

UNIVERSITY OF WISCONSIN - MADISON
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A MODEL OF SET THEORY WITH A UNIVERSAL SET

Emerson Mitchell

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ABSTRACT

Let \underline{T} be the theory of the language of set theory saying that:

- 1) Sets are extensional.
- 2) Every set has a universal complement, i.e. given a set x there is a set y such that every set z is a member of y if and only if it is not a member of x .
- 3) Every set x has a power set containing exactly the subsets of x .
- 4) The result of replacing every member of a well-founded set by some set is a set.
- 5) The well-founded sets form a model of Zermelo-Fraenkel set theory.

Then within the universe V of Zermelo-Fraenkel set theory there is a definable internal model of \underline{T} . The members of the internal model are chosen by an inductive definition within V , and then a new membership relation is inductively defined such that the members of the internal model with the defined membership relation satisfy \underline{T} . It happens that the members of the internal model which are well-founded on the defined membership relation form an isomorphic copy of V . Thus one can regard the construction as an extension of V to a model of \underline{T} .

Corollary: \underline{T} is consistent if and only if Zermelo-Fraenkel set theory is consistent.

This paper constitutes the author's dissertation to be submitted to the University of Wisconsin-Madison. It was inspired by the author's hearing ALONZO CHURCH give his paper Set Theory with a Universal Set, Proceedings of the Tarski Symposium, Proceedings of Symposia in Pure Mathematics, XXV, pp 297-308, AMS (1974).

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*critique
abstract
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Section 0 - Introduction

Ordinary Zermelo-Fraenkel set theory (Z-F) is based on two heuristic principles of set theory. These are:

- 1) The principle of specialized comprehension, that set theory is constructed by assuming as many cases as consistently possible of the naive and inconsistent comprehension axiom scheme, and
- 2) The principle of limitation of size, that only sets which are too small to be placed in a many-one (or one-one) relation with the universe should be allowed.

In [1] Alonzo Church asserts that there is no justification for the principle of limitation of size, and indeed constructs an extension of the universe of Z-F containing a universal set. Church regards his theory in [1] as a preliminary attempt, and encourages further research toward axioms of set theory with a universal set. This paper is a further consistency result in the program of investigating set theories more comprehensive than Z-F.

Church's construction in [1] and the construction in this paper both create extensions of any model of Z-F, but neither is an extension of the other. Church's construction is closed under unions and intersections, which the construction in this paper is not. The construction in this paper contains the power set of any set, and Church's does not. Additionally, Church's construction contains

*entire intro
could be removed*

objects called j -cardinal numbers which this construction does not. Those j -cardinal numbers are too complicated to describe here.

There are several ways of looking at the construction in this paper. Informally one adds ideal elements to the original universe of Z-F until one has sufficient elements to satisfy the new axioms. One can also think of adding new objects to any model of Z-F to create a model of the new theory. Formally the procedure used here is to provide a relative interpretation of the new theory within Z-F.

All work will be within Z-F. Let \mathcal{L} be the language of Z-F: predicate logic with equality (=) and membership (ϵ). For the purposes of this paper, the class model $\mathfrak{U} = \langle A, E \rangle$ of \mathcal{L} (to be constructed) is a predicate A and a binary relation E , both definable in Z-F.

Any sentence ϕ of \mathcal{L} can be relativized to \mathfrak{U} in the following fashion: relativize all quantifiers to A and replace all occurrences of ϵ by occurrences of E . Note that the relation of equality (=) is not changed. The result will be denoted $\phi^{\mathfrak{U}}$, and we will write

$$\vDash_{\mathfrak{U}} \phi \text{ to mean } \vDash_{Z-F} \phi^{\mathfrak{U}}.$$

The most familiar example of a class model is the constructible universe L with standard membership. ($xEy \equiv x \epsilon y$).

Another example of a class model is Church's construction.

The simplest case of his construction is defined by $\mathfrak{C} = \langle C, F \rangle$ where $C(x)$ is the predicate $x = x$ and F is the binary relation defined by:

$$x F y \equiv_{DF} \begin{cases} x \in \langle y_0, i \rangle \text{ if } y = \langle y_0, i + 1 \rangle \text{ for some} \\ \text{set } y_0, \text{ for some } i \in \omega \\ x \notin y_0 \text{ if } y = \langle y_0, 0 \rangle \text{ for some set } y_0 \\ x \in y \text{ in any other case .} \end{cases}$$

Note that the effect of Church's construction is to replace each ordered pair $\langle y_0, i \rangle$ by the ordered pair $\langle y_0, i + 1 \rangle$, thus freeing the ordered pairs $\langle y_0, 0 \rangle$ to play the role of complements to the rest of the sets.

The purpose of this paper is to define a particular class model $\mathfrak{U} = \langle A, E \rangle$ and show that it has certain properties I through V defined below. These are the properties that \mathfrak{U} satisfies extensionality, has for every set a complement and a power set, can form arbitrary sets by replacing each member of a well-founded set by an arbitrary set, and contains a model of Z-F which is precisely the well-founded sets.

Before giving properties I thru V formally, some definitions and notational conventions are in order. We define the predicate "u is well-founded" by:

$$WF(u) \equiv_{DF} \forall z u \in z \Rightarrow (\exists x x \in z \wedge (\forall y y \in z \Rightarrow y \not\in x)) .$$

The greek letter ϕ is a meta-language variable standing for arbitrary formulae in \mathcal{L} and the notation $\phi(x, z, u, \vec{v})$ means

x, z, u, \vec{v} may (not must) occur free in ϕ . The vector notation $\vec{v} \dots \vec{v} \dots$ stands for $\forall v_1, \dots, \forall v_n, \dots, v_1, \dots, v_n, \dots$.

For the purposes of V below, \mathfrak{B} is the class model defined by $\mathfrak{B} = \langle WF, \epsilon \rangle$ so that

$$\models_{\mathfrak{A}} \models_{\mathfrak{B}} \phi \equiv_{DF} \models_{\mathfrak{A}} \phi^{\mathfrak{B}}$$

where $\phi^{\mathfrak{B}}$ is formed from ϕ by relativizing all quantifiers and free variables (if any) to WF .

