Why Set theory without Foundation?

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1 Introduction

Some years ago Frank Oles came to Cambridge and gave us a talk on the desirability of set theory without the axiom of foundation. Although the cause is good, the reasons he gave for it were not.

The case was made as follows.

1. We wish to implement streams.
2. The correct way to implement streams is as ordered pair of head and tail.
3. The correct way to implement ordered pair is Wiener-Kuratowski.
4. The stream X of zeroes is therefore equal to $\{\{0\}, \{0, X\}\}$.
5. This contradicts the axiom of foundation, which must therefore be dropped.

Item 3 is the one to concentrate on. Too often students are taught that the ordered pair of x and y *is* $\{\{x\}, \{x, y\}\}$. This is a misrepresentation. $\{\{x\}, \{x, y\}\}$ is (quite often) a jolly good way to *implement* ordered pairs but it is not the only way. (2 is debatable too, but we will go along with it for the sake of argument).

It is elementary to show that there are pairing functions whose use in this context does not conflict with foundation. This is elementary, important, and not generally appreciated. As happens too often, the ideas needed are quite old, and the literature by now obscure. See Quine and Rosser *op. cit.*.

2 Flat pairing functions in the style of Quine

A flat pairing function is one such that $\langle x, y\rangle$ is in some suitable sense the same type as x and y. 'Type' is vague here: "flat" is a piece of slang not a term of art. One thing it could mean is that assertions like $(\exists z)(\exists x)(x = \langle z, x\rangle)$ do not contradict foundation, or that formulæ like $x = \langle z, x\rangle$ are stratified, or well-typed. Wiener-Kuratowski pairs are in no sense flat. Quine pairs are flat in both senses.

Suppose we can find two disjoint classes A and B each the same size as V, the collection of all sets. Let $\theta_1 : V \longleftrightarrow A$ and $\theta_2 : V \longleftrightarrow B$. Since the ranges of θ_1 and θ_2 are disjoint, nothing is both a value of θ_1 and a value of θ_2. Therefore when we are confronted with a set x some of whose members are values of θ_1 and some are values of θ_2 we can recover the set of z such that $\theta_1`z \in x$ and likewise the set of $z$ such that $\theta_2`z \in x$. Now we can obviously use this to implement disjoint union $(x \sqcup y)$ of x and y as $\theta_1``x \cup \theta_2``y$.[1]

But any implementation of disjoint union gives rise to an implementation of ordered pair, because we can take $\langle x, y\rangle$ to be $x \sqcup y$.

In fact in all realistic cases where we can find disjoint classes A and B both the same size as V we can actually find disjoint classes A' and B' both the same size as V satisfying the further condition that $A' \cup B' = V$. (The only difficulty is to ensure that the bijections between V and the new sets are definable in the appropriate sense). In these circumstances *every* set is an ordered pair.

REMARK **2.1** *(Rosser-