E. D.

Corollary V. $\vdash_T C_T \supset C_Z$ if and only if T and Z are both inconsistent.

Proof: If T inconsistent, then all w.f.f. are theorems, $\therefore \vdash_T C_T \supset C_Z$.

If $\vdash_T C_T \supset C_Z$, then $\vdash_Z C'_T \supset C'_Z$ (Cor. II.)

$\vdash_Z C'_Z$ (Cor. 3.)

Z is inconsistent by Gödel's theorem.

$\vdash_Z W_{m_0}$ and $\vdash_Z W_{Neg(m_0)}$

$\vdash_T \mathfrak{B}ew_Z(m_0) \,\&\, \mathfrak{B}ew_Z(Neg(m_0))$

$\vdash_T \sim C_Z$

$\vdash_T \sim C_T$

T is inconsistent by Rosser's generalization of Gödel's theorem.

Q. E. D.

Corollary VI. The set of integers Tr is not definable in T.

Proof: Suppose there were a w.f.f. of T which expressed the property of belonging to Tr. Then there would be a w.f.f. expressing the property of not belonging to Tr. We could then find a W_m which expresses that m does not belong to Tr.[30] That is $W'_m \equiv \sim [m \in Tr]$. But by Theorem I., $W'_m \equiv [m \in Tr]$. Contradiction. Hence there is no such w.f.f. in T.

Q. E. D.[31]

The above theorems are true even if we weaken both systems by replacing (5*) by (5). (5*) is used only in theorem 2, and there it was shown that it is needed in Z only if it is used in T.

3. Extensions of T and Z.

"Nearly there!" the Queen repeated.
"Why we passed it ten minutes ago."[32]

Due to Gödel's theorem, no one logical system is adequate for Mathematics. We must always consider stronger and stronger systems. So it is natural to ask what the relation is between extensions of T and Z. But it will turn out that the methods of the last chapter are sufficient to answer this question. (Except for an additional complication due to the use of only one-placed predicates.)

The essential difference between T and Z is in the method adopted to avoid the vicious circle paradoxes. In T this is done by going to a higher type when we define a set of things from a certain type. Thus larger sets are introduced in higher types. The natural way to extend T is by the inclusion of additional types. We consider a series of systems T_γ, for ordinals γ, $0 < \gamma < \omega^2$. In Z we hope to avoid the paradoxes by restricting carefully the existence-axioms. Larger sets are here introduced by new axioms. Our series of extensions will differ from Z only in the addition of new axioms. The new axiom will guarantee the existence of the least cardinal which cannot be proven to exist in the previous system. We get a series of systems Z_k, $k = 1, 2, \ldots$.

I have seen only one system of transfinite type-theory so far which was worked out sufficiently in detail to convince me that it is adequate. The system is based on axioms and definitions due to A. Church, and the system is developed in great detail by E. Bustamante in his Ph.D. thesis.[33] Unfortunately this thesis has not been published.

The Bustamante system is a type-theory with variables ranging over all types $< \omega^2+2$. Our systems T_γ will be subsystems of this one; and in no T_γ will we admit variables of type ω^2, because there are some questions as yet unanswered on this level. But it is clear that our results can be extended, probably to all types which are constructive ordinals.[34]

We shall make a few other inessential changes to bring the Bustamante system into a form closer to that of T. The most important changes are caused by the fact that we use axiom schemata and the ι-operator. Furthermore, in our system type 0 is empty, and hence type 1 corresponds to Bustamante's type 0. This simplifies the correspondence between T_γ and the Z_k. We now construct simultaneously the systems T_γ for all ordinals, γ, $0<\gamma<\omega^2$. In the following system, γ must be understood to be a fixed ordinal.

This system is described here in detail, because I feel that it is interesting in itself, quite aside from its relationship to this thesis.

__The logical system T_γ.__[35]

Primitive symbols: $x_\alpha, y_\alpha, \ldots$ (variables), (,), [,], $\sim$, $\supset$, $\forall$, ι, (where α is an ordinal $<\gamma$.)

w.f.f. and terms of type α : (Definition by recursion.)

1. If a_α is a variable with subscript α, then (a_α) is a term of type α.
2. If a_α is a variable with sbbscript α, and $\mathcal{A}$ is a w.f.f., then $(\iota a_\alpha \mathcal{A})$ is a term of type α.
3. If $\mathcal{A}$, $\mathcal{B}$ are w.f.f. and a_α is a variable, then $[\sim\mathcal{A}]$, $[\mathcal{A}\supset\mathcal{B}]$, $[\forall a_\alpha \mathcal{A}]$ are w.f.f.
4. If A_{α_1}, B_{α_2} are terms of type α_1, α_2, then $[A_{\alpha_1} B_{\alpha_2}]$ is a w.f.f.
5. The sets of w.f.f. and of terms of type α are the smallest sets having the four above properties.

