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The Consistency of NF

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Summary. The consistency of Quine's set theory NF is proved by the construction of an ambiguous model of type theory. Starting with a model of type theory over a sufficiently large set of urelements, and where every set is describable by an infinitary formula, the ambiguous model results by deleting the first type. A modification of the procedure yields a model of Oberschelp's system ZF + NF.

Zusammenfassung. Die Widerspruchsfreiheit der Quineschen Mengenlehre NF wird durch die Angabe eines den Ambiguitätsaxiomen genügenden Modells der Typentheorie gezeigt. Ausgehend von einem Modell der Typentheorie über einem hinreichend großen Urelementebereich, in welchem alle Mengen durch infinitäre Ausdrücke beschreibbar sind, erreicht man die Erfüllung der Ambiguitätsaxiome durch Weglassen der untersten Schicht. Eine Abänderung der Konstruktion liefert ein Modell für das System ZF + NF von Oberschelp.

Резюме. Доказывается непротиворечивость Куайновской теории множеств NF построением модели теории типов удовлетворяющей аксиомам типовой неопределенности. Исходя из модели теории типов над достаточно большой областью индивидов, где все множества определяемые бесконечными формулами, получается выполнение аксиом типовой неопределенности отрезая самый низкий уровень. Модель Обершельпской системы ZF + NF получается изменением конструкций...

0. Introduction. We will give a model for Quine's set theory NF; to be more precise: a model for a transfinite extension of NF presented at the end of § 2. The model construction is made within a model of "usual" classical (naive) set theory not intended to be of a special system; but a formalization in one of the common systems, especially Zermelo-Fraenkel set theory ZFC with axiom of choice, will be possible at least in the case of the existence of a strongly inaccessible cardinal. We begin with a certain model of homogeneous type theory over a sufficiently large set of urelements. The sets of this model will be exactly those describable by a transfinitely composed formula. We identify those sets in the model which are describable by the same formula without constants. The classes of identified sets resp. the describing formulae are the objects of the NF model. The other sets of the different types, with only describing formulae containing constants (for the urelements), are necessary to define the membership relation, by suitable representatives. The major part of this note consists in proving that after deleting the lowest type, the rest of the model of type theory satisfies the axioms of ambiguity. As extensionality and the comprehension scheme remain valid, the consistency of NF follows by a result of Specker's [8]. Though the construction might be formalized within ZFC + existence of a strongly inaccessible cardinal, of course we will not do so. It seems that the construction might be formalized even in ZFC, but at the expense of considerable complications in the presentation. For the sake of a better readability of this paper we will use the stronger supposition.

At the end, we will modify our construction to get a model for Oberschelp's system ZF + NF.

The present note is a more detailed and enlarged version of [7].

1. A convention concerning classical set theory, notations. In what follows, all objects may be supposed to be elements of a model say of ZFC + existence of a strongly inaccessible cardinal α_0. If the cardinality card a of a set a is less than α_0, then we shall call the set small. Large sets will be

sets a with card $a = \alpha_0$. Sets with a cardinality greater than α_0 will seldom be considered and need no special name. Thus, the singleton of a large set will be small, the elements of small sets may possibly be large. A *cosmall* subset of a large set has a small complement, a *colarge* subset has a large relative complement. Large sets have *many* elements, small sets have *few* elements.

Let f, g be binary relations. Then may hold $g \circ f = \{(a,c) : \exists b((a,b) \in f \wedge (b,c) \in g\}$, dom $f = \{a : \exists b (a,b) \in f\}$, rng $f = \{b : \exists a (a,b) \in f\}$, $f\langle c \rangle = \{b : \exists a((a,b) \in f \wedge a \in c)\}$, $f|a = f \wedge (a \times \text{rng } f)$. Let $a \setminus b$ be the set $\{c : c \in a \wedge \neg c \in b\}$, and Pa be the power set of a. In the case a is large, we shall mostly consider no subsets of Pa with a cardinality greater than α_0.

i, j, k, l, m, n may generally be variables for natural numbers (elements of ω), unless marked otherwise. (Sometimes i and j will also be used as variables for elements of certain index sets.) $[m,n]$ will be the set $\{k : m \leq k \leq n\}$. - Every ordinal number is the set of the smaller ordinal numbers; but this convention is not used before § 12.

2. Stratified formulae. First we constitute an infinitary set-theoretical language, then the domain of the formulae of this language will be restricted by the stratification requirement.

Let $\underline{C} = \{\underline{c}_\alpha : \alpha < \alpha_0\}$ be a set with $\underline{c}_\alpha \neq \underline{c}_\beta$ for $\alpha \neq \beta$. The elements of $\underline{C}$ will be called *constants*. Likewise, let $\underline{X} = \{x_\alpha : \alpha < \alpha_0\}$ be a set of *variables*, with $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Let $\underline{C}$ and $\underline{X}$ be disjoint. Moreover, let $\underline{\in}$, $\underline{=}$, $\neg$, $\bigwedge$, $\forall$ be five different objects (elements of the underlying classical set theory) beyond $\underline{C} \cup \underline{X}$.

Definition. The set of the *set-theoretical formulae* is the least set satisfying F1), ..., F5):

F1) Every triple $(a, \underline{\in}, b)$ with $a \in \underline{X} \cup \underline{C}$, $b \in \underline{X}$ is a set-theoretical formula.

F2) Every triple $(a, \underline{=}, b)$ with $a, b \in \underline{X} \cup \underline{C}$ is a set-theoretical formula.

F3) If h is a set-theoretical formula, then so is the couple $(\neg, h)$.

F4) If A is a small nonempty set of set-theoretical formulae, then the couple $(\bigwedge, A)$ is a set-theoretical formula.

F5) If V is a small nonempty set of variables, and h is a set-theoretical formula, then the triple $(\forall, V, h)$ is a set-theoretical formula. ∎

Some conventions:

$(a, \underline{\in}, b)$	will be abbreviated by	$a \underline{\in} b$;
$(a, \underline{=}, b)$	will be abbreviated by	$a \underline{=} b$;
$(\neg, h)$	will be abbreviated by	$\neg h$;
$(\bigwedge, \{h_i\}_{i \in I})$	will be abbreviated by	$\bigwedge_{i \in I} h_i$;
$(\bigwedge, \{h_1, h_2\})$	will be abbreviated by	$(h_1 \wedge h_2)$;
$(\forall, V, h)$	will be abbreviated by	$\forall V h$;
$(\forall, \{x_\alpha\}, h)$	will be abbreviated by	$\forall x_\alpha h$.

These writings will be justified by the interpretation of those set-theoretical formulae which are stratified (§ 3). $\vee$, $\bigvee$, $\rightarrow$, $\leftrightarrow$, $\exists$ are introduced as usually. Though the logical signs will as well be used semantically to formulate our assertions on our constructions, therefrom no confusion will arise.

We shall often write x, y, ..., y_0, y_1, ... instead of x_0, x_1, ... We will not generally distinguish variables and names of variables or variables for variables, with one exception: In the definition of H^x below exactly one variable x is designated. - <u>Subformulae</u> may be defined as usually. Especially, every h_i with $i \in I$ is a subformula of a set-theoretical formula $\bigwedge_{i \in I} h_i$, but $\bigwedge_{i \in J} h_i$ with $J \subseteq I$ is no subformula of $\bigwedge_{i \in I} h_i$. The occurrence of <u>terms</u> (elements of $\underline{C} \cup \underline{X}$) in a set-theoretical formula may be defined as usually. A variable z occurs <u>bound</u> at a certain place in a set-theoretical formula if that place belongs to a subformula $\forall V h$ with $z \in V$. All other occurrences of variables in set-theoretical formulae are called <u>free</u>. Sometimes, we will also note terms beyond $\underline{C} \cup \underline{X}$, for example terms like $\underline{\{} x_1, x_2 \underline{\}}$, $\underline{\{} u\colon h(u) \underline{\}}$. Set-theoretical formulae containing such terms may be conceived in the traditional way as abbreviations of longer set-theoretical formulae; for example $\{x_1,x_2\} \underline{\in} y$ is to be understood as an abbreviation for $\forall z(\forall x(x \underline{\in} z \leftrightarrow x \underline{=} x_1 \vee x \underline{=} x_2) \rightarrow z \underline{\in} y)$.

We graphically distinguish $\underline{\in}$ as a constituent of set-theoretical formulae from $\in$ referring to the underlying classical set theory, as well $\underline{=}$ from $=$.

If in a sequence $(h_1,h_2,...)$ of set-theoretical formulae every occurring h_{i+1} is a proper subformula of h_i, then by the definition of set-theoretical formulae this sequence is necessarily finite. Therefore proofs of properties of set-theoretical formulae will be possible by transfinite order-theoretical induction on the composition of the formulae. (The few subformulae of a given set-theoretical formula are well-orderable compatibly with the subformula relation.)