The properties I through V can now be formally stated:

$$I) \models_{\mathfrak{A}} \forall x \forall y (\forall z (z \in x \iff z \in y) \implies x = y$$

(extensionality)

$$II) \models_{\mathfrak{A}} \forall x \exists y \forall z (z \in y \iff z \notin x$$

(complements)

$$III) \models_{\mathfrak{A}} \forall x \exists y \forall z (z \in y \iff z \subseteq x$$

(power sets).

$$IV) \models_{\mathfrak{A}} \vec{v} \forall u (WF(u) \implies [(\forall x \in u \exists ! z \phi(x, z, u, \vec{v})) \implies$$

$$\exists y \forall z [z \in y \iff \exists x \in u \phi(x, z, u, \vec{v})]]]$$

(where ϕ is any formula of \mathcal{L} not containing y free)

(well-founded replacement).

$$V) \models_{\mathfrak{A}} \models_{\mathfrak{B}} \phi \quad (\text{where } \phi \text{ is any axiom of Z-F})$$

(end extension of Z-F).

The original contribution of the author is to combine the power set axiom (III) with the other properties. Church's class model defined above satisfies I, II, IV and V as well as being complete under unions and intersections.

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A couple of comments on complements in the sense of II are in order. First, no model of II can be well-founded.

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Proof: Assume by II, for some a, b in the model that $\forall z z \in a \iff z \notin b$. If the model were well-founded then we would have $a \notin a$ and $b \notin b$. But then we would have $a \in b$ and $b \in a$. \dashv

Second, II is inconsistent with the subset comprehension schema in the presence of either an empty set or an axiom of finite unions.

Proof: Given an empty set \emptyset , II yields a universe u such that $\forall x x \in u$. On the other hand, by II we get sets x and y such that $\forall z z \in x \iff z \notin y$, and if we had an axiom of finite unions then we would get that $\forall z z \in u$ for $u = x \cup y$. In either case the subset comprehension schema applied to u would yield the inconsistent comprehension schema $\exists x \forall z z \in x \iff \phi(z)$.

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(ϕ any formula of \mathcal{L} not containing x free). \dashv

Note that this last paragraph does not imply the inconsistency of I-V. V does imply the existence of an empty set, but I-V together are not strong enough to imply the subset comprehension schema.

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Section 1 - Construction

The entire construction is pure bookkeeping & could be replaced by a more intuitive notation if we assume the proof.

For the construction of the class model $\mathfrak{U} = \langle A, E \rangle$ it is necessary to give definitions of A and E . To define A , first a four-place predicate $A_{ij\alpha}(x)$ is defined and then A is defined by "union":

$$A(x) \equiv_{DF} \exists i \exists j \exists \alpha A_{ij\alpha}(x).$$

The definition that will be given for $A_{ij\alpha}(x)$ will be in the form of an inductive definition. The reader may check that the inductive definition actually yields an $A_{ij\alpha}(x)$ definable by a formula of Z-F. The thing to note is that each induction step depends only on a fixed set of "previous" steps in the induction, in fact on a set of objects of lower rank as Z-F sets than the object to which the induction is being applied.

This is prevented by clause i.

The reader should note that for any set x , $A(x)$ will imply that x is an ordered four-tuple $\langle x', i_x, j_x, \alpha_x \rangle$, where x' is some set, i_x is a member of $4 = \{0, 1, 2, 3\}$, j_x is a natural number and α_x is an ordinal. We define therefore the following notation: if x is any four-tuple then

$$x =_{DF} \langle x', i_x, j_x, \alpha_x \rangle.$$

This notation will be used throughout this paper.

The motivation of the definition of A is to provide a class of sets which will serve as the objects of the class model \mathfrak{U} . Each clause provides objects to serve a specific role in \mathfrak{U} . Clauses ii and iii provide two objects for every set in Z-F, one to serve as

the copy of that set in \mathfrak{U} , the other as its complement. Clause viii and ix provide objects to serve as the power set and complement of the power set of objects for which no previously generated object would serve as a power object. Clauses x and xi serve the same purpose for sets required by the replacement axiom scheme IV. After A is defined to provide the required objects, E will be defined so that they play the specified role in \mathfrak{U} .

The definition is as follows:

i) $(i \neq 4 \vee j \neq \omega \vee -(\alpha \text{ is an ordinal})) \Rightarrow$

$$(A_{ij\alpha}(x) \equiv_{DF} x \neq x)$$

ii) $A_{000}(x) \equiv_{DF} \exists y x = \langle y, 0, 0, 0 \rangle$

*original ZF sets Y
their complements \overline{Y}*

iii) $A_{100}(x) \equiv_{DF} \exists y x = \langle y, 1, 0, 0 \rangle$

iv) $A_{20\alpha}(x) \equiv_{DF} x \neq x$

v) $A_{30\alpha}(x) \equiv_{DF} x \neq x$

*unused pieces of the construction,
purely for convenience*

vi) $A_{0j+1\alpha}(x) \equiv_{DF} x \neq x$

vii) $A_{1j+1\alpha}(x) \equiv_{DF} x \neq x$

viii) $A_{2j+1\alpha}(x) \equiv_{DF} \exists y \exists i \in \{1, 2, 3\} A_{ij\alpha}(y)$

$$\wedge x = \langle y, 2, j+1, \alpha \rangle$$

$$\wedge y \neq \langle 0, 1, 0, 0 \rangle$$

$$x = \mathcal{P}(Y)$$

ix) $A_{3j+1\alpha}(x) \equiv_{DF} \exists y \exists i \in \{1, 2, 3\} A_{ij\alpha}(y)$

$$\wedge x = \langle y, 3, j+1, \alpha \rangle$$

$$\wedge y \neq \langle 0, 1, 0, 0 \rangle$$

$$x = \overline{Y}$$

for $Y \in \text{power set}$

x is the 'replacement' object whose 'members' are the members of objects which are members of Y

$$\begin{aligned}
 \text{x) } \alpha > 0 \Rightarrow (A_{00\alpha}(x) \equiv_{DF} \exists y [x = \langle y, 0, 0, \alpha \rangle \\
 \wedge [\forall z \in y \exists i \exists j \exists \beta \in \alpha A_{ij\beta}(z)] \\
 \wedge \text{L.U.B. \{Ordinals } (\beta + 1) : \exists i \exists j \exists z \in y A_{ij\beta}(z) \wedge \neg A_{000}(z) \} \\
 = \alpha])
 \end{aligned}$$

x = \bar{Y} for Y a replacement object

$$\text{xi) } \alpha > 0 \Rightarrow (A_{10\alpha}(x) \equiv_{DF} \exists y [x = \langle y, 1, 0, \alpha \rangle \wedge A_{00\alpha}(\langle y, 0, 0, \alpha \rangle)]) .$$

Before going on to the definition of E, some more terminology and notation must be defined. The objects in A are naturally classified by the clause which generated them, or equivalently by their second co-ordinate i_x . This classification will often be referred to according to the intent of the particular clause involved, as follows:

- x is a replacement object $\equiv_{DF} i_x = 0$
- x is a complement object $\equiv_{DF} i_x = 1$ or $i_x = 3$
- x is a power object $\equiv_{DF} i_x = 2$.