Convention:

a_α, b_α, ...	are used to stand for variables with subscript α.
A_α, B_α, ...	are used to stand for terms of type α.
$\mathcal{A}$, $\mathcal{B}$, ...	are used to stand for w.f.f.
α^0	stands for $\begin{cases}\alpha+1 & \text{if } \alpha \text{ is of the 1st kind or 0} \\ \alpha & \text{if } \alpha \text{ is of the 2nd kind}\end{cases}$

We introduce all the usual abbreviations. In particular:

$a_{\alpha_1} = b_{\alpha_2}$ stands for $[C_{\alpha_3} a_{\alpha_1}] \equiv_{C_{\alpha_3}} [C_{\alpha_3} b_{\alpha_2}]$, where $\alpha_3 = max(\alpha_1,\alpha_2)+1$.

$(\hat{a}_\alpha A)$ stands for $(\iota b_{\alpha+1}.\,[b_{\alpha+1}a_\alpha] \equiv_{a_\alpha} \mathcal{A})$
($b_{\alpha+1}$ not free in $\mathcal{A}$)

θ " " $(\iota x_1.\,\forall y_0.\sim[x_1 y_0])$

Axiom schemata:

(1) $\mathcal{A}$ where $\mathcal{A}$ is a substitution instance of a tautology

(2) $\mathcal{A} \supset_{a_\alpha} \mathcal{B} \supset.\, \mathcal{A} \supset \forall a_\alpha \mathcal{B}$ a_α not free in $\mathcal{A}$

(3) $[\forall a_{\alpha_1} \mathcal{A}] \supset [\dot{S}^{(a_{\alpha_1})}_{B_{\alpha_2}} \mathcal{A}|]$ $0<\alpha_2\leq\alpha_1$ and no free variable of B_{α_2} bound in $\mathcal{A}$

(4) $[(b_{\alpha_1^\circ})(a_{\alpha_1}) \equiv_{a_{\alpha_1}} (c_{\alpha_2})(a_{\alpha_1})] \supset [b_{\alpha_1^\circ} = c_{\alpha_2}]$ α_1

(5*)[36] $[\exists a_\alpha \mathcal{A}] \supset [\dot{S}^{(a_\alpha)}_{(\iota a_\alpha \mathcal{A})} \mathcal{A}|]$ no variable free and bound in $\mathcal{A}$

(6) $[\forall a_\alpha.\sim\mathcal{A}] \supset [(\iota a_\alpha \mathcal{A}) = \theta]$

(7) $[\mathcal{A} \equiv_{a_\alpha} \mathcal{B}] \supset [(\iota a_\alpha \mathcal{A}) = (\iota a_\alpha \mathcal{B})]$

(8) $\exists b_{\alpha_2}.\,[b_{\alpha_2} a_{\alpha_1}] \equiv_{a_{\alpha_1}} \mathcal{A}$ b_{α_2} not free in $\mathcal{A}$ $\alpha_1 < \alpha_2$

(9) $\sim \exists a_0.\, a_0 = a_0$

(10) $[a_{\alpha_1^\circ} b_{\alpha_2}] \supset \exists c_{\alpha_1}.\, c_{\alpha_1} = b_{\alpha_2}$ $\alpha_1 < \alpha_2$

(11) $\forall a_\alpha \exists b_\alpha.\,[b_\alpha c_\alpha] \equiv_{c_\alpha} [c_\alpha \subseteq a_\alpha]$ α of 2nd kind

(12)[37] $b_{\alpha_2} \subseteq a_{\alpha_1} \supset \exists c_{\alpha_1}.\, c_{\alpha_1} = b_{\alpha_2}$ α_1 of 2nd kind, $\alpha_1 < \alpha_2$

$n>0$: (13) $[a_{\omega n+1}(\hat{e}_{\omega(n-1)}[e_{\omega(n-1)} = e_{\omega(n-1)}])] \supset.$
$[a_{\omega n+1} b_{\omega n} \supset_{b_{\omega n} c_{\omega n}}.\, c_{\omega n} d_{\omega n} \supset_{d_{\omega n}} d_{\omega n}$
$b_{\omega n} \supset a_{\omega n+1} c_{\omega n}] \supset.\, (\hat{f}_{\omega n}[f_{\omega n} = f_{\omega n}]) \subseteq a_\omega$

Rules of inference:

[I] From $\mathcal{A}$ and $[\mathcal{A} \supset \mathcal{B}]$ infer $\mathcal{B}$.

[II] From $\mathcal{A}$ infer $[\forall a_\alpha \mathcal{A}]$.