Definition. The length of an atomic set-theoretical formula $a \underline{\in} b$ or $a \underline{=} b$ is the cardinal number ω; if the length of a set-theoretical formula h is α, then the length of $\neg h$ is α, and the length of any set-theoretical formula $\forall V h$ is $\max(\mathrm{card}\, V, \alpha)$; the length of a set-theoretical formula $\bigwedge_{i\in I} h_i$ is the cardinal sum of the lengths of the set-theoretical formulae h_i for $i \in I$. ■

Definition. A set-theoretical formula h is a stratified formula iff there exists a function s from the set of the terms occurring in h and into a finite (!) set of natural numbers, satisfying S1), S2), S3):

S1) $s(t_1) = s(t_2) + 1$ if $t_1 \underline{\in} t_2$ is a subformula of h.

S2) $s(t_1) = s(t_2)$ if $t_1 \underline{=} t_2$ is a subformula of h.

S3) $s(c) = \max \mathrm{rng}\, s$ if $c \in \underline{C} \cap \mathrm{dom}\, s$. ■

s depends on h, therefore we shall write s_h if necessary. - We have inverted the stratification function s relatively to the usual one, and count the types from top to bottom. The cause for this is the generalization given in § 12. The condition for $\mathrm{rng}\, s$ to be finite is important for the simplicity of our following constructions and will be replaced by a weaker one in § 12.

Definition. Let H be the set of all stratified formulae. Let x be a designated element of $\underline{X}$ - say x_o -, and H^x be the set of those $h \in H$ containing free exactly the variable x. H_o^x may be the set of those $h \in H^x$ with $s(x) = 0$ for an appropriate stratification function s as above. (Type-theoretically, the unique variable x occurring free in h is one of the upmost ones.) Let K, K^x, K_o^x be respectively the sets of those $h \in H$, H^x, H_o^x containing no constants (elements of $\underline{C}$). For $h \in H^x$ and $y \in \underline{X}$, by $h(y)$ we denote the result of a substitution of y for x in h, the usual precautions assumed to avoid variable confusions and to preserve stratifiedness. If, for $h \in H$, V is the set of the variables free in h, Gen h shall be the stratified formula $\forall V h$. ∎

The extension of Quine's system NF the consistency of which we shall prove, is established by the axiom of extensionality
$\forall z(z \underline{\in} x \leftrightarrow z \underline{\in} y) \leftrightarrow \forall u(x \underline{\in} u \leftrightarrow y \underline{\in} u)$ $\quad (\leftrightarrow x \underline{=} y)$
and the axiom scheme of comprehension
$\exists y \forall x(x \underline{\in} y \leftrightarrow h)$ for $h \in K$, y not occurring in h,
with standard-interpreted logical constants. In the original system, only stratified formulae of a finitary language are admitted. - In the following section we shall introduce another interpretation of the quantifiers and of $\underline{\in}$. That interpretation will have a type-theoretical character.

3. Relative validity. The validity of stratified formulae will be defined to be analogous to validity in models of homogeneous type theory, but with ambiguity of the type assignments. That ambiguity will cause some technical difficulties, but it will be necessary in the generalization in § 13.

Let X be a binary relation with dom $X \subseteq \omega$ for the set ω of the natural numbers. Let X_n^- be the set $X\langle\{n\}\rangle$ and X_n be the restriction $X|\{n\}$, for $n<\omega$. Thus we have
$X_n^- = \text{rng } X_n$.

X will be conceived as a subset of a model of homogeneous type theory; we have indexed its elements to make disjoint the sets X_n^- for different natural numbers n, especially the empty sets possibly existing in those sets are thus distinguished from one another. The minus-exponents make forget the indices of the various types. The metatheoretical variable X for certain sets (certain relations) must not be confounded with the name $\underline{X}$ of the fixed set of the variables! The metatheoretical variable X will stand for several sets of our fixed model of the underlying classical set theory, especially the sets Y and Z defined in § 5, and some of their subsets.

Before going to our truth-value stipulation, we define an isomorphy relation for subsets of X, which later on will often be used. That isomorphy is the usual set-theoretical one for $\operatorname{rng} X$ with a possible uniform shifting of the indices:

Definition. Let X' and X'' be subsets of X. X' and X'' are isomorphic iff there are a biunique function η from X' onto X'' and an integer g with

$$\eta((i,a) = (j,b) \rightarrow j = i + g \quad \text{and}$$
$$\eta((i,a_1) = (i+g,b_1) \wedge \eta((i+1,a_2)) = (i+g+1,b_2) \rightarrow$$
$$\rightarrow ((a_1 \in a_2) \leftrightarrow (b_1 \in b_2)),$$

for every (i,a), (i,a_1), $(i+1,a_2) \in X'$ and the corresponding η-values. ∎

Definition. Let h be a stratified formula. A function φ, defined at least on the variables occurring free in h, is an X-admissible valuation of h iff there are a stratification function s as in § 2, a function ψ defined at least on $\operatorname{dom} s$ and with $\psi \supseteq \varphi$, and a natural number m, satisfying V1) and V2) for every $t \in \operatorname{dom} s$ and every $c \in \underline{C} \cap \operatorname{dom} s$:

V1) $\psi(t) \in X_{m-s(t)}$.

V2) $m = s(c) \wedge (0,c) \in X_0$. ∎

Consequently, the valuation φ has to respect the stratification of the formula h. An X-admissible valuation of $h \in K$ exists if $\operatorname{dom} X$ includes a sufficiently long interval of natural numbers; in that case the condition V2) is void. In the case of $h \in H \setminus K$ the interval must begin with 0, and X_0^- must contain as elements the constants occurring in h. Clearly, if $T \supseteq X$, then every X-admissible valuation of h is as well a T-admissible valuation of h.

We go to define the X-values of terms and stratified formulae. X-Val(h,φ) shall denote the truth-value of the stratified formula h under the "valuation" φ, X being the domain for the bound variables. φ itself need not be X-admissible, but the truth-value X-Val(h,φ) will not exist but in the case there are a function φ' defined exactly on the variables bound in h and with $\operatorname{rng}\varphi' \subseteq X$ and a set T such that $(\varphi \setminus \varphi') \cup \varphi'$ is a T-admissible valuation of h defined on all variables occurring in h. Thus, φ has to respect the stratification of the formula h and nothing more; the values of φ may possibly be elements of $T \setminus X$.

Definition. Let φ be a function. We define:

T1) X-Val$(y,\varphi) = a$ iff $y \in \operatorname{dom}\varphi$ and there exists an n with $\varphi(y) = (n,a)$.

T2) X-Val$(c,\varphi) = c$ for every $c \in \underline{C}$.

T3) X-Val$(t\underline{\in}y,\varphi) = \underline{T}$ iff $t \in \underline{C} \cup \operatorname{dom}\varphi$, $y \in \operatorname{dom}\varphi$, φ is a $(\underline{C} \cup \operatorname{rng}\varphi)$-admissible valuation of $t\underline{\in}y$, and X-Val$(t,\varphi) \in$ X-Val(y,φ),

$= \underline{F}$ iff $t \in \underline{C} \cup \operatorname{dom}\varphi$, $y \in \operatorname{dom}\varphi$, φ is a $(\underline{C} \cup \operatorname{rng}\varphi)$-admissible valuation of $t\underline{\in}y$, and not X-Val$(t,\varphi) \in$ X-Val(y,φ).

T4) X-Val$(t_1\underline{=}t_2,\varphi) = \underline{T}$ iff $t_1, t_2 \in \underline{C} \cup \operatorname{dom}\varphi$, φ is a $(\underline{C} \cup \operatorname{rng}\varphi)$-admissible valuation of $t_1\underline{=}t_2$, and X-Val$(t_1,\varphi) =$ X-Val(t_2,φ),

$= \underline{F}$ iff $t_1, t_2 \in \underline{C} \cup \operatorname{dom}\varphi$, φ is a $(\underline{C} \cup \operatorname{rng}\varphi)$-admissible valuation of $t_1\underline{=}t_2$, and X-Val$(t_1,\varphi) \neq$ X-Val(t_2,φ).

T5) X-Val$(\neg h,\varphi) = \underline{T}$ iff X-Val$(h,\varphi) = \underline{F}$,

$= \underline{F}$ iff X-Val$(h,\varphi) = \underline{T}$.

T6) X-Val$(\bigwedge_{i\in I} h_i,\varphi) = \underline{T}$ iff X-Val$(h_i,\varphi) = \underline{T}$ for every $i \in I$,

$= \underline{F}$ iff X-Val$(h_i,\varphi) \in \{\underline{T},\underline{F}\}$ for every $i \in I$ but X-Val$(\bigwedge_{i\in I} h_i,\varphi) = \underline{T}$ does not hold.

T7) $X\text{-Val}(\forall Vh, \varphi) = \underline{F}$ iff there is an $(X \cup \operatorname{rng} \varphi)$-admissible valuation ψ of h with $X\text{-Val}(h, \psi) = \underline{F}$, $\psi(z) = \varphi(z)$ for every variable z free in $\forall Vh$, $\psi(v) \in X$ for every $v \in V$,

$X\text{-Val}(\forall V\neg h, \varphi) = \underline{F}$ iff there is an $(X \cup \operatorname{rng} \varphi)$-admissible valuation ψ of h with $X\text{-Val}(h, \psi) = \underline{T}$, $\psi(z) = \varphi(z)$ for every variable z free in $\forall Vh$, $\psi(v) \in X$ for every $v \in V$,

$X\text{-Val}(\forall Vh, \varphi) = \underline{T}$ iff $X\text{-Val}(\forall V\neg h, \varphi) = \underline{F}$ but not $X\text{-Val}(\forall Vh, \varphi) = \underline{F}$,

$X\text{-Val}(\forall V\neg h, \varphi) = \underline{T}$ iff $X\text{-Val}(\forall Vh, \varphi) = \underline{F}$ but not $X\text{-Val}(\forall V\neg h, \varphi) = \underline{F}$.