The reader should not confuse the above terminology, in which "complement" and "power" occur as adjectives modifying "object", with set-theoretic use of "complement" and "power set" as in "x is the complement of y". Three further notational definitions should be given. If x is a replacement object, then m_x is a Z-F set:

the set of A-objects which will be \mathfrak{U} -members of x. We define m_x by

$$i_x = j_x = 0 \Rightarrow m_x =_{DF} \begin{cases} x' & \text{if } \alpha_x > 0 \\ \{y : A_{000}(y) \wedge y' \in x'\} & \text{if } \alpha_x = 0 . \end{cases}$$

If x is any A-object, then \bar{x} will be the complement of x in \mathcal{U} , where \bar{x} is defined by

$$\bar{x} = \text{DF} \begin{cases} \langle x', 1, j_x, \alpha_x \rangle & \text{if } i_x = 0 \\ \langle x', 0, j_x, \alpha_x \rangle & \text{if } i_x = 1 \\ \langle x', 3, j_x, \alpha_x \rangle & \text{if } i_x = 2 \\ \langle x', 2, j_x, \alpha_x \rangle & \text{if } i_x = 3. \end{cases}$$

If x is any A-object, then $\mathcal{P}x$ is the object which will be the power set of x in the sense of \mathcal{U} . We can define $\mathcal{P}x$ on all A-objects by:

$$\mathcal{P}x = \text{DF} \begin{cases} x & \text{if } x = \langle 0, 1, 0, 0 \rangle \\ \langle x, 2, j_x + 1, \alpha_x \rangle & \text{if } i_x = 1, 2 \text{ or } 3 \\ & \text{and } x \neq \langle 0, 1, 0, 0 \rangle \\ \langle \{z : z \subseteq x'\}, 0, 0, 0 \rangle & \text{if } i_x = 0, \alpha_x = 0 \\ \langle \{z : i_z = 0 \wedge \alpha_z \leq \alpha_x \wedge m_z \subseteq m_x\}, 0, 0, \alpha_x \rangle & \text{if } i_x = 0, \alpha_x > 0. \end{cases}$$

$\mathcal{P}(u) = u$ universal set

powers of replacement objects are supplied by subsequent replacement objects.

The reader should note that at this point it remains to prove that the functions m_x , \bar{x} and $\mathcal{P}x$ actually serve the intended purpose. Nevertheless they are well defined.

From here on
to work on page 2d
is the actual proof.
It is a tour-
de-force of
case-by-case
bookkeeping
arguments, and
not particularly
interesting in
itself.

If you include
it, you had
best include it
all, as it
does not condense
well.

The definition of E will be given by induction using the intended properties of the various kinds of A -objects for various cases.

There is a difficulty in this plan. The power objects are intended to have as E -members all E -subsets of x . But if, for example, x is the complement of a replacement object, then arbitrarily far out in the construction of A we keep getting new objects which should be E -subsets of x , because their E -members will all be E -members of x . This is apparently circular. To evade this problem, E will be defined by induction simultaneously with another relation S .

One case will specify that $y \in P x$ iff $y S x$ has been determined "previously" in the induction. Upon completing the inductive definition it will be shown that $y S x$ iff $A(x) \wedge A(y) \wedge \forall z z E y \Rightarrow z E x$.

The following eighteen clauses constitute the inductive definition of E and S :

- a) $\forall x \forall y - A(x) \vee - A(y) \Rightarrow [-(x S y) \wedge -(x E y)]$
- b) $A_{000}(y) \Rightarrow x E y \equiv_{DF} A_{000}(x) \wedge x' \in y'$
- c) $A_{000}(y) \Rightarrow x S y \equiv_{DF} A_{000}(x) \wedge x' \subseteq y'$
- d) $\alpha > 0 \Rightarrow A_{00\alpha}(y) \Rightarrow x E y \equiv_{DF} A(x) \wedge x \in y'$
- e) $\alpha > 0 \Rightarrow A_{00\alpha}(y) \Rightarrow x S y \equiv_{DF} i_x = 0 \wedge m_x \subseteq m_y$
- f) $A_{10\alpha}(y) \Rightarrow x E y \equiv_{DF} - (x E \bar{y})$
- g) $A_{2j\alpha}(y) \Rightarrow x E y \equiv_{DF} x S y'$
- h) $A_{3j\alpha}(y) \Rightarrow x E y \equiv_{DF} - (x E \bar{y})$
- j) $A_{10\alpha}(y) \Rightarrow A_{00\beta}(x) x S y \equiv_{DF} m_x \cap m_y^- = \emptyset$

- k) $A_{10\alpha}(y) \Rightarrow A_{10\beta}(x) \Rightarrow xSy \equiv_{DF} m_y^- \subseteq m_x^-$
- l) $A_{10\alpha}(y) \Rightarrow A_{2j\beta}(x) \Rightarrow xSy \equiv_{DF} \forall w \in m_y^- - (wEx)$
- m) $A_{10\alpha}(y) \Rightarrow A_{3j\beta}(x) \Rightarrow xSy \equiv_{DF} \bar{ySx}$
- n) $A_{2j\alpha}(y) \Rightarrow (A_{10\beta}(x) \vee A_{3k\delta}(x)) \Rightarrow xSy \equiv_{DF} x \neq x$
- o) $A_{2j\alpha}(y) \Rightarrow A_{00\beta}(x) \Rightarrow xSy \equiv_{DF} \forall w \in m_x wSy'$
- p) $A_{2j\alpha}(y) \Rightarrow A_{2k\beta}(x) \Rightarrow xSy \equiv_{DF} x'Sy'$
- q) $A_{3j\alpha}(y) \Rightarrow A_{00\beta}(x) \Rightarrow xSy \equiv_{DF} \forall w \in m_x - (wE\bar{y})$
- r) $A_{3j\alpha}(y) \Rightarrow (A_{10\beta}(x) \vee A_{2k\delta}(x)) \Rightarrow xSy \equiv_{DF} x \neq x$
- s) $A_{3j\alpha}(y) \Rightarrow A_{3k\beta}(x) \Rightarrow xSy \equiv_{DF} \bar{ySx}$.