Next we construct the systems Z_k. They differ from Z only slightly. To get Z_k we first of all enlarge the list of variables of Z. We allow as subscripts for the variables any ordinal α, $0 < \alpha < \omega k + 1$.[38] We define w.f.f. and terms in a manner analogous to that used in Z. The rules and most of the axiom schemata remain unchanged (if we remember that we must allow the new variables to occur in these). Only (9) is changed as follows:

We introduce a formal definition which expresses the following recursion in the system:[39]

$$\Psi(0) = 0$$

$$\Psi(\alpha+1) = \mathcal{P}(\Psi(\alpha))$$

$$\Psi(\alpha) = (\iota a.\, b \in a \equiv_b \exists \beta.\, \beta < \alpha \wedge b \in \Psi(\beta)) \quad \text{if } \alpha \text{ of 2nd kind.}$$

In Z_1 we simply drop (9). In Z_{k+1} we add the axiom $\Psi(\omega k) \neq 0$ to the axioms of Z_k. This guarantees the existence of $\aleph_{\omega(k-1)}$ the least cardinal which cannot be proven to exist in Z_k.

We can easily see that the system T is quivalent to $T_{\omega 2}$ and Z to Z_2. The results of the previous chapter generalize as follows:

For any $k \geq 2$, we can map any T_γ, $\gamma < \omega k+1$, into Z_k and prove the analogues of Theorems I, 2.[40]

The mapping is defined as follows:

$\overline{a_\alpha}$ stands for $[a_\alpha \in \Psi(\alpha)]$

1., $(a_\alpha)^*$ is (a_α)

2., $(\iota a_\alpha \mathcal{A})^*$ is $(\iota a_\alpha . \overline{a_\alpha} \wedge \mathcal{A}^*)$

3., $[\sim \mathcal{A}]^*$ is $[\sim \mathcal{A}^*]$

$[\mathcal{A} \supset \mathcal{B}]^*$ is $[\mathcal{A}^* \supset \mathcal{B}^*]$

$[\forall a_\alpha \mathcal{A}]^*$ is $[\forall a_\alpha . \overline{a_\alpha} \supset \mathcal{A}^*]$

4., $[A_{\alpha_1} B_{\alpha_2}]^*$ is $[B^*_{\alpha_2} \in A^*_{\alpha_1}]$

For every w.f.f. $\mathcal{A}$ of T_γ($\gamma < \omega k+1$), we define a w.f.f. $\mathcal{A}'$ of Z_k, its translation:

If $\mathcal{A}$ has the free variables $a_{\alpha_1}, \ldots, a_{\alpha_i}$, then $\mathcal{A}'$ is $[[\overline{a_{\alpha_1}} \wedge \cdots \wedge \overline{a_{\alpha_i}}] \supset_{a_{\alpha_1} \ldots a_{\alpha_i}} \mathcal{A}^*]$. If, in particular, $\mathcal{A}$ has no free variables, then $\mathcal{A}'$ is $\mathcal{A}^*$.

This definition is the complete analogue of the definitio[n] given for T. The formal proof of the analogues of theorems I, 2 is therefore very close to the one given in the previous chapter; it will suffice to indicate what changes are necessar[y]

The systems we are considering are T_γ ($0 < \gamma < \omega k+1$) and Z_k, instead of T and Z. Thus the abbreviations must be changed accordingly. For example, T_l now stands in Z_k for the type of the l^{th} variable of T_γ, and $\mathcal{B}ew_T(m)$ must be replaced by $\mathcal{B}ew_{T_\gamma}(m)$ which expresses in Z_k that w.f.f. number m of T_γ is provable in T_γ. A further change must be made to account for the fact that the type α now corresponds to $\Psi(\alpha)$, not to $\Psi(\omega+\alpha)$. Among the abbreviations this is handled by letting Σ (k) stand in Z_k for the set of k-tuples whose l^{th} member is an element of $\Psi(T_l)$. And these are the only changes necessary to get a truth-definition. S(k) and $\mathcal{T}r$ then give us a correct definition of satisfaction and truth.

In all theorems and lemmas we have to make the same changes in the abbreviated terms and w.f.f. Furthermore, terms like $\Psi(\omega+T_l)$ must be replaced by the corresponding $\Psi(T_l)$. But these changes are also sufficient to get correct proofs in all cases except theorem 2. In theorem 2 we must also consider the new axiom schemata. Let us consider this proof.