Here $\underline{T}$ and $\underline{F}$ are two distinct objects (truth values). ∎

The existence requirement in the first and second part of T7) means that φ can be extended to a valuation of all variables in h (including the bound occurrences) in such a wise that the stratification of h is respected and the new values of the valuation are elements of X. That is also required in the third and fourth part of T7), but indirectly. Therefore, $X\text{-Val}(\text{Gen } h, \emptyset)$ exists iff there is an X-admissible valuation of all variables of h iff $\emptyset$ is an X-admissible valuation of Gen h. If $\varphi(x) \in X_o$, then, e.g., $X\text{-Val}(\forall y\ y \leq x, \emptyset)$ does not exist.

Definition. The stratified formula h is X-valid (resp. X-valid relatively to a fixed valuation φ of some variables) iff $X\text{-Val}(\forall Vh, \varphi) = \underline{T}$ for V being the set of the variables free in h (resp. the set of the variables free in h but beyond dom φ). (I.e., especially, h is X-valid iff $X\text{-Val}(\text{Gen } h, \emptyset) = \underline{T}$.) The stratified formulae h_1 and h_2 are X-equivalent iff the set-theoretical formula $h_1 \leftrightarrow h_2$ is stratified and X-valid. ∎

4. Lemma (essentially Löwenheim-Skolem). Let h be a stratified formula, $X\text{-Val}(\text{Gen } h, \emptyset)$ may exist (i.e., dom X may be large enough), let X' be a small subset of X. Then there

exists a small set X^* with $X' \subseteq X^* \subseteq X$ such that X^*-Val(Gen h, $\emptyset$) also exists and X^*-Val(h', φ) = X-Val(h', φ) holds for every subformula h' of h and every X^*-admissible valuation φ of h'.

Proof. The lemma is clearly true if the range of the subformulae h' is restricted to the quantifierless subformulae of h, for then T-Val(h', φ) does not depend on T. (T is the domain for the bound variables only and independent of rng φ.) In this case we may set $X^* = X' \cup \operatorname{rng} \overset{\cup(\underline{C} \cap \operatorname{dom} s_h)}{\varphi}$ for an arbitrary X-admissible valuation φ of all variables of h (bound or not); such a valuation φ exists by the presupposed existence of X-Val(Gen h, $\emptyset$). X^*-Val(Gen h, $\emptyset$) then also exists by $X^* \supseteq \operatorname{rng} \varphi \cup (\underline{C} \cap \operatorname{dom} s_h)$.

As an induction hypothesis, we assume the lemma to be true for the range of the stratified formulae h' restricted to a (small!) set $H' = \{h_i : i \in I\}$ of subformulae h_i of h, i.e., that for every small subset X' of X there exists a small set X^* with $X' \subseteq X^* \subseteq X$ such that X^*-Val(Gen h, $\emptyset$) exists and X^*-Val(h_i, φ) = X-Val(h_i, φ) holds for every $h_i \in H'$ and every X^*-admissible valuation φ of h_i. We have just seen that the assertion is true if we choose for H' the set of the quantifierless subformulae of h. Therefore we assume H' to contain as elements all quantifierless subformulae of h. Without changing X^*, we can extend H' in such a way that $\neg h'$ is an element of H' if only $\neg h'$ is a subformula of h and $h' \in H'$, and that $\bigwedge_{i \in J} h_i$ is an element of H' if only $\bigwedge_{i \in J} h_i$ is a subformula of h and $J \subseteq I$. We assume therefore H' to be already closed with respect to the partial operations indicated. Now let $H'' = \{\forall V_j h_j : j \in J\}$ be the set of all subformulae of h of the form $\forall V_j h_j$ with $j \in I$. We shall show that then again there exists a small set X^* with $X' \subseteq X^* \subseteq X$ and such that X^*-Val(Gen h, $\emptyset$) exists and X^*-Val(h', φ) = X-Val(h', φ) holds for every $h' \in H'' \cup H'$ and every X^*-admissible valuation of h'. (The new set X^* need not be equal to the previous one.) The bound variables of h may be renamed so that the sets V_j are mutually disjoint. We shall procure the small set X^{*} belonging to $H' \cup H''$ by a further transfinite induction.

Let $X''(0)$ be a small set with $X''(0) \supseteq X'$ and large enough to grant the existence of an $X''(0)$-admissible valuation of the stratified formula h. Let β be a small ordinal number. We assume the small subsets $X''(\alpha)$ of X to be given for $\alpha \leq \beta$. Let $X''(\beta)'$ be a small set with $X''(\beta) \subseteq X''(\beta)' \subseteq X$ and the following property:

For every $I' \subseteq I$, every $J' \subseteq J$, and any two X-admissible valuations ψ_1, ψ_2 of the stratified formula $\bigwedge_{i \in I'} h_i$ such that $\psi_1 \cap \psi_2$ is an $X''(\beta)$-admissible valuation of the stratified formula $\bigwedge_{j \in J'} \forall V_j h_j$ and $X\text{-Val}(h_j, \psi_1) \neq X\text{-Val}(h_j, \psi_2)$ for any $j \in J'$, there may also exist $X''(\beta)'$-admissible valuations φ_1, φ_2 of $\bigwedge_{i \in I'} h_i$ with $\varphi_1 \cap \varphi_2 = \psi_1 \cap \psi_2$ and $X\text{-Val}(h_j, \varphi_1) \neq X\text{-Val}(h_j, \varphi_2)$ for the same j.

We have to prove the existence of such a small set $X''(\beta)'$: Since $X''(\beta)$ and the number of the variables free in $\bigwedge_{j \in J'} \forall V_j h_j$ are small for every $J' \subseteq J$, the number of $X''(\beta)$-admissible valuations of exactly the variables free in $\bigwedge_{j \in J'} \forall V_j h_j$ is small. For every $J' \subseteq J$, every $j \in J'$, and every $X''(\beta)$-admissible valuation of $\bigwedge_{j \in J'} \forall V_j h_j$, we have to add at most card V_j new elements to $X''(\beta)$, and the result will satisfy the condition for $X''(\beta)'$. Since card V_j is small for every $j \in J$, and J is small as well, indeed $X''(\beta)'$ may be assumed to be small.

Substituting the set $X''(\beta)'$ for X' in the assumption on H', we obtain a small set $X''(\beta+1)$ with $X''(\beta)' \subseteq X''(\beta+1) \subseteq X$ and $X''(\beta+1)\text{-Val}(h_i, \varphi) = X\text{-Val}(h_i, \varphi)$ for every $i \in I$ and every $X''(\beta+1)$-admissible valuation φ of h_i. For small limit numbers λ we set $X''(\lambda) = \bigcup_{\alpha<\lambda} X''(\alpha)$. Now let γ be a small cardinal (or initial) number greater than the length of h.

We will show now that $X''(\gamma)$ is the set X^{*} belonging to $H' \cup H''$, i.e., that $X''(\gamma)\text{-Val}(\text{Gen } h, \emptyset)$ exists and $X''(\gamma)\text{-Val}(h', \varphi) = X''(\beta)\text{-Val}(h', \varphi) = X\text{-Val}(h', \varphi)$ holds for every stratified formula $h' \in H' \cup H''$, every $X''(\gamma)$-

admissible valuation φ of h' and an appropriate ordinal number $\beta < \gamma$ (dependent on h' and φ).

The existence assertion is true by $X''(\gamma) \supseteq X''(0)$. For the rest, we begin a new transfinite induction. The equations shurely hold for the quantifierless $h' \in H'$. If they hold for $h' \in H'$ or for every h_i with $i \in I'$ with respect to a subset I' of I, then they also hold for stratified formulae of the form $\neg h'$ or $\bigwedge\limits_{i \in I'} h_i$. Thus, we have to show that they hold for $\forall Vh' \in H' \cup H''$ provided they hold for $h' \in H'$. Let φ be an $X''(\gamma)$-admissible valuation of exactly the variables free in $\forall Vh'$. Since γ is greater than the length of h, every $X''(\gamma)$-admissible valuation ψ of exactly the variables free in h' with $\psi \supseteq \varphi$ is at the same time an $X''(\beta)$-admissible valuation of h' for an ordinal number $\beta < \gamma$, and we may assume

$X\text{-}Val(h',\psi) = X''(\gamma)\text{-}Val(h',\psi) = X''(\beta)\text{-}Val(h',\psi)$,

by the last induction hypothesis. This entails by construction

$X\text{-}Val(\forall Vh',\varphi) = X''(\beta+1)\text{-}Val(\forall Vh',\varphi)$,

and likewise

$X''(\gamma)\text{-}Val(\forall Vh',\varphi) = X''(\beta+1)\text{-}Val(\forall Vh',\varphi)$

for the same β. This means that $X''(\gamma)$ has the required property, indeed.