It may not be immediately obvious that clauses a thru s constitute a well-founded induction and not a circular definition. An analogous problem is encountered in the definition of forcing. The problem here is handled similarly.

The relations xEy , xSy are defined by simultaneous induction on the lexicographic order of

$$\sigma_A(x, y) =_{DF} \langle \max(\alpha_x, \alpha_y), \min(\alpha_x, \alpha_y), j_x, j_y, i_x, i_y \rangle.$$

It is then straightforward to check that in each clause xSy or xEy is defined in terms of S, E for a set of pairs with lower values of $\sigma_A(x, y)$. This removes any question of circularity.

The reader may be bothered by the fact that the induction is apparently being accomplished on proper classes. That is,

$\{\langle x, y \rangle : \sigma_A(x, y) = \langle \beta, \gamma, j, j', i, i' \rangle\}$ is usually a proper class for fixed β, γ, j, j', i and i' . This caused the author a problem when

Dr. Ken Kunen pointed out that induction on sequences of proper classes is not in general formalizable in Z-F. That this particular induction is formalizable in Z-F can be seen in two ways. Observe that every clause defines xSy or xEy in terms of pairs of objects of no greater set-theoretic rank. Thus for any α the induction defines sets $S \cap R(\alpha)^2$, $E \cap R(\alpha)^2$, giving S and E by union over all α . Alternatively, the fact that each clause depends on a set of previous pairs allows us to construct sets deciding the value of S, E for certain pairs such that if any pair is decided, all pairs on which its value depends are also decided by the same set. One then proves that all such decision sets are compatible and any pair is decided by some decision set. This last argument is similar to the argument that the rank function is definable.

It now must be shown that within A xSy means x is an E -subset of y , as promised.

Theorem:

$$\forall x \forall y (A(x) \wedge A(y)) \implies [xSy \iff \forall z zEx \implies zEy] .$$

Proof: Start by observing that by clauses c and e , if y is a replacement object ($i_y = 0$) then for all x

$$\begin{aligned} xSy &\iff i_x = 0 \wedge m_x \subseteq m_y \\ &\iff i_x = 0 \wedge \forall z z \in m_x \implies z \in m_y \\ &\iff i_x = 0 \wedge \forall z zEx \implies zEy \\ &\quad \text{(by definition of } m_x \text{)} . \end{aligned}$$

The case $i_y = 0$ will be complete if we can show that for all non-replacement A-objects x there is a z such that $zEx \wedge z\bar{E}y$, since clause c implies $x\bar{E}y$ for any non-replacement $x(i_x \neq 0)$.

LEMMA: If x is a non-replacement A-object, there is a proper class of objects z such that zEx .

Proof of lemma: By induction on j_x . $j_x = 0 \Rightarrow i_x = 1 \Rightarrow x$ is the complement of a replacement object $\bar{x} \Rightarrow \exists$ a set $m_{\bar{x}}$ of objects z such that $z\bar{E}x \Leftrightarrow z \in m_{\bar{x}}$. Now $zEx \Leftrightarrow z\bar{E}\bar{x}$ for all z such that $A(z)$. And A defines a proper class since A_{000} contains a "copy" of every set in the universe. Hence zEx for all z in $A - m_{\bar{x}}$, which is certainly a proper class. That completes the base step of the induction.

Now assume the lemma holds whenever $j_x \leq n$. We must show it is true when $j_x = n + 1$. Now $j_x = n + 1 \Rightarrow i_x = 2$ or 3 . Assume $i_x = 2$, so $x = \mathcal{P}x'$ is a power object. Then $j_{x'} = n$ and $i_{x'} = 1, 2$ or 3 by clause viii, and hence by hypothesis there is a proper class of those z such that zEx' . Consider all the replacement objects of the form $y_1 = \langle \{z\}, 0, 0, \alpha_z + 1 \rangle$. There is a proper class of such, and they all satisfy $\forall z \in m_{y_1} = \{z\}, zEx'$.

Now if $i_{x'} = 1$, we have $\forall z \in m_{y_1} zEx'$

$\Leftrightarrow \forall z \in m_{y_1} z \notin \overline{Ex'}$ by clause f

$\Leftrightarrow \forall z \in m_{y_1} z \notin m_{\overline{x'}}$ by definition of $m_{\overline{x'}}$,
(noting $\overline{x'}$ is a replacement object)

$\Leftrightarrow m_{y_1} \cap m_{\overline{x'}} = \emptyset$

$\Leftrightarrow y_1 Sx'$ by clause j

$\Leftrightarrow y_1 Ex$ by clause g

and if $i_{x'} = 2$, we have $\forall z \in m_{y_1} zEx'$

$\Leftrightarrow \forall z \in m_{y_1} zSx''$ by clause g

$\Leftrightarrow y_1 Sx'$ by clause o

$\Leftrightarrow y_1 Ex$ by clause g

and if $i_{x'} = 3$, we have $\forall z \in m_{y_1} zEx'$

$\Leftrightarrow \forall z \in m_{y_1} z \notin \overline{Ex'}$ by clause h

$\Leftrightarrow y_1 Sx'$ by clause q

$\Leftrightarrow y_1 Ex$ by clause g .

Hence, whatever $i_{x'}$ is we have a proper class of y_1 such that $y_1 Ex$, which completes the case where $i_x = 2$. Now

assume $i_x = 3$, so $x = \overline{Px'}$ is the complement of a power

object, and by clause ix we have $x' \neq \langle 0, 1, 0, 0 \rangle$. This requires a

Claim: under the given hypotheses, $\exists z$ such that zEx' .