We will have 12 cases corresponding to the 12 schemata. (The rules are the same as for T.[41]) Cases 1, 2, 3, 5*, 6, and 7 are the same as before.[41] In case 4, $T_{l_3} \leqslant T^o_{l_1} = T_{l_2}$ instead of $T_{l_1}+1 = T_{l_2} = T_{l_3}$. But this does not change the proof; as a matter of fact, some such condition is necessary for the steps

$$t \in M(\ell_2, K_m, X) \supset_t t \in \Psi(T_{\ell_1})$$

$$t \in M(\ell_3, K_m, X) \supset_t t \in \Psi(T_{\ell_1})$$

in case T_{ℓ_2}, T_{ℓ_3} are of the second kind.

In case 8, $T_{\ell_1} > T_{\ell_2}$ instead of $T_{\ell_1} = T_{\ell_2} + 1$ but the proof still holds.

Case 9., W_m is $\sim \exists v_\ell . v_\ell = v_\ell.$ $\qquad T_\ell = 0.$

Let W_{m_1} be $v_\ell \neq v_\ell$

x satisfies W_m (on level K_m) if and only if it satisfies $\forall v_\ell W_{m_1}$ or

$$t \in \Psi(0) \supset_t . \langle D(K_m, x, \ell, t), m_1 \rangle \in S(K_m)$$

But $\forall t. \sim [t \in \Psi(0)]$

$\therefore m \in Tr$ as usual.

Case 10., W_m is $[v_{\ell_1} v_{\ell_2}] \supset . \sim \forall v_{\ell_3} . v_{\ell_3} \neq v_{\ell_2}.$

$$T_{\ell_1} = T^{\circ}_{\ell_3}\ ; \quad T_{\ell_2} > T_{\ell_1}.$$

Let W_{m_1} be $\forall v_{\ell_3} . v_{\ell_3} \neq v_{\ell_2}$

x satisfies W_m (on level K_m) if and only if

$$[M(\ell_2, K_m, X) \in M(\ell_1, K_m, X)] \supset . \sim [\langle X, m_1 \rangle \in S(K_m)]$$

or $[M(\ell_2, K_m, X) \in M(\ell_1, K_m, X)] \supset . \sim [t \in \Psi(T_{\ell_3}) \supset_t . t \neq M(\ell_2, K_m, X)]$

or $[M(\ell_2, K_m, X) \in M(\ell_1, K_m, X)] \supset \exists t . [t \in \Psi(T_{\ell_3}) \,\&\, . \, t = M(\ell_2, K_m, X)]$

but $M(\ell_1, K_m, X) \in \Psi(T^{\circ}_{\ell_3})$; hence the first clause is satisfied only if $M(\ell_2, K_m, X) \in \Psi(T_{\ell_3})$; then $M(\ell_2, K_m, X)$ satisfies the second clause.

$\therefore m \in Tr$ as usual.

The proof of cases 11, 12 is somewhat lengthy, because we have to make use of the properties of Ψ quite heavily. But once the properties of Ψ have been developed, these two cases give us no difficulty.[39] Case 13 also depends on the properties of Ψ . The simplest method in this case is the one used to prove case 9 in the previous chapter.

These few brief remarks should suffice to show that the results of the previous chapter can be extended to the systems T_γ and Z_k.

For the sake of completeness we include an informal discussion of certain other truth-definitions for the above systems. First of all we note that the systems Z_k correspond very closely to the systems $T_{\omega k+1}$. And, as the reader can easily see, their intended models are the same. But somehow $T_{\omega k+1}$ is too weak to represent the intended model; this is due to the fact that axiom-schemata (12),(13) function properly for type $\omega \cdot k$ only if we have variables with subscripts $\omega k+1$. This seems unavoidable in a purely type-theoretical system. If we attempt to correct this by adding other axioms, we are led to a system which is almost exactly Z_k. It therefore seems natural to replace $T_{\omega k+1}$ by Z_k. So we define:

$$L_\gamma \text{ is } \begin{Bmatrix} Z_k \text{ if } \gamma = \omega k+1 \\ T_\gamma \text{ otherwise} \end{Bmatrix};$$

We now write all the variables of Z_k with subscripts ωk; then Z_k becomes a sub-system of $T_{\omega k+2}$, and the naturalness of our series is seen more clearly. We thus get a series of systems each one stronger than the previous ones. Indeed, we shall demonstrate that a truth-definition can be given for any system in the following one, if the latter system is adequate for recursive arithmetic.[42] In view of these facts I would say that type-theory and set-theory are not two fundamentally different kinds of systems, but that set-theory is the first transfinite type-theory, and that the extensions of set-theory are simply the "stepping-stones" of the type-theories, i.e., the systems introducing new kinds of transfinite variables.

Only the main ideas of the truth-definition will be given.