Now we take $H' \cup H''$ instead of the set hitherto named H' and repeat, if necessary, the procedure. It is sufficient to make that passage from H' to $H' \cup H''$ less than γ times, and the union of the successive sets H' will be the small set of all subformulae of h. (If the procedure is repeated λ times, λ a limit number, the new set H' is of course the union of the old sets H'.) With that the proof of the lemma is completed.

<u>Remark.</u> The lemma remains valid if φ is relativized by the condition $\varphi \supseteq \varphi_0$, φ_0 a fixed T-admissible valuation of some variables free in h, for any T (not necessarily $T \subseteq X'$). Moreover, the small sets $X^{*} = X^{*}(\varphi_0, X')$ corresponding to different valuations φ_0 (not necessarily X'-admissible for a fixed X') and different small sets X' can be constructed isomorphically, if only the cardinalities of the sets X' are

limited by a fixed small cardinal.

The first part of the remark is a trivial consequence of the lemma. One proof of the second part parallels the above proof completely, but simpler is the following reduction: Let α be a small cardinal. Then there is a small cardinal β such that for every $X' \subseteq X$ with card $X' < \alpha$ a small set $X^{\#}(X')$ exists with card $X^{\#} < \beta$ and satisfying the conditions of the lemma with respect to X, X', h; the proof of the lemma gives us the possibility to calculate a small upper bound for β. For any given small ordinal number β, again, the cardinalities of small sets A of mutually non-isomorphic sets $X^{\#}(X')$ with card $X^{\#}(X') < \beta$ have a small upper bound. Hence there are maximal small sets A with that property; let X'' be the union of such a maximal small set A. X'' is small and contains as subsets up to isomorphy all sets $X^{\#}(X')$ as above with card $X^{\#}(X') < \beta$. Then there is a small set $X^{\#}$ satisfying the conditions of the lemma with respect to X, X'' (instead of X'), h; this small set $X^{\#}$ and its isomorphic images satisfy the conditions of the remark for $X^{\#}(\varphi_o, X')$ for every φ_o as above and every X' as above with card $X' < \alpha$; $\alpha >$ card rng φ_o can be assumed.

Notations. The assertion of the lemma for h, X, X', $X^{\#}$ will be abbreviated by $\Phi(h, X, X', X^{\#})$. The assertion of the remark including the isomorphy condition will accordingly be written $\Phi_{\varphi_o}(h, X, S, X^{\#})$ if the domain of the sets X' is relativized by the conditions $X' \supseteq S$ and card $X' \leq \max$ (card S, length of h), S supposed to be small. In the case of $\varphi_o = \emptyset$ we shall write $\Phi_o(h, X, S, X^{\#})$ instead of $\Phi_{\emptyset}(h, X, S, X^{\#})$. As a consequence of our relativization for the sets X', for any valuation $\varphi \supseteq \varphi_o$ of all variables of h a corresponding set $X^{\#}(\varphi)$ with $\Phi(h, X, S \cup \text{rng}\, \varphi, X^{\#}(\varphi))$ is isomorphically contained in $X^{\#}$ as a subset, and $X^{\#}$ is not an exceptionally "poor" Löwenheim-Skolem set. - $\Phi_o(h, X, S, X^{\#})$ clearly entails $\Phi(h, X, S, X^{\#})$, but $\Phi_{\varphi_o}(h, X, S, X^{\#})$ does not necessarily entail it. - Later, in 7.1, some stronger conditions, marked by $\Phi^{\#}$, Φ', Φ^{o}, will be introduced; in the moment, they are not yet definable.

5. Definition of the ambiguous model of type theory. Let be

$Y_o^- = \underline{C}$, $Y_o = \{0\} \times \underline{C}$, ...,

$Y_{n+1}^- = \{a\colon a \subseteq Y_n^- \wedge \exists h(h \in H_o^x \wedge$ (the stratified formula "$\forall x(x\underline{\in}(n+1,a) \leftrightarrow h)$" is $Y_o \cup \ldots \cup Y_n$-valid$))\}$,

$Y_{n+1} = \{n+1\} \times Y_{n+1}^-$, ..., $Y = \bigcup\{Y_n\colon n \in \omega\}$, $Z = Y \setminus Y_o$. ∎

Remarks. 1) The phrase

"the stratified formula "$\forall x(x\underline{\in}(n+1,a) \leftrightarrow h)$" is $Y_o \cup \ldots \cup Y_n$-valid"

has to be conceived as

"the stratified formula "$\forall x(x\underline{\in}y \leftrightarrow h)$" is $Y_o \cup \ldots \cup Y_n$-valid relatively to the valuation $\{(y, (n+1,a))\}$",

if y does not occur in h.

2) "$Y_o \cup \ldots \cup Y_n$-valid" can, as a rule, equivalently be replaced by "Y-valid", as because of the condition $h \in H_o^x$ and of the fixed valuation of y with $(n+1,a)$, the variable x in the stratified formula "$\forall x(x\underline{\in}y \leftrightarrow h)$" cannot be Y-admissibly valuated but in Y_n, and, as a rule, the variables bound in h cannot be Y-admissibly valuated but in $Y_o \cup \ldots \cup Y_n$. There are exceptions: Take for example the stratified formula $x \underline{\equiv} x \rightarrow \forall y\, y \underline{\equiv} y$, here the stratification is disconnected. However, those exceptions are not important, since in subformulae as $\forall y\, y \underline{\equiv} y$ causing the irregularity, the variable x does not occur; those subformulae are therefore easily eliminable.

3) Y_1^- consists exactly of the small and the cosmall (§ 1) subsets of Y_o^-. As a rule, in Y_n with $n > 1$, the elements are neither small nor cosmall. The deleting of Y_o in the definition of Z will avoid consequences of this peculiarity of Y_1 with respect to the Z-validity of stratified formulae. (Cf. the beginning of § 7.)

4) One may not overlook the following: With respect to Z, we will of course be interested in the Z-validity of stratified formulae, but the requirement for the stratified formulae defining the elements of Z to be Y-valid cannot generally be equivalently replaced by the requirement of Z-validity of other stratified formulae, even in the case the formulae are

elements of K_o^x. Thus, as a rule, the elements of Z are externally definable, but not so internally; and, as long as we are but interested in the Z-validity of stratified formulae, we can say that the defining stratified formulae for the elements of Z remain essentially secret.

Notations. For $h \in H_o^x$, $a \in Z_k$, and if the stratified formula $\forall x(x \underline{\in} a \leftrightarrow h)$ is Y-valid, we denote h by h_a, a by $r(h,k)$. (The definition of h_a is unique only up to Y-equivalence, but that will mostly do for our purposes; if necessary, we shall choose appropriate representatives for h_a.) If T is a subset of Y, we denote by T_n^- the set $T\langle\{n\}\rangle$, by T_n the set $T|\{n\}$, in accordance with the notation of § 3.

A further remark. Given $h \in H_o^x$ and a stratification function s_h with $s_h(x) = i$, max rng $s_h = j$, there exists an $r(h, j-i+1)$; Y_o^- and with it Y_o are in the definition supposed to be a set (of the underlying classical set theory). We assume that $Y_o, \ldots, Y_{j-i}$ are sets. h is also a (small) set, and the condition "the stratified formula "$\forall x(x \underline{\in} a \leftrightarrow h)$" is $Y_o \cup \ldots \cup Y_{j-i}$-valid" is describable by an expression (in the metatheoretical variable a) of the underlying classical set theory. Hence for example in ZFC, there exists a set b with $r(h, j-i+1) = (j-i+1, b)$ as a subset of Y_{j-i}^- by the axiom of subsets; b is therefore an element of Y_{j-i+1}^-. By the axioms of subsets and of power-sets, Y_{j-i+1}^- and Y_{j-i+1} are then sets, too. In a similar way, all sets discussed later on can be proved to exist. - Besides we remark that every small subset a of Y_k^- is an element of Y_{k+1}^-, for we can set $h_{(k+1,a)} = \bigvee_{b \in a} \forall y(y \underline{\in} x \leftrightarrow h_{(k,b)}(y)) \in H_o^x$ for $k > 0$ resp. $h_{(1,a)} = \bigvee_{b \in a} x \equiv b \in H_o^x$ for $k = 0$; analogously for cosmall subsets a of Y_k^-.