Proof of claim: If $i_{\bar{x}'} = 1, 2$ or 3 we need only observe that

$j_{\bar{x}'} = j_{x'} = j_x - 1$, and hence by the lemma induction hypothesis

the claim is trivial. If $i_{\bar{x}'} = 0$, the claim is that $m_{\bar{x}'} \neq \emptyset$.

Now for $\alpha_{\bar{x}'} = 0$, we must have $\alpha_{\bar{x}'} = j_{\bar{x}'} = i_{\bar{x}'} = 0$, so if

$m_{\bar{x}'} = 0$, we would have $\bar{x}' = \langle 0, 0, 0, 0 \rangle$ and hence

$x' = \langle 0, 1, 0, 0 \rangle$, which is ruled out. And if $\alpha_{\bar{x}'} > 0$, the l.u.b.

part of clause x rules out having $m_{\bar{x}'} = \emptyset$. That completes the

proof of the claim.

Given the claim, there is then a proper class of replacement

objects of the form $y_1 = \langle \{z, w\}, 0, 0, \alpha_{y_1} \rangle$ such that zEx' .

For any such y_1 , the following holds

$$\exists z \in m_{y_1} \ni zEx'$$

The claim is that all such y_1 satisfy y_1Ex . By clauses viii

and ix, it is the case that $i_{x'} = 1, 2$ or 3 . Now if $i_{x'} = 1$

then

$$\exists z \in m_{y_1} \ni zEx'$$

$$\Leftrightarrow m_{y_1} \cap m_{\bar{x}'} \neq \emptyset$$

by definition of $m_{\bar{x}'}$, noting

that \bar{x}' is a replacement

object

$$\begin{aligned} &\Leftrightarrow y_1 \bar{S}x' && \text{by clause j} \\ &\Leftrightarrow y_1 \bar{E}Px' && \text{by clause g} \\ &\Leftrightarrow y_1 E\bar{P}x' = x && \text{by clause h.} \end{aligned}$$

And if $i_{x'} = 2$ then

$$\begin{aligned} &\exists z \in m_{y_1} \ni z \bar{E}x' \\ &\Leftrightarrow \exists z \in m_{y_1} \ni z \bar{Z}x' && \text{by clause h} \\ &\Leftrightarrow -(\forall z \in m_{y_1} zEx') && \text{by quantifier theory} \\ &\Leftrightarrow -(\forall z \in m_{y_1} zSx'') && \text{by clause g} \\ &\Leftrightarrow -(y_1 Sx') && \text{by clause o} \\ &\Leftrightarrow -y_1 EPx' && \text{by clause g} \\ &\Leftrightarrow y_1 E\bar{P}x' = x && \text{by clause h.} \end{aligned}$$

And if $i_{x'} = 3$ then

$$\begin{aligned} &\exists z \in m_{y_1} \ni z \bar{E}x' \\ &\Leftrightarrow -(\forall z \in m_{y_1} - (z \bar{E}x')) && \text{by quantifier theory} \\ &\Leftrightarrow -(y_1 Sx') && \text{by clause q} \\ &\Leftrightarrow -(y_1 EPx') && \text{by clause g} \\ &\Leftrightarrow y_1 E\bar{P}x' = x && \text{by clause h.} \end{aligned}$$

Hence, for any value of $i_{x'}$, we have that $y_1 Ex$ for all of a proper class of y_1 , completing the proof of the lemma for $i_x = 3$, which is the last case for the lemma. \dashv

Given the lemma, we complete the case $i_y = 0$ of the theorem by observing that m_y is a set, and hence by the lemma any non-replacement x will contain a z not in m_y . Such a z satisfies zEx and $z\bar{E}y$. The next case of the theorem is to assume $i_y = 1$, so y is the complement of a replacement object. This divides into four subcases depending on the value of i_x .

$$i_y = 1, i_x = 0$$

$$xSy \equiv_{DF} m_x \cap m_y^- = \emptyset$$

by clause j

$$\Leftrightarrow \forall z z \in m_x \Rightarrow z \notin m_y^-$$

by definition of \cap

$$\Leftrightarrow \forall z zEx \Rightarrow z\bar{E}y$$

by definition of m_x, m_y^-

$$\Leftrightarrow \forall z zEx \Rightarrow zEy$$

by clause f.

$$i_y = 1, i_x = 1$$

$$xSy \equiv_{DF} m_y^- \subseteq m_x^-$$

by clause k

$$\Leftrightarrow \forall z zE\bar{y} \Rightarrow zE\bar{x}$$

by definitions of \subseteq, m_y^-, m_x^-

$$\Leftrightarrow \forall z z\bar{E}y \Rightarrow z\bar{E}x$$

by clause f

$$\Leftrightarrow \forall z zEx \Rightarrow zEy$$

by contraposition.

$$i_y = 1, i_x = 2$$

$$xSy \equiv_{DF} \forall w w \in m_y^- \Rightarrow w\bar{E}x$$

by clause l

$$\Leftrightarrow \forall w w\bar{E}y \Rightarrow w\bar{E}x$$

by definition of m_y^- and

clause f

$$\Leftrightarrow \forall w wEx \Rightarrow wEy$$

by contraposition.

$$i_y = 1, i_x = 3$$

$$xSy \equiv_{DF} \bar{y}S\bar{x}$$

by clause m

$$\Leftrightarrow \forall w \in m_y^- wSx'$$

by clause o

$$\Leftrightarrow \forall w wE\bar{y} \Rightarrow wE\bar{x}$$

by clause g and definition

of m_y^-

$$\Leftrightarrow \forall w wE\bar{y} \Rightarrow wE\bar{x}$$

by clauses f and h

$$\Leftrightarrow \forall w wEx \Rightarrow wEy$$

by contraposition.

That proves the theorem for $i_y = 1$. The cases $i_y = 2$ and $i_y = 3$ will proceed jointly by induction of j_y and j_x . That is, we will assume as our induction hypothesis that

$$\forall v j_v < j_x \vee j_v < j_y \Rightarrow \forall w wSv \Leftrightarrow \forall z zEw \Rightarrow zEv.$$

Note that we have already argued the base case where $j_x = j_y = 0$ and hence $i_x = 0$ or 1 and $i_y = 0$ or 1. Proceeding by cases of $i_y \in \{2, 3\}$ and $i_x \in 4$, let $i_y = 2$ and $i_x = 0$ and get

$$xSy \equiv_{DF} \forall w \in m_x wSy'$$

by clause o

$$\Leftrightarrow \forall w wEx \Rightarrow wEy$$

by definition of m_x and clause g.