Let us choose a fixed system L_γ for which we are to construct a truth-definition within $L_{\gamma+1}$, where $L_{\gamma+1}$ is adequate for recursive arithmetic. We shall further assume that $\gamma \geqslant \omega+1$, for reasons given later. If $\gamma = \omega k$ (then $k \geqslant 2$), then $\gamma+1 = \omega k+1$, $L_{\gamma+1} = Z_k, k \geqslant 2$, hence we can give a truth-definition as shown above. So we may assume that γ is of the first kind; hence $\gamma-1$ is the highest type term in L_γ. In $L_{\gamma+1}$ we have terms of type γ.

The most important trick is to be able to represent k-tuples of type $\gamma-1$ within type $\gamma-1$. We now proceed to outline one method by which this can be done.[43] Since we assume that $L_{\gamma+1}$ is adequate for arithmetic, we will feel free to use arithmetical expressions without explicit definition. Suppose $\gamma-1 = \omega l + m$. We represent k-tuples of type ωl as classes of type ωl which can be interpreted as one-many[44] mappings of the set of k members upon the set $\{1,2,\ldots, k\}$. The i^{th} member of the k-tuple is the set corresponding to i in this mapping. Now suppose we have accomplished this definition up to type $\omega l + n$, then k-tuples of type $\omega l+n+1$ will be classes all of whose elements are k-tuples of type $\omega l+n$ (a concept already defined). The i^{th} member of such a k-tuple will be the set of all i^{th} members of its elements (a concept already defined). More precisely:

1., $\{a_{\omega l}, b_{\omega l}\}$ stands for $(\iota c_{\omega l} . [c_{\omega l} d_{\omega l}] \equiv_{d_{\omega l}} . d_{\omega l} = a_{\omega l} \vee d_{\omega l} = b_{\omega l})$

2., $\langle a_{\omega e}, b_{\omega e}\rangle$ stands for $\{\{a_{\omega e}, a_{\omega e}\}, \{a_{\omega e}, b_{\omega e}\}\}$

3., "$b_{\omega e}$ is a one-many correspondence"

stands for $[b_{\omega e}\, a_{\omega e} \supset_{a_{\omega e}} \exists c_{\omega e} \exists d_{\omega e}.$

$a_{\omega e} = \langle c_{\omega e}, d_{\omega e}\rangle] \,\&\, [b_{\omega e} \langle c_{\omega e}, d_{\omega e}\rangle$

$\supset_{c_{\omega e} d_{\omega e}} .\ b_{\omega e}\langle e_{\omega e}, d_{\omega e}\rangle \supset_{e_{\omega e}} .\ c_{\omega e} = e_{\omega e}]$

4., "$b_{\omega e}$ is a k-tuple" stands for "$b_{\omega e}$ is a one-many correspondence" $\&.\ \exists a_{\omega e}\, [b_{\omega e}\langle a_{\omega e}, c_{\omega e}\rangle] \equiv_{c}$ "$c_{\omega e}$ is a positive integer $\leq$ k"

5., $M(i, k, b_{\omega e})$ stands for $(\iota a_{\omega e}.$ "$b_{\omega e}$ is a k-tuple" $\&$ $[b_{\omega e}\langle a_{\omega e}, i\rangle])$

Scheme 6., "$b_{\omega e+n+1}$ is a k-tuple" stands for $[b_{\omega e+n+1}\, a_{\omega e+n}$ $\supset_{a_{\omega e+n}}$ "$a_{\omega e+n}$ is a k-tuple"

Scheme 7., $M(i, k, b_{\omega e+n+1})$ stands for $(\iota a_{\omega e+n+1}.$ "$b_{\omega e+n+1}$ is a k-tuple" $\&.\ [a_{\omega e+n+1}\, c_{\omega e+n}] \equiv_{c}$ $\exists d_{\omega e+n}.\ [b_{\omega e+n+1}\, d_{\omega e+n}] \,\&\, M(i, k, d_{\omega e+n}) = c_{\omega e+n}$

By m applications of schemes 6, 7, we get a definition for "$b_{\gamma-1}$ is a k-tuple" and for $M(i, k, b_{\gamma-1})$.

We can now construct the truth-definition in analogy to that given in the previous chapter (remembering, however, that slight changes have been made in the definition of w.f.f.)

K_m, ϵ_1, ϵ_2, D, etc. are defined in analogy to the previous chapter. k, l, m, n are used in place of variables of type ω, the type of the integers.