Explanation. Z will be the ambiguous model of type theory. A restricted scheme of comprehension ($h \in K_o^x$ instead of $h \in K^x$ resp. more generally $h \in K$) by definition holds in Z. We shall prove in § 8 that the restriction is only a seeming one. Thus, in extension of the notations introduced above, $r(h,k+1)$

exists not only for $h \in H_o^x$ but also for $h \in K^x$ and every k, even for $h \in H^x$, $h \in K$, $h \in H$, for every k in the case of $h \in K$, for at least one k in the case of $h \in H \setminus K$. A model for NF is built from Z by identification of the various $r(h,k+1)$ for every fixed $h \in K^x$. As every set a of a given type with $h_a \in K^x$ must be identifiable with one set of each higher type, it is plausible to choose those sets as candidates for elements of $\mathrm{rng}\ Z$ which are describable by certain stratified formulae. However, in most cases the describing stratified formulae must be interpreted essentially beyond Z, externally or "secretly" as indicated in remark 4, in order to make our proof work and the model of type theory really satisfy the axioms of ambiguity; for this reason we have deleted Y_o. On the other hand, Z must not be chosen too large; for every k the type Y_k^- must not be too comprehensive when seen from Y_{k+1}^-. In standard models, e.g., of type theory, we have the axiom of choice resp. the theorem of well-ordering, among the elements of the higher types are some which "overlook" the lower types perfectly, i.e., which well-order them. But the theorem of well-ordering contradicts NF and thus contradicts the axioms of ambiguity, too. With regard to our construction, a well-ordering of $\underline{C} = Y_o^-$ is certainly no element of $\mathrm{rng}\ Y$ because there is no $h \in H_o^x$ (with a small length!) to describe it. The situation in the higher types is analogous. Therefore, also in Y and Z only special large subsets are fully comprehensive in their structure by means of small sets of stratified formulae. - If, in the definition of H_o^x with respect to its application in the definition of Y and in our proof, the transfinite composition of set-theoretical formulae is really indispensable, is an open question.

6. <u>Lemma (Orderability of the bound variables).</u> For every stratified formula h there is a Y-equivalent stratified formula h' such that for every subformula $\bigvee V h''$ of h', for every $y \in V$, and for every z bound in h'' holds $s_{h'}(y) \le s_{h'}(z)$. If $h \in K$, then also $h' \in K$ can be assumed.

So in h' the variables with the greater stratification index, if at all quantified, are quantified before those with the lower stratification index, u before v if $u \in v$ occurs in h'.

Definition. A stratified formula is ordered iff it fulfills the condition imposed on h' in this lemma.

Proof of the lemma. The idea of this proof consists in relativizing the outer quantified variables with great stratification indices on singletons with small stratification indices, and then, in the interior, in relativizing on the unique elements of those singletons. In this manner, the stratification indices of those outer quantified variables are lowered in comparison with the other variables. – We assume that a subformula $\forall Vh''$ of h has not the required property, i.e., that there are a $y \in V$ and a variable z bound in h'' such that $s_{h'}(y) > s_{h'}(z)$. Let $V_1 = \{y_i : i \in I\}$ be the small set of those variables $y \in V$ such that $s_{h'}(y) > s_{h'}(z)$ holds for at least one variable z bound in h''. By our assumption, V_1 is not empty. Let $V_2 = \{v_i : i \in I\}$ be a set of variables not occurring in $\forall Vh''$, the relation $\{(y_i, v_i) : i \in I\}$ may be biunique. Let $\forall V'h^+$ be a subformula of h'', there may exist no subformula $\forall V''h^{+\prime}$ of h'' such that $\forall V'h^+$ is in the same position also a subformula of $h^{+\prime}$. We can say that $\forall V'h^+$ is maximal among the subformulae of h'' beginning with $\forall$. Renaming the variables if necessary, we may assume V_1 and V' to be disjoint. We now substitute in $\forall Vh''$ simultaneously every subformula $\forall V'h^+$ as above by $\forall V' \cup V_1 (\bigwedge_{i \in I} y_i \in v_i \rightarrow h^+)$, and every atomic subformula $h^{*\prime}$ of h'' with no variables bound in h'' analogously by $\forall V_1 (\bigwedge_{i \in I} y_i \in v_i \rightarrow h^*)$; after that we replace the resulting stratified formula $\forall Vh^o$ by $\forall (V \setminus V_1) \cup V_2 (\bigwedge_{i \in I} \exists u(u \in v_i \wedge \forall w(w \in v_i \leftrightarrow w = u)) \rightarrow h^o)$.
The result is Y-equivalent to $\forall Vh''$. The stratification indices of the elements of $(V \setminus V_1) \cup V_2$, in comparison with those of V_1, are by at least 1 lower. If every one of the subformulae $\forall V'h^+$ as above has already the property required above for h', then also a stratified formula resulting from $\forall Vh''$ by a finite number of repetitions of the described procedure has

the same property. In this case, the procedure has an end after a finite number of repetitions, as $\operatorname{rng} s_h$ is finite. Generally, we have to repeat the transformation procedure as often as necessary, starting with the innermost subformulae beginning with $\forall$ and proceding successively to outermore subformulae. After a, as a rule, infinite, but small, number of steps (the length of the stratified formula h is small and hence also the number of subformulae; that number is an upper bound for the number of steps necessary) we are done. If $\operatorname{rng} s_h \subseteq [0,n]$, then $\operatorname{rng} s_{h'} \subseteq [0,n]$ can be assumed, for the variables newly introduced can get stratification indices which the old ones already have. The requirement for $\operatorname{rng} s_{h'}$ to be finite (in the definition of H) is thus respected. The last clause of the lemma is obviously valid, as the transformations did not insert constants. In this case apart from Y-equivalence also Z-equivalence holds.

<u>7. Proposition (Ambiguity).</u> For $n \geq m > 0$, a stratified formula $h \in K$ is $Y_m \vee \ldots \vee Y_n$-valid iff $Y_{m+1} \vee \ldots \vee Y_{n+1}$-valid.

<u>Remark 1.</u> The proposition states the validity of the <u>axioms of ambiguity</u> for Z (cf. [1], [8]). – Following Grišin [3], cf. also [2], it is sufficient to consider the case $m = 1$, $n = 3$. But that reduction does not simplify our proof, and we shall not use Grišin's result. On the other hand, $n = 3$ seems to be the borderline up to which the following constructions remain to be sufficiently imaginable.

<u>Remark 2.</u> The proposition is not true for $m = 0$. Let, e.g., h be the stratified formula

$$\exists x \exists y (\forall z \exists u_1 \exists u_2 (z \underline{\in} y \to z \underline{=} \underline{\{} u_1, u_2 \underline{\}}) \wedge \forall u_1 \forall u_2 \forall u_3 (\underline{\{} u_1, u_2 \underline{\}} \underline{\in} y \wedge \wedge \underline{\{} u_1, u_3 \underline{\}} \underline{\in} y \to u_2 \underline{=} u_3) \wedge \forall u_1 \exists u_2 (\underline{\{} u_1, u_2 \underline{\}} \underline{\in} y \wedge \neg (u_1 \underline{\in} x \leftrightarrow \leftrightarrow u_2 \underline{\in} x))),$$

which means "there is an x equivalent to its complement (relative to the same type)". This equivalence relation is established by a biunique set y of unordered pairs. h is not $Y_0 \vee Y_1 \vee Y_2$-valid (for every element $x = (1, x')$ of Y_1 the set x' is either a small or a cosmall subset of Y_0^-; a small x' and its large Y_0^--complement – or vice versa – cannot be equivalent in the underlying classical set theory, x can all the less be equivalent to its complement in the sense of Y-validity), but shurely $Y_1 \vee Y_2 \vee Y_3$-valid: Take for x' the set $\{u: \underline{c}_0 \in u\}$ ($\underline{c}_0 \in \underline{C}$, fixed!), which is at the same time large and colarge, set $x = (2, x')$, and take for y the set $(3, \{\{u_1, u_2\}: u_1, u_2 \in Y_1^{-} \wedge u_1 = u_2 \vee \{\underline{c}_0\} \wedge \neg\, \underline{c}_0 \in u_2\})$.

The proposition will be proved in sections 7.1 through 7.6. We will sometimes intersperse plausibility considerations to make more clear our procedure; of course, these considerations shall not replace any part of the proof. - Some parts of the proof will be separated from the main proof and made up in special subsections, for the sake of a better lucidity of the procedure.

<u>7.1. Preliminaries, definition of Z^{**}.</u> The goal of the proof of proposition 7 consists in the following: Start with a small set Z^{*} with $\Phi(h, Y_{m+1} \vee \dots \vee Y_{n+1}, \emptyset, Z^{*})$ and a further property to be precised later and shift it one type downwards isomorphically without the Φ-property being lost, i.e., find a small set $Z^{0} \subseteq Y_m \vee \dots \vee Y_n$ isomorphic to Z^{*} and with $\Phi(h, Y_m \vee \dots \vee Y_n, \emptyset, Z^{0})$.

We procede by induction on n. For $n = 1$ the proposition is certainly true as Y_1 and Y_2 both are large sets of equal cardinalities, and in stratified formulae h for which Y_1-Val(Gen h, $\emptyset$) is defined, no $\in$ can occur; therefore the formulae h are constructed from identities only, which are interpreted standard. The proposition may be proved for $1 \leq n < n_0$, we will prove it for $n = n_0$; we shall write n instead of n_0 in future. Let h be a stratified formula for which $Y_{m+1} \vee \dots \vee Y_{n+1}$-Val(Gen h, $\emptyset$) is defined. Without loss of generality, we assume h to be ordered and $m = 1$.

Before going on in the main line, we have to note some definitions.