$i_y = 2$ and $i_x = 2$ gives

$$xSy \equiv_{DF} x'Sy'$$

by clause p

$$*) \quad \Leftrightarrow \forall w wEx' \Rightarrow wEy'$$

by induction hypothesis.

In order to show that

$$xSy \iff \forall z zEx \implies zEy$$

the two implications will be argued separately.

\implies) Assume xSy and choose any z such that zEx , must show zEy .

Now since x is a power object, $x = \rho x'$, then

$$zE\rho x'$$

$$\text{iff } zSx'$$

by clause g

$$\text{iff } \forall w wEz \implies wEx'$$

by induction hypothesis.

Hence $zEx \implies \forall w wEz \implies wEy'$ from * above.

Now $\forall w wEz \implies wEy'$

$$\iff zSy'$$

by induction hypothesis

applied to y'

$$\iff zE\rho y'$$

by clause g

$$\iff zEy$$

since y is a power

object, $y = \rho y'$.

By the generality of z , we have shown the forward implication

$$xSy \implies \forall z zEx \implies zEy .$$

\Leftarrow) Assume now that

$$\forall z zEx \implies zEy. \quad \text{but then}$$

$$\forall z zSx' \implies zSy'$$

by clause g

so $x'Sx' \implies x'Sy'$

by univ. spec.

but by induction hypothesis

$$x'Sx' \iff \forall z zEx' \iff zEx' .$$

The right hand formula is a tautology, so we get

$x'Sx'$ whence we get

$x'Sy'$ by modus ponens,

which gives xSy by clause p. That completes the induction

step for $i_y = 2$ and $i_x = 2$. Assume $i_y = 2$ and $i_x = 3$

or $i_x = 1$. By clause n we have that $x\cancel{S}y$ so it remains only

to show that $\exists z zEx \wedge z\cancel{E}y$. For $i_x = 1$ a previous lemma

showed there is a proper class of z such that $z\cancel{E}y$, but

there is only a set m_x^- of z such that $z\cancel{E}x$, so trivially

there is some z such that $zEx \wedge z\cancel{E}y$. If now $i_x = 3$,

consider the single object $u = \langle 0, 1, 0, 0 \rangle$. We have that for

any z

$zEu \iff z\cancel{E}u$ by clause f

$\iff z\cancel{E} \langle 0, 0, 0, 0 \rangle$

$\iff z \notin \emptyset$.

Hence $\forall z zEu$. The claim is that uEx and $u\cancel{E}y$. Recall

that in the current case $x = \overline{Px'}$ and $y = Py'$. So by clauses

h and g the claim is that $u\cancel{S}x'$ and $u\cancel{S}y'$. By the induction

hypothesis,

$uSx' \iff \forall z zEu \implies zEx'$

$\iff \forall z zEx'$.

Now it was shown previously that either \bar{x}' has a proper class of E-members or is a replacement object. So $\forall z zEx'$ is only possible if x' is the complement of a replacement object \bar{x}' with $m_{\bar{x}'} = \emptyset$. But $m_{\bar{x}'} = \emptyset$ cannot happen in clause x by the L. U. B. requirement, since L. U. B. $\phi = 0$. The form $\bar{x}' = \langle \bar{x}'^1, 0, 0, 0 \rangle = \langle 0, 0, 0, 0 \rangle$ is the only remaining possibility. But then x' would be $u = \langle 0, 1, 0, 0 \rangle$, and we specifically ruled out this possibility in clauses viii and ix. Hence $\exists z z\bar{E}x'$ and $u\bar{E}x'$ as claimed. By the same argument, $u\bar{E}y'$. We have left to argue only the induction step for $i_y = 3$, with 4 sub-cases depending on i_x .

Case $i_y = 3, i_x = 0$. Then

$$\begin{aligned}
 xSy &\equiv_{DF} \forall w \in m_x - (wE\bar{y}) && \text{by clause q} \\
 &\iff \forall w wEx \implies wEy && \text{by definition of } m_x \\
 &&& \text{and clause f.}
 \end{aligned}$$

$i_y = 3, i_x = 3$. Then

$$\begin{aligned}
 xSy &\equiv_{DF} \bar{y}S\bar{x} && \text{by clause s} \\
 &\iff \forall z zE\bar{y} \implies zE\bar{x} && \text{by induction hypothesis} \\
 &\iff \forall z z\bar{E}y \implies z\bar{E}x && \text{by clause h} \\
 &\iff \forall z zEx \implies zEy && \text{by contraposition.}
 \end{aligned}$$

For $i_y = 3$, $i_x = 1$ or $i_x = 2$ clause r gives that xSy , so it remains only to prove that

$$\exists z zEx \wedge zEy.$$

If $i_x = 1$, observe that by the previous argument \bar{y} has a proper class of E-members and \bar{x} does not. Hence

$$\exists z zE\bar{y} \wedge zE\bar{x} \text{ and so by clauses f and h } \exists z zEx \wedge zEy.$$

Finally, if $i_x = 2$ the claim is that $\phi_E = \langle 0, 0, 0, 0 \rangle$ satisfies

$$\phi_E Ex \wedge \phi_E Ey. \text{ By clauses g and h the claim reduces to}$$

$$\phi_E Sx' \wedge \phi_E Sy', \text{ observing that } x = \rho x' \text{ and } y = \overline{\rho y'}. \text{ By}$$

the induction hypothesis, this reduces to

$$(\forall z zE\phi_E \Rightarrow zEx') \wedge (\forall z zE\phi_E \Rightarrow zEy').$$

But this is trivially true, since $\forall z zE\phi_E$ because $m_{\phi_E} = \phi$.

That completes the proof that

$$\forall x \forall y (A(x) \wedge A(y)) \Rightarrow [xSy \Leftrightarrow \forall z zEx \Rightarrow zEy]. \quad \dashv$$

Section 2 - Proof

It now fairly obviously follows that

$$\mathfrak{U} = \langle A, E \rangle \models I, II, III, IV \text{ and } V.$$

They will be proven one at a time.