8., $\mathrm{Rec}^k_1\, a_\gamma$ stands for $[[\exists l_1 \exists l_2.\ \epsilon_1(M(2, 2, \mathfrak{z}_{\gamma-1}),$ $l_1, l_2) \,\&.\ M(l_1, k, M(1, 2, \mathfrak{z}_{\gamma-1}))$ $M(l_2, k, M(1, 2, \mathfrak{z}_{\gamma-1}))] \supset [a_\gamma\, \mathfrak{z}_{\gamma-1}]]$

9., $\mathrm{Rec}^k_2\, a_\gamma$ stands for $[[\exists l_1 \exists l_2 \exists m.\ \epsilon_2(M(2, 2, \mathfrak{z}_{\gamma-1}),$ $l_1, m, l_2) \,\&.\ (\iota t_{\gamma-1}.\ \exists z_{\gamma-1}.\ M(1, 2, z_{\gamma-1}) =$ $D(k, M(1, 2, \mathfrak{z}_{\gamma-1}), l_1, t_{\gamma-1}) \,\&.\ M(2, 2, z_{\gamma-1}) = m$ $\&\, [a_\gamma\, z_{\gamma-1}])\, M(l_2, k, M(1, 2, \mathfrak{z}_{\gamma-1}))] \supset [a_\gamma\, \mathfrak{z}_{\gamma-1}]]$

10.,-14., $\mathrm{Rec}^k_3\, a_\gamma, \ldots, \mathrm{Rec}^k_7\, a_\gamma$ in analogy to previous chapter, just like 8 and 9.

15., $\mathrm{Rec}^k\, a_\gamma$ stands for $[[$ "$M(1, 2, \mathfrak{z}_{\gamma-1})$ is a k-tuple" $\&.$ "$M(2, 2, \mathfrak{z}_{\gamma-1})$ is an integer" $\&.\ K_{M(2,2,\mathfrak{z}_{\gamma-1})} \leq k] \supset_{\mathfrak{z}_{\gamma-1}}.$ $\mathrm{Rec}^k_1\, a_\gamma \,\&\, \cdots \,\&\, \mathrm{Rec}^k_7\, a_\gamma]$

16., S^k stands for $\hat{\mathfrak{z}}_{\gamma-1}.\ \mathrm{Rec}^k X_\gamma \supset_{X_\gamma} [X_\gamma\, \mathfrak{z}_{\gamma-1}]$

17., Tr_γ stands for $\hat{m}.$ "m is an integer" $\&.$ "$X_{\gamma-1}$ is a K_m-tuple" $\supset_{X_{\gamma-1}} \exists \mathfrak{z}_{\gamma-1}.\ M(1, 2, \mathfrak{z}_{\gamma-1}) = X_{\gamma-1}$ $\&.\ M(2, 2, \mathfrak{z}_{\gamma-1}) = m \,\&\, [S^{K_m}\, \mathfrak{z}_{\gamma-1}]$

The proof that this truth-definition is correct (i.e., the proof of the analogues of theorems I, 2) is beyond the scope of this thesis.

This proves that we can give a truth-definition for L_γ within $L_{\gamma+1}$, if $L_{\gamma+1}$ is $\geqslant \omega+2$ and adequate for recursive arithmetic. Obviously this can be generalized to: We can give a truth-definition for L_{γ_2} in L_{γ_1} if $\gamma_1 > \gamma_2$ and $\gamma_1 \geqslant \omega+2$ and L_{γ_1} adequate for recursive arithmetic. (Since each system is an extension of the previous systems.)

We now proceed to show that L_γ is adequate for recursive arithmetic if $\gamma \geqslant \omega+2$. For this it is sufficient to show that $L_{\omega+2}$ is adequate. If $L_{\omega+2}$ contains many-place predicates, this is well-known; however, our $L_{\omega+2}$ is also adequate for recursive arithmetic.

In order to show that a system is adequate for recursive arithmetic, we must show that natural numbers can be defined and that we can define addition and multiplication so as to have the usual properties; and that is all we need to show.[45] We define the set of natural numbers first.

18., a'_ω stands for $(\iota b_\omega . [b_\omega c_\omega] \equiv_{c_\omega} . b_\omega c_\omega \vee . b_\omega = c_\omega)$

19., $N_{\omega+1}$ stands for $(\iota z_{\omega+1} . [[y_{\omega+1}\, \theta] \,\&.\, [y_{\omega+1}\, x_\omega]$

$$\supset_{x_\omega} [y_{\omega+1}\, x'_\omega]] \supset_{y_{\omega+1}} . \; z_{\omega+1} \subseteq y_{\omega+1})$$

20., "a_ω is an integer" stands for $[N_{\omega+1}\, a_\omega]$

21., "a_ω is a positive integer $\leqslant k$" stands for

$$[N_{\omega+1}\, a_\omega] \,\&.\, a_\omega \neq \theta \,\&\, [k\, a_\omega]$$

We then use definitions 4 and 5 to define "a_ω is a k-tuple" and $M(i,k,a_\omega)$. Using these we can define addition and multiplication:

22., $[a_{\omega+1} \langle x_\omega, y_\omega, z_\omega \rangle]$ stands for $\exists t_\omega$. "t_ω is a 3-tuple"

$$\&.\, M(1,3,t_\omega) = x_\omega \,\&.\, M(2,3,t_\omega) = y_\omega$$

$$\&.\, M(3,3,t_\omega) = z_\omega \,\&\, [a_{\omega+1}\, t_\omega]$$

23., $[x_\omega + y_\omega = z_\omega]$ stands for $[[\forall p_\omega\, [a_{\omega+1} \langle p_\omega, \theta, p_\omega \rangle]$

$$\&.\, [a_{\omega+1} \langle q_\omega, r_\omega, s_\omega \rangle] \supset_{q_\omega r_\omega s_\omega}$$

$$[a_{\omega+1} \langle q_\omega, r'_\omega, s'_\omega \rangle]] \supset_{a_{\omega+1}} [a_{\omega+1} \langle x_\omega, y_\omega, z_\omega \rangle]]$$

24., $[x_\omega \cdot y_\omega = z_\omega]$ stands for $[[\forall p_\omega\, [a_{\omega+1} \langle p_\omega, \theta, \theta \rangle]$

$$\&.\, [a_{\omega+1} \langle q_\omega, r_\omega, s_\omega \rangle] \supset_{q_\omega r_\omega s_\omega} .$$

$$[s_\omega + q_\omega = t_\omega] \supset_{t_\omega} [a_{\omega+1} \langle q_\omega, r'_\omega, t_\omega \rangle]] \supset_{a_{\omega+1}} [a_{\omega+1} \langle x_\omega, y_\omega, z_\omega \rangle]$$

We again omit the proofs that our definitions are adequate, but they are close enough to standard definitions to make the proofs easy.

This proves that any T_γ, $\gamma \geqslant \omega+2$, is adequate for recursive arithmetic.[47] So we can now sharpen our previous result to read: We can give a truth-definition for L_{γ_2} in L_{γ_1}, if $\gamma_1 > \gamma_2$ and $\gamma_1 \geq \omega+2$. But we can also show that these conditions are necessary.

If $\gamma_1 < \omega+2$, then L_{γ_1} has models (e.g., the intended model[48]) in which all sets are finite. Hence Tr_γ (for any γ) and the set of all natural numbers cannot be defined in L_{γ_1}, since they are both infinite. So this system is not adequate for recursive arithmetic or for a truth-definition. If $\gamma_2 \geq \gamma_1$, then it is well-known that no truth-definition is possible. (Assuming throughout this paragraph that all the L_γ are consistent.)[49]

We now get the following theorem:

Theorem III. If all our L_γ are consistent, then we can give a truth-definition for L_{γ_2} in L_{γ_1} if and only if $\gamma_1 > \gamma_2$ and $\gamma_1 \geq \omega+2$.

Q. E. D.

Footnotes.

1. See [4], p. 214.

2. For a good discussion of the history of these systems see [9]. Although the systems there described differ somewhat from T and Z, the references there given are the ones that apply to this thesis too. (Especially [12], [13], [15], [16].) This paper is also the only paper previous to this thesis to make a valuable contribution to the relation between systems like T and Z. The essential difference is that Quine's systems contain no axiom of infinity or of choice. There is also the inessential difference that Quine's set-theory has "Urelemente."

3. The basic ideas of T are taken from a system due to Tarski, see [13]. T differs from this system in that it contains axioms of infinity and choice, and it has a description operator.

4. These are individual and functional variables (i.e. set-variables); thus no propositional variables are used.

5. "W. f. f." is an abbreviation for "well-formed formula" or for the plural of this phrase.

6. In this, and similar definitions some convention, only too well known, must be adopted as to which variable c_{n_3} is.

7. This axiom, the choice axiom, may be weakened into a description axiom:

(5) $[\exists a_n . \mathcal{A} \& . S^{(a_n)}_{(b_{n_1})} \mathcal{A} | \supset_{b_{n_1}} . b_{n_1} = a_n] \supset [S^{(a_n)}_{(\iota a_n \mathcal{A})} \mathcal{A} |]$ where no variable is both free and bound in $\mathcal{A}$, $n_1 \leqslant n$, and b_{n_1} does not occur in $\mathcal{A}$.

8. For the history of this system see [9].

9. Set-variables only.

10. These schemata are, in order, tautology, quantifier, quantifier, extensionality, choice, conventional, extension of description, subset, pair, sumset, powerset, and infinity axioms.