<u>Definition of the Φ^{*}-property.</u> Let S be a subset of a small subset T of $Y \setminus (Y_0 \vee Y_1)$. We assume that there exists a small set $W \subset Y_0 \vee Y_1$ such that

$\Phi_0(h_k^{*}, Y_0 \vee Y_1 \vee Y_2 \vee \dots \vee Y_{k-1}, \emptyset, W \vee (T \cap (Y_2 \vee \dots \vee Y_{k-1})))$

holds for every $k \geq 2$, any biunique function ψ_k with $\mathrm{dom}\,\psi_k = V_k \subseteq X$, $\mathrm{rng}\,\psi_k = S_k$, $h_k^{*} = \bigwedge_{v \in V_k} h_{\psi_k(v)}(v)$, provided h_k^{*} is an ordered stratified formula (i.e., the defining stratified formulae h_a for $a \in S$ and V_k have been suitably chosen). The fulfilment of these conditions for S and T may be marked by $\bar{\Phi}^{*}(S,T)$; we shall write $\bar{\Phi}^{*}(T)$ instead of $\bar{\Phi}^{*}(T,T)$. Correspondingly, $\bar{\Phi}^{*}(h, Y_2 \cup \dots \cup Y_{n+1}, S, Z^{*})$ shall be an abbreviation for $\Phi(h, Y_2 \cup \dots \cup Y_{n+1}, S, Z^{*}) \wedge \bar{\Phi}^{*}(Z^{*})$. In this case let additionally be $\mathrm{card}\, Z_k^{*} >$ length of h, for every $k \geq 2$. $\bar{\Phi}_{\varphi}^{*}$ resp. $\bar{\Phi}_{o}^{*}$ instead of $\bar{\Phi}^{*}$ in the definiendum shall have the analogous meaning with Φ_{φ} resp. Φ_o in the first conjunctive of the definiens. ∎

First we will see that for $S \subset Y_2 \cup \dots \cup Y_{n+1}$, S small, the conditions $\bar{\Phi}_o^{*}(h, Y_2 \cup \dots \cup Y_{n+1}, S, Z^{*})$ and with that the conditions $\bar{\Phi}^{*}(S,T)$ can be realized; here the realizability of the cardinality condition needs no words. – There is a small set $W^{(n+1)}$ with $\Phi_o(h, Y_2 \cup \dots \cup Y_{n+1}, S, W^{(n+1)})$. We set $Z_{n+1}^{*} = W_{n+1}^{(n+1)}$. We assume the sets $Z_{n+1}^{*}, \dots, Z_k^{*}$ to be already defined for $2 \leq k \leq n+1$. Then there is a small set $W^{(k-1)}$ with

$$\Phi_o(h_k^{*}, Y_o \cup Y_1 \cup Y_2 \cup \dots \cup Y_{k-1}, W^{(k)} \cap (Y_o \cup \dots \cup Y_{k-1}), W^{(k-1)}),$$

$$\Phi_o(h, Y_2 \cup \dots \cup Y_{k-1} \cup Z_k^{*} \cup \dots \cup Z_{n+1}^{*},$$
$$(W^{(k)} \cap (Y_o \cup \dots \cup Y_{k-1})) \cup Z_k^{*} \cup \dots \cup Z_{n+1}^{*}, W^{(k-1)}).$$

Then let be $Z_{k-1}^{*} = W_{k-1}^{(k-1)}$; at last let be $W = W_1^{(2)} \cup W_o^{(2)}$. Since the stratified formulae h and h_a used in the definition are supposed to be ordered, the resulting small set $Z^{*} = Z_2^{*} \cup \dots \cup Z_{n+1}^{*}$ meets all requirements.

To be shure, the $\bar{\Phi}^{*}$-conditions cannot refer but to a fixed Z^{*}; for an arbitrary Z-admissible valuation φ_o of h we can find a small set S isomorphic to Z^{*} and with $\Phi_o(h, Y_2 \cup \dots \cup Y_{n+1}, \mathrm{rng}\, \varphi_o, S)$, but $\bar{\Phi}^{*}(S)$ does not necessarily hold.

<u>Supplement to the definition.</u> Let $\bar{\Phi}'(S,T)$ be defined precisely as $\bar{\Phi}^{*}(S,T)$ but with respect to $W \subset Y_o$ instead of

$W \subset Y_0 \cup Y_1$ and $Y \setminus Y_0$ instead of $Y \setminus (Y_0 \cup Y_1)$. Let $\mathfrak{F}^o(S,T)$ be defined precisely as $\mathfrak{F}^{\ast}(S,T)$ but with respect to $W = \emptyset$ and Y instead of $Y \setminus (Y_0 \cup Y_1)$. The further conditions composed with $\mathfrak{F}'$, $\mathfrak{F}^o$ instead of $\mathfrak{F}^{\ast}$ may be defined analogously. ∎

h and arbitrary small sets $T \subset Z$ in the following way induce equivalence relations $J(T)$ in a subset of PY, which can be conceived as relations of indiscernability with respect to T and the truth-value calculation of $Z\text{-Val}(\mathrm{Gen}\ h, \emptyset)$. Sometimes we shall write $AJ(T)B$ instead of $(A,B) \in J(T)$.

<u>Definition of $J(T)$.</u> For $A, B \subset Y$ let be $AJ(T)B$ iff $\mathrm{dom}\ \{a\} = \mathrm{dom}\ \{b\}$ for every $a \in A$, $b \in B$ (i.e., $\mathrm{dom}\ A$ and $\mathrm{dom}\ B$ are the same singleton, A and B are subsets of the same Y_k), $\mathrm{card}\ A \leq \mathrm{card}\ T_k$ for $A \subset Y_k$, and there are small sets T_A and T_B and an isomorphism η from T_A onto T_B such that $\mathfrak{F}(h, Z, T \cup A, T_A \cap Z)$, $\mathfrak{F}(h, Z, T \cup B, T_B \cap Z)$, $\mathfrak{F}^o(T, T_A)$, $\mathfrak{F}^o(T, T_B)$, $\eta\langle A\rangle = B$, $\eta(c) = c$ for every $c \in T$. ∎

Let be $\mathfrak{F}^{\ast}_o(h, Y_2 \cup \ldots \cup Y_{n+1}, \emptyset, Z^{\ast})$.

In the following, we define p_k, $p^{\ast}_k$, $Z^{\ast+}(k)$, $Z^{\ast\ast}(k)$ by induction on k backward from $k = n$ to $k = 1$. $Z^{\ast\ast}$ will contain as elements representatives of large $J(T)$-equivalence classes for certain small sets T (certain extensions of $Z^{\ast}$), the small ones will be subsets of $Z^{\ast+}$. The labelling "definition" is somewhat liberal, for $Z^{\ast+}(k)$ and $Z^{\ast\ast}(k)$ are not determined uniquely below; however the concrete choice is not important.

<u>Definition of p_k, $p^{\ast}_k$, $Z^{\ast+}(k)$, $Z^{\ast\ast}(k)$, $Z^{\ast+}$, $Z^{\ast\ast}$.</u>
For $A \subset Y_n$ let be $p_n(A) = J(Z^{\ast})\langle\{A\}\rangle$ iff $J(Z^{\ast})\langle\{A\}\rangle$ is a nonvoid small set, but $p_n(A) = \emptyset$ iff $J(Z^{\ast})\langle\{A\}\rangle$ is a large set, p_n not be defined elsewhere.

We remark: $\mathrm{rng}\ p_n$ is a small set; for, by the remark to § 4, there is a small set T such that for every $A \subset Y_n$ with $\mathrm{card}\ A \leq \mathrm{card}\ Z^{\ast}_n$ there is a set T_A isomorphic to T with $\mathfrak{F}(h, Z, Z^{\ast} \cup A, T_A \cap Z)$ and $\mathfrak{F}^o(Z^{\ast}, T_A)$; therefore $\mathrm{card}\ \mathrm{rng}\ p_n$ is not greater than $\mathrm{card}\ PT$. $p_n(A)$ and $\bigcup p_n(A)$ being small for every $A \in \mathrm{dom}\ p_n$, also $\bigcup\bigcup(\mathrm{rng}\ p_n)$ is small.

Let $Z^{**+}(n)$ be a small set with
$\Phi_o(h,\ Y_2 \cup \ldots \cup Y_n \cup Z^{*}_{n+1},\ Z^{*} \cup \bigcup\bigcup(\mathrm{rng}\ p_n),\ Z^{*+}(n))$,
$\Phi^{*}(Z^{*}_{n+1},\ Z^{*+}(n))$.

We now suppose the functions p_i and the small sets $Z^{*+}(i)$ to be defined for $2 \leq k < i \leq n$, we define p_k and $Z^{*+}(k)$.

For $A \subset Y_k$ let be $p_k(A) = J(^{*+}(k+1))\langle\{A\}\rangle$ iff $J(Z^{*+}(k+1))\langle\{A\}\rangle$ is a nonvoid small set; let be $p_k(A) = \emptyset$ iff $J(Z^{*+}(k+1))\langle\{A\}\rangle$ is a large set, p_k may not be defined elsewhere. Then $\mathrm{rng}\ p_k$ is again, as $\mathrm{rng}\ p_n$, a small set because $Z^{*+}(k+1)$ is supposed to be a small set. Then $\bigcup\bigcup(\mathrm{rng}\ p_k)$ is a small set, too.