Theorem: (I) EXTENSIONALITY

$$\mathfrak{U} \models \forall x \forall y (\forall z (z \in x \iff z \in y) \implies x = y).$$

Proof: This translates into

$$\forall x \forall y (A(x) \wedge A(y)) \implies [(\forall z (zEx \iff zEy)) \implies x = y]$$

which is equivalent to

$$\forall x \forall y (A(x) \wedge A(y)) \implies [x \neq y \implies \exists z (zEx \wedge \neg zEy) \vee (\neg zEx \wedge zEy)].$$

So we assume $A(x) \wedge A(y) \wedge x \neq y$ and show $\exists z (zEx \not\iff zEy)$.

The argument breaks into familiar cases depending on the values of i_x and i_y . By the proper class lemma, we have already dealt with the cases where one of x or y is a replacement object or the complement of a replacement object and the other satisfies $i_x \neq i_y$. And the arguments using u and ϕ_E have already dealt with the cases where one of x or y is a power object and the other is the complement of a power. Hence the only cases left are those where $i_y = i_x$.

Case $i_x = i_y = 0$. We must show

$$x \neq y \implies m_x \neq m_y.$$

Now if $\alpha_x \neq \alpha_y$, this is trivial by the l. u. b. part of clause x.

And if $\alpha_x = \alpha_y$ we have $i_x = i_y = 0$ and hence $j_x = j_y = 0$, so

$$m_x = m_y \Rightarrow x = y.$$

Case $i_x = i_y = 1$.

$$\begin{aligned} x \neq y &\Rightarrow \bar{x} \neq \bar{y} \Rightarrow m_x^- \neq m_y^- && \text{by previous argument} \\ &\Rightarrow \exists z z \in m_x^- \not\subset z \in m_y^- && \text{by ext. in Z-F} \\ &\Rightarrow \exists z z E \bar{x} \not\subset z E \bar{y} && \text{by definition of } m_x^- \text{ and } m_y^- \\ &\Rightarrow \exists z z E x \not\subset z E y && \text{by clause f.} \end{aligned}$$

The cases where $i_x = i_y = 2$ and $i_x = i_y = 3$ proceed by joint induction on j_x and j_y as before. If $i_x = i_y = 2$,

$$x = \rho x' \neq y = \rho y' \Rightarrow x' \neq y' \quad \text{since } \rho \text{ is 1-1 in this case}$$

$$\Rightarrow \exists z z E x' \not\subset z E y' \quad \text{by induction hypothesis}$$

$$\Rightarrow \exists z \langle \{z\}, 0, 0, \alpha_z + 1 \rangle S x' \not\subset \langle \{z\}, 0, 0, \alpha_z + 1 \rangle S y'$$

by the theorem on S

$$\Rightarrow \exists w w E x \not\subset w E y \quad \text{by clause g.}$$

Finally, if $i_x = i_y = 3$ then

$$x \neq y \Rightarrow \bar{x} \neq \bar{y} \quad \text{by definition of } x \rightarrow \bar{x}$$

$$\Rightarrow \exists w w E \bar{x} \not\subset w E \bar{y} \quad \text{by previous case}$$

$$\Rightarrow \exists w w E x \not\subset w E y \quad \text{by clause g. } \quad \text{---}$$

The reader may note that the next two theorems show that the functions $x \mapsto \rho x$ and $x \rightarrow \bar{x}$ perform as expected.

Theorem: (II) COMPLEMENTS

$$\mathcal{U} \models \forall x \exists y \forall z z \in y \Leftrightarrow z \notin x.$$

Proof: This translates to

$$\forall x \text{ in } A \exists y \text{ in } A \forall z \text{ in } A zEy \iff zEx$$

which is trivially satisfied by $y = \bar{x}$ due to clauses f and h. -|

Theorem: (III) POWERS

$$\mathfrak{U} \models \forall x \exists y \forall z z \in y \iff z \subseteq x .$$

Proof: This translates to

$$\forall x \text{ in } A \exists y \text{ in } A \forall z \text{ in } A zEy \iff zSx$$

using the theorem on S. For any x which is not a replacement object and not u , we constructed Px explicitly to satisfy this by clause g. Now if $x = u$, we have trivially that $\forall z zEu$ and $\forall z zSu$, so $Pu =_{DF} u$ satisfies the theorem. And if x is a replacement object, then any subobject z of x is also, and was constructed by the stage $i = 0, j = 0, \alpha = \alpha_x$. Furthermore, there is only a set of such z since each corresponds to a different subset of m_x . Hence at stage $i = 0, j = 0, \alpha = \alpha_x + 1$ there is a replacement object y such that

$$\forall z z \in m_y \iff m_z \subseteq m_x .$$

Hence $\forall z zEy \iff zSx$ and the theorem is done. -|

Theorem: (IV) WELL FOUNDED REPLACEMENT

$$\mathfrak{U} \models \bar{\forall} \forall u WF(u) \implies$$

$$[(\forall x \in u \exists ! z \phi(x, z, u, \bar{v})) \implies$$

$$\exists y \forall z (z \in y \iff \exists x \in u \phi(x, z, u, \bar{v}))] .$$

Proof: It is clear that the replacement objects satisfy this axiom scheme if the following lemma holds:

$$(\vDash_{\mathfrak{U}} \text{WF}(x)) \text{ if and only if } A_{000}(x) .$$

The proof of the lemma follows below. \dashv

Theorem: (V) END EXTENSION OF Z-F

$\mathfrak{U} \vDash$ The well-founded (WF) objects are a class model of Z-F.

Proof: This is also a trivial corollary of the following lemma. \dashv

Lemma:

$$[\mathfrak{U} \vDash \text{WF}(x)] \text{ iff } A_{000}(x) \text{ for all } x \text{ in } A.$$

Proof: The lemma divides into two sublemmas. The first says that \mathfrak{U} thinks x is well-founded if and only if Z-F thinks E is well-founded at x . The second says that Z-F thinks E is well-founded at x iff x is an A_{000} object. The lemma obviously follows.