11. We can again weaken (5*) to

(5) $[\exists a . \mathcal{A} \& S^{(a)}_{(b)} \mathcal{A} | \supset_b . b = a] \supset [S^{(a)}_{(\iota a \mathcal{A})} \mathcal{A} |]$
where no variable of $\mathcal{A}$ is both free and bound, and b does not occur in $\mathcal{A}$.

All the theorems proved in chapter 2 are true if we replace (5*) by (5) in both systems. (See fn. 7.)

12. See [7].

13. See [2], p. 66. Bernays gives a definition of $\Psi(\alpha)$ for all ordinals α, and develops the most important properties of the sets $\Psi(\alpha)$, some of which we shall use later on.

14. See [14].

15. See [4], p. 213.

16. See, for example, [1].

17. In all these it is intended that if k, l, m are not integers, or if x, t are not sets of the proper kind, then the symbol on the left stands for 0. E.g. $M(l,k,x) = 0$ unless l, k are integers and x is an element of $\Sigma(k)$ and $l \leqslant k$.

18. This really is a set since $\Sigma(k) \subseteq \Psi(\omega + 2k - 2 + \sum_{i=1}^{k} T_i)$.

19. In all these it is intended that if m, m_1, m_2, l_1, l_2 are not integers, then the relation does not hold.

20. Since these proofs are all in Z, "$\vdash$" will mean "it is a theorem <u>of Z</u>."

21. This was defined earlier. See p. 4.

22. Where $\ulcorner D \urcorner$ of course stands for $\ulcorner D(k,x,l_1,t) \urcorner$.

23. It is convenient to use the letters k, l, m, x, y, z with or without subscripts as variables of Z.

24. See [14].

25. A good illustration is [10].

26. Since $\forall(m_3, 1_1, m_5)$. But these obvious remarks will be omitted from now on.

27. This proof is actually much simpler and could have been used in case 5., but we want to avoid using the choice axiom if it is not used in T. Compare with fn. 11.

28. We could also use corollary II to supply the proof, but in this particular case it is simpler to find the proof directly.

29. This is the crucial step of the proof. This can be proved only by recursion on the no. of the w.f.f. For this we have to talk about W'_x with a variable x. I.e. we need a w.f.f. with a free variable x, say $\mathfrak{A}_{[x]}$, such that for every integer m, $\vdash_Z \mathfrak{A}_{[m]} \equiv W'_m$. This is precisely the role played by Tr . (See theorem I.) The recursion is then carried out in theorem 2. If we do not have Tr, all we can hope to prove is corollary II, which talks about constant m. From this we get only that $Bew'_T(Neg(m_0)) \supset Bew'_Z(Neg(m_0))$. This is too weak to prove C'_T. We would also need C'_Z, as in corollary 4.

30. See lemma 1 of [11].

31. This proof reproduces the well known paradox of "The liar." It's use was suggested to me by Dr. L. Henkin.

32. See [4], p. 165.

33. See [3].

34. For the theory of constructive ordinals see the papers of A. Church and S.C. Kleene. A summary of results and a good bibliography can be found in [8] and in its footnotes.

35. The model intended is as follows: The type 0 is empty. Type $\alpha+1$ contains all sets of subsets of sets of type α. Type α, α of 2nd kind, contains all sets of lower type. Thus, e.g., since T_ω has variables of all finite types, its model contains all the sets formable from 0 by a finite number of taking sets of subsets; but they will occur in different types. In the model of $T_{\omega+1}$ the same sets occur, but they all occur in type ω. Etc. Clearly in every T_γ, each type is contained in all previous types.

36. We can again weaken this to (5), as in chapter 1; all the theorems of this chapter remain true if this change is made both in the T_γ and in the Z_k. Compare with fn. 11.

37. The independence of this axiom is unsettled sofar, according to Bustamante. I believe that it is independent.

38. This is a trivial change introduced only to simplify later definitions.

39. See fn. 13.

40. Z_1' is too weak for a truth-definition. This will be provéd later on.

41. Except that subscripts now have a wider range, but this does not change the proof.

42. By recursive arithmetic we mean the branch of arithmetic dealing with primitive recursive functions. We mean a system strong enough to serve as a syntax language in the sense of [5].

43. I am indebted to both Prof. Church and Prof. Gödel for many valuable suggestions in connection with the following proof. The basic idea I finally adopted is due to Prof. Gödel.

44. They are one-many mappings to allow the same set to occur more than once as a member of a givén k-tuple.

45. See [6].

46. That there really is such a set, or more precisely that $x_\omega' \neq 0$, is proven in theorem 82 of [3].

47. It is interesting to note that L_γ, for $\gamma \geq \omega+2$, (hence any transfinite type-theory of this kind) is as strong as the corresponding system with many-place variables.

48. See fn. 35.

49. See [14].

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