Let $Z^{*+}(k)$ be a small set with
$\Phi_o(h,\ Y_2 \cup \ldots \cup Y_k \cup Z^{*+}(k+1),\ Z^{*+}(k+1) \cup \bigcup\bigcup(\mathrm{rng}\ p_k),\ Z^{*+}(k))$,
$\Phi^{*}(Z^{*},\ Z^{*+}(k))$.

Finally, let be $Z^{*+} = Z^{*+}(2)$.

The construction of Z^{**} largely parallels the construction of Z^{*+}, but there will be an additional condition.

Let be $Z^{**}(n+1) = Z^{*+}$.

Let be $2 \leq k \leq n$. We now suppose the small sets $Z^{**}(i)$ to be defined for $k < i \leq n+1$. We define the function p^{*}_k, then we shall define the small set $Z^{**}(k)$.

For $A \subset Y_k$ let be $p^{*}_k(A) = J(Z^{**}(k+1))\langle\{A\}\rangle$ iff $J(Z^{**}(k+1))\langle\{A\}\rangle$ is a nonvoid small set; let be $p^{*}_k(A) = \emptyset$ iff $J(Z^{**}(k+1))\langle\{A\}\rangle$ is a large set; p^{*}_k may not be defined elsewhere. Then $\mathrm{rng}\ p^{*}_k$ is again, as $\mathrm{rng}\ p_n$, a small set because $Z^{**}(k+1)$ is supposed to be a small set. Then $\bigcup\bigcup \mathrm{rng}\ p^{*}_k$ is a small set, too.

Let $Z^{**}(k)$ be a small set with
$\Phi^{*}_o(h,\ Y_2 \cup \ldots \cup Y_k \cup Z^{**}(k+1),\ Z^{**}(k+1) \cup \bigcup\bigcup(\mathrm{rng}\ p^{*}_k),\ Z^{**}(k))$,
$p^{*}_k(A) = \emptyset \rightarrow J(Z^{**}(k+1))\langle\{A\}\rangle \cap P(Z^{**}(k) \setminus Z^{**}(k+1)) \neq \emptyset$
for every $A \in \mathrm{dom}\ p^{*}_k \cap P(Y_k \setminus Z^{**}(k+1))$.

Finally, let be $Z^{**} = Z^{**}(2)$. ▫

Since h is supposed to be ordered, we have by our construction $\Phi_o(h, Y_2\cup\ldots\cup Y_{n+1}, Z^*, Z^{*+})$, $\Phi^*(Z^*, Z^{*+})$, $\Phi_o^*(h, Y_2\cup\ldots\cup Y_{n+1}, Z^{*+}, Z^{**})$. The cardinalities of Z_k^*, Z_k^{*+}, Z_k^{**} are strongly increasing in this sequence.

To illustrate the notions, we remark:

As a consequence of the definition of Z^*, for every $a = (k+1, a') \in Z_{k+1}^*$, $2 \leq k \leq n$, the family of the truth-values $Z\text{-Val}(b \underline{\in} a, \emptyset)$ for $b \in Z_k^*$, resp. the family of the truth-values $b' \in a'$ for $b = (k, b') \in Z_k^*$, can be said to represent the set a within Z^* resp. the set a' within Z_{k+1}^{*-}, at least as far as their properties play a rôle in the truth-value calculation of $Y_2\cup\ldots\cup Y_{n+1}\text{-Val}(\text{Gen } h, \emptyset)$, by the assumption $\Phi^*(Z^*)$ even somewhat farther: a defining stratified formula h_a is characterized, too, absolutely and not only with respect to Z^*. The family of the truth-values $Z\text{-Val}(b \underline{\in} a, \emptyset)$ for $b \in Z_k^{**} \setminus Z^{*+}$ can be said to describe the asymptotic behaviour of a or the behaviour of a in the large; among the isomorphy types (of the small sets $T_{\{b\}}$ in the definition of $J(Z^*)$) used in the calculation of $Y_2\cup\ldots\cup Y_{n+1}\text{-Val}(\text{Gen } h, \emptyset)$, that family represents the often (a great number of times) occurring ones. The family of the truth-values $Z\text{-Val}(b \underline{\in} a, \emptyset)$ for $b \in Z_k^{*+} \setminus Z^*$ can be said to describe the behaviour of a in an area of transition, between the initial in a narrower sense (relative to Z^*) and the asymptotic ones. The initial behaviour characterizes a in totu; the asymptotic behaviour, the behaviour almost everywhere (beyond Z_k^{*+}), does not change the characteristics of a and has been concentrated on Z_k^{**}. If, in the definition of $J(Z^*)$, we have taken small sets T_B instead of $T_{\{b\}}$, we have done so because generally in h not single variables but certain small sets of variables are quantified; also the (generally necessary) cardinality bounds in the definitions of $J(Z^*)$ and $\Phi^*(Z^*)$ have been chosen with regard to this. While the concentration of those properties of the large set $Z \setminus Z^*$ on the small set Z^{**} which are essential for the truth-value calculation of

$Y_2 \cup \dots \cup Y_{n+1}$-Val(Gen h, $\emptyset$) has a crucial importance, the area $Z^{**+} \setminus Z^{*}$ of transition has a merely technical importance. That importance lies in the circumstance that the property of a set to be large or small is expressable by the Z-validity of stratified formulae. The very fact that small sets can be arbitrarily extensive prevents us from supposing $Z^{*+} \subseteq Z^{*}$. Sometimes, we shall utilize the property of Z^{*} to be able to be shifted isomorphically in the horizontal line (without raising or lowering types) without loosing the Φ-property, but the corresponding set Z^{*+} cannot be shifted together with Z^{*} preserving the characterizing properties. – In the next subsection, we shall shift the small set Z^{*} one type downward; while doing so, the borderlines between large and small must be preserved (otherwise especially the function f^{+} of 7.4 would hardly be definable), all the more as certain large sets describable by means of constants for elements of Z^{**} will have to be partitioned several times in the future into large subsets.

<u>7.2. Construction of the function f.</u> We shall define a function f which shifts Z^{*} one type downward, a function f from Z^{*} into $Y_1 \cup \dots \cup Y_n$ which is an isomorphism and for which additionally $\Phi(\text{Gen h}, Y_1 \cup \dots \cup Y_n, \emptyset, \text{rng f})$ holds, and therefore also

$$Y_2 \cup \dots \cup Y_{n+1}\text{-Val(Gen h, }\emptyset) = Z^{*}\text{-Val(Gen h, }\emptyset) =$$
$$= (\text{rng f})\text{-Val(Gen h, }\emptyset) = Y_1 \cup \dots \cup Y_n\text{-Val(Gen h, }\emptyset).$$

In order to make, in this sense, rng f a complete analogon of Z^{*} one type lower, we shall provide that for $a \in \text{dom f}$ the image f(a) will have, as far as possible, the same initial and the same asymptotic behaviour as the counterimage a (cf. the explanation at the end of 7.1), viz. the same behaviour "in the large" and "in the small" with respect to the truth-value calculation of Z-Val(Gen h, $\emptyset$). To realize that, we have to shift Z^{*} downward in such a manner that the relation $J(Z^{*})$ is shifted down together with Z^{*}. To be able to do so, that relation has been condensed into Z^{**}.

The function f will essentially (apart from the indexing of its arguments and values) be the union of functions f_k

for $2 \leq k \leq n+1$; for the "transitional behaviour" we also need auxiliary functions q_k for $2 \leq k \leq n$. Every function value $f_k(a)$ will be the union of a function value $f_k'(a)$ and of a function value $f_k''(a)$. We first define the functions f_2, q_2, f_3 separately.

Let χ be a biunique function from $(Z^{**} \setminus Z^{*+})_2^{-}$ into $\underline{C}$, f_2 be a biunique function from Z_2^{*-} into Y_1^{-}, and q_2 be a biunique function from $(Z^{*+} \setminus Z^{*})_2^{-}$ into $Y_1^{-} \setminus \operatorname{rng} f_2$. Let f_3'' be the function from Z_3^{*-} into $P(Y_1^{-})$ with

$f_3''(a) = (f_2 \cup q_2)\langle a\rangle$ for every $a \in Z_3^{*}$.

By this, the sets $Z_2^{*+} \cup Z_3^{*}$ and $(\{1\} \times \operatorname{rng}(f_2 \cup q_2)) \cup (\{2\} \times \operatorname{rng} f_3'')$ are isomorphic by an isomorphism η with $\eta((2,a)) = (1, (f_2 \cup q_2)(a))$ for $a \in \operatorname{dom}(f_2 \cup q_2)$, $\eta((3,a)) = (2, f_3''(a))$ for $a \in \operatorname{dom} f_3''$. So we can say that f_3'' shifts downward the initial behaviour of a. To be able to define the function f_3' ruling the shifting of the asymptotic behaviour, we have to make some preparations.

Let be $g = \operatorname{dom} \chi = (Z^{**} \setminus Z^{*+})_2^{-}$, and F be a principal ultrafilter in Pg. Let $\Psi_2(x,y)$ be the expression

$\exists c(c \in g \cap x \wedge \chi(c) \in y \wedge \forall d(d \in g \wedge \chi(d) \in y \rightarrow c = d))$

of the underlying classical set theory. It means: $y \cap \operatorname{rng} \chi$ has exactly one element, and the χ-counterimage of it is an element of x.

Let f_3' be the function from $P(Y_2^{-})$ into $P(Y_1^{-})$ with

$$f_3'(a) = \{b : b \in Y_1^{-} \setminus \operatorname{rng}(f_2 \cup q_2) \wedge (a \cap g \in Pg \setminus F \rightarrow \Psi_2(a,b)) \wedge \wedge (a \cap g \in F \rightarrow \neg \Psi_2(g \setminus a, b))$$

for $a \subseteq Y_2^{-}$. That means: If $a \cap g$ does not belong to the ultrafilter, or in this sense has only few elements, then $f_3'(a)$ contains almost all (up to a small number of exceptions in $\operatorname{rng}(f_2 \cup q_2)$) elements $b \in Y_1^{-}$ with $\Psi_2(a,b)$, i.e. almost all those elements b of Y_1^{-} the unique common element of which with $\operatorname{rng} \chi$ belongs to a. In the other case, if $a \cap g$ belongs to the ultrafilter, and in this sense has many elements, $f_3'(a)$ contains not only almost all elements $b \in Y_1^{-}$ with $\Psi_2(a,b)$ but moreover almost all those elements $b \in Y_1^{-}$ which have not exactly one element common with $\operatorname{rng} \chi$.

Before continuing the definition, we moreover remark with respect to $f_3^!$ that for every $a \subseteq Y_2^-$, $A \subseteq P(Y_2^-)$ the following properties hold:

B1) $f_3^!(\bigcup A) = \bigcup\{f_3^!(a_1)\colon a_1 \in A\}$,

B2) $f_3^!(\bigcap A) = \bigcap\{f_3^!(a_1)\colon a_1 \in A\}$,

B3) $f_3^!(Y_2^- \setminus a) = Y_1^- \setminus (\mathrm{rng}\,(f_2 \cup q_2) \cup f_3^!(a))$,

B4) $f_3^!(a)$ is large iff $a \cap g$ is not empty.

By this, the restriction of $f_3^!$ on Pg defines an isomorphism from the generalized Boolean algebra $(Pg, \bigcap, \bigcup)$ into the generalized Boolean algebra $(P(Y_1^-), \bigcap, \bigcup)$. $f_3^!$ itself is yet a generalized Boolean homomorphism, turning to the isomorphism by the identification of elements with equal images $(f_3^!(a_1) = f_3^!(a_2) \leftrightarrow a_1 \cap g = a_2 \cap g$ for $a_1, a_2 \in \mathrm{dom}\, f_3^!)$.

Proof of the properties B1), ..., B4). Let the principal ultrafilter be generated by $c_0 \in g$.

Ad B1). Case B1a): $c_0 \notin \bigcup A$. Then $a_1 \cap g \in Pg \setminus F$ holds for every $a_1 \in A$, and further $\Psi_2(\bigcup A, b) \leftrightarrow \exists a_1(a_1 \in A \wedge \Psi_2(a_1, b))$, independently of b, which immediately entails the assertion. Case B1b): $c_0 \in \bigcup A$. Let be $c_0 \in a_2 \in A$. Now for $b \in Y_1^- \setminus \mathrm{rng}\,(f_2 \cup q_2)$ either $\neg\exists! c(c \in g \wedge \chi(c) \in b)$ holds, thence $b \in f_3^!(a_2)$, $b \in f_3^!(\bigcup A)$; or $\exists! c(c \in g \wedge \chi(c) \in b)$ holds, then $b \in f_3^!(\bigcup A)$ holds iff $c_1 \in \bigcup A$ for $c_1 = \iota c(c \in g \wedge \chi(c) \in b)$, in which case $\exists a_1(c_1 \in a_1 \in A)$, and that is equivalent to $\exists a_1\ b \in f_3^!(a_1)$.

Ad B2). B2) is a consequence of B1) and B3).

Ad B3). Let be, without loss of generality, $c_0 \in a$. Case B3a): Assume $\neg\exists! c(c \in g \wedge \chi(c) \in b)$ for an element b of $Y_1^- \setminus \mathrm{rng}\,(f_2 \cup q_2)$. Then $\neg\Psi_2(g \setminus a, b)$, therefore $b \in f_3^!(a)$ by $a \cap g \in F$; on the other hand we have $b \notin f_3^!(Y_2^- \setminus a)$ by $(Y_2^- \setminus a) \cap g = g \setminus a \in Pg \setminus F$ and $\neg\Psi_2(g \setminus a, b)$. Case B3b): Let be $\exists! c(c \in g \wedge \chi(c) \in b)$ for the same b. Then for $c_1 = \iota c(c \in g \wedge \chi(c) \in b)$ either holds $c_1 \in a$ or $c_1 \in g \setminus a$ and hence either $b \in f_3^!(a)$ or $b \in f_3^!(g \setminus a) = f_3^!(Y_2^- \setminus a)$.

Ad B4). The $\rightarrow$-direction is trivial. $\leftarrow$: Let be $c_2 \in g \cap a$. The set $B = \{\{\chi(c_2), c\}\colon c \in \underline{C}\}$ is a large subset of Y_1^-. As f_2, q_2, χ are all small, all elements of a cosmall subset of B belong to $Y_1^- \setminus \mathrm{rng}\,(f_2 \cup q_2)$ and to $f_3^!(a)$, independently of being $a \cap g$ an element of the ultrafilter or not. – The supposition $g \cap a \neq \emptyset$ can be weakened to $\neg a \subseteq Z_2^{\aleph +-}$ if $a \in Z_3^{\aleph -}$, because then a is large (by construction of $Z^{\aleph +}$), and that entails $g \cap a \neq \emptyset$ by construction of $Z^{\aleph\aleph}$. As by B3) the two truth-values are symmetric, we have as well: If $a \in Z_3^{\aleph -}$ and $\exists b(b \in Y_2^- \wedge b \notin \mathrm{dom}\ (f_2 \cup q_2) \wedge b \notin a)$, then $f_3'(a)$ is colarge.

We said that $Z_2^{\aleph\aleph} \setminus Z^{\aleph +}$ can be taken to characterize the asymptotic behaviour of $a \in Z_3^{\aleph}$. Apart from the type indices, f_3' has thus transferred that behaviour lattice-isomorphically into $P(Y_1^-)$.

We go to continue the definition: Finally let be $f_3(a) = f_3''(a) \cup f_3'(a)$ for every $a \in Z_3^{\aleph -}$. Now f_3 can be taken as a synthesis of the isomorphism f_3'' transferring the initial behaviour of the elements $a \in Z_3^{\aleph -}$ and the lattice isomorphism f_3' transferring the asymptotic behaviour of the elements $a \in Z_3^{\aleph -}$. – That holds $f_3(a) \in Y_2^-$ for $a \in \mathrm{dom}\ f_3$ will be proved later (in 7.3).

We now assume the functions f_i (biunique from $Z_i^{\aleph -}$ into Y_{i-1}^-) to be defined for $2 \leq i \leq k \leq n$ and $k > 2$, and the functions q_i (biunique from $(Z^{\aleph +} \setminus Z^{\aleph})_i^-$ into $Y_{i-1}^- \setminus \mathrm{rng}\ f_i$) to be defined for $2 \leq i < k$. We are going to define the functions q_k and f_{k+1}.

To be able to define q_k, f_{k+1}', f_{k+1}'' (by which f_{k+1} will later be defined), we first define relations Im_k, I_k similar to the relation $J(Z^{\aleph})$ defined in 7.1.

<u>Definition of Im_k.</u> For $B \subset Y_{k-1}^- \setminus \mathrm{rng}\ f_k$, $A \subset (Z^{\aleph\aleph} \setminus Z^{\aleph})_k^-$ we set $B Im_k A$ iff card $A \leq$ card $Z_k^{\aleph}$ and there are a small set S and a function η such that

I1) the condition

$\Phi_\varphi(h', Y_1 \cup \ldots \cup Y_{k-1},$
$\quad \bigcup\{\{i-1\} \times \mathrm{rng}\ (f_i \cup q_i) : 2 \leq i < k\} \cup (\{k-1\} \times \mathrm{rng}\ f_k) \cup B,\ S)$

holds for every subformula h' of h and a $Y_1 \cup \ldots \cup Y_n$-admissible valuation φ of all variables in h with $y \in \mathrm{dom}\ \varphi \rightarrow \exists i\ \varphi(y) = (i, \emptyset)$, i.e., the valuation φ of all variables in h with empty sets,

I2) η is an isomorphism from S onto $Z^{\aleph\aleph} \cap (Y_2 \cup \ldots \cup Y_k)$ with $\eta\langle\{k-1\} \times B\rangle = \{k\} \times A$, $\eta((i-1, f_i(d))) = (i, d)$ for every