Sublemma 1:

$$[\mathfrak{U} \vDash \text{WF}(x)] \text{ iff } \forall z x \in z \Rightarrow \exists y y \in z \wedge \forall w w \in z \Rightarrow w \notin y .$$

Proof: The contrapositive

$$[\mathfrak{U} \vDash \neg \text{WF}(x)] \text{ iff } \exists z x \in z \wedge \forall y y \in z \Rightarrow \exists w w \in z \wedge w \in y$$

is much easier to prove. We will argue each implication separately.

\Rightarrow) Assume $\mathcal{U} \models \neg WF(x)$. Then we have

$$\mathcal{U} \models \exists z x \in z \wedge \forall y y \in z \Rightarrow \exists w w \in z \wedge w \in y$$

which says

$$\exists z \text{ in } A \ x \in z \wedge \forall y \text{ in } A \ y \in z \Rightarrow \exists w \text{ in } A \ w \in z \wedge w \in y .$$

Choose any such z in A , and then form the following sequence of sets: (for $i \in \omega$)

$$\sigma_0 = \{x\}$$

$$\sigma_{i+1} = \bigcup_{y \in \sigma_i} \{w \text{ in } A, w \text{ of least rank in } Z\text{-F, such that } w \in z \text{ and } w \in y\}.$$

This is possible because of our assumption, noticing that each σ_{i+1} is a set because the previous σ_i is and because our sets of w 's are subcollections of the set of some rank. Now consider $\sigma = \bigcup_{i \in \omega} \sigma_i$. For this σ we can prove

$$x \in \sigma \wedge \forall y y \in \sigma \Rightarrow \exists w w \in \sigma \wedge w \in y ,$$

since each $y \in \sigma$ is in some σ_i and the required w is therefore in σ_{i+1} . We hence have

$$\exists z x \in z \wedge \forall y y \in z \Rightarrow \exists w w \in z \wedge w \in y . \quad \dashv$$

\Leftarrow) Conversely, assume that

$$\exists z x \in z \wedge \forall y y \in z \Rightarrow \exists w w \in z \wedge w \in y . \quad \dashv$$

Choose any such z . It is a set of A -objects since E can

only hold between A-objects. Hence it gives rise to a replacement object z° such that $m_{z^{\circ}} = z$ by the definition of A. But then z° satisfies

$$xEz^{\circ} \wedge \forall y \text{ in } A yEz^{\circ} \Rightarrow \exists w \text{ in } A wEz^{\circ} \wedge wEy .$$

Which gives immediately that

$$\mathfrak{A} \models -WF(x) . \quad \dashv$$

We have completed sublemma 1. \dashv

Sublemma 2:

$$\forall x \text{ in } A [A_{000}(x) \text{ iff } \forall z x \in z \Rightarrow \exists y y \in z \wedge \forall w w \in z \Rightarrow wEy] .$$

Proof: Fix some A-object x. Again argue the two implications separately.

\Rightarrow) If x is in A_{000} then the assertion that E is well-founded at x is exactly equivalent to the set m_x being well-founded in Z-F, which it is. \dashv

\Leftarrow) Hence we need only prove that if x is not in A_{000} then E is not well-founded at x. Slightly more can be proven: if the A-object x is not in A_{000} then there is a descending E-chain from x ending in a finite E-loop

$$x_{i+m} E x_{i+m-1} E \dots E x_{i+1} E x_i = x_{i+m} .$$

We prove this by cases of i_x .

If $i_x = 1$ then x is the complement of a replacement object \bar{x} .

But then we have that $x \notin \bar{x}$, since $\alpha_x \geq \alpha_{\bar{x}}$. Hence $x \in x$ by the clause for complements. That completes the argument for this case.


If $i_x = 3$ then x is the complement of a power object, $x = \overline{\rho x'}$.

By an earlier argument, we know that the universe $u = \langle 0, 1, 0, 0 \rangle$ satisfies $u \notin x'$ and hence $u \notin \overline{\rho x'}$ and hence $u \in \overline{\rho x'}$. This case is completed by noting that $u \in u$.

If $i_x = 2$ we proceed by induction on j_x , noting that $x = \rho x'$ and $j_x = j_{x'} + 1$ and that $x' \in \rho x'$.

Finally, if $i_x = 0$ and x is not in A_{000} , then the least upper bound part of clause x) guarantees that some y with $\alpha_y < \alpha_x$ satisfies $y \in x$ and y is not in A_{000} , so this final case can be completed by induction on α_x . \dagger

The reader may note that if Church's axioms D and E of union and intersection in [1] are restricted to well-founded objects x , and his axiom J of power sets has the same restriction removed, then the model \mathfrak{U} satisfies the modified axioms. It is still an unsolved problem whether a model with all of D, E and J unrestricted is possible.

end of proof

The rest of the paper is philosophy, not mathematics.
Inclusion optional.

As Church observes, whether or not the axiom of choice holds in set theories with a universal set depends very much on which form of the axiom is considered. If a given form of choice holds in the universe V of Z-F, then that same form holds in \mathfrak{U} if it is restricted to apply to small sets - those in 1-1 correspondence to a well-founded set. Zorn's lemma for sets holds in \mathfrak{U} unrestricted if it holds in V , since complement objects and power objects have trivial maximal members: If a is a replacement object, then \bar{a} is a maximal member of \bar{a} , x is always the maximum member of $\mathcal{P}x$, and the universe is a member of any complement of a power object. Finally, if V has a global well-ordering then the construction of \mathfrak{U} yields an obvious well-ordering of \mathfrak{U} .

The eventual aim of the research program in which this paper is a small step is to provide a setting for set theory more general than Z-F. Conceivably a consistency proof for a system like Quine's New Foundations may result. One also hopes to reach a system powerful enough to provide a foundation for Category Theory in which, e.g., the "Category of all Categories" is a normal object.

The primary difficulty in such a program is connected with the following observation: both the proof in Church's paper and

that in this involve constructing exactly one name for each set in the new model. Thus equality does not need to be redefined. It is easy to construct classes of names for objects satisfying more powerful axioms than II-V (e.g. II-V + unions), but in general extensionality (I) forces one to set various names equal to each other, which makes other names have the same "members", etcetera. Since this kind of model is the opposite of well-founded, there is great difficulty proving that this process converges.

BIBLIOGRAPHY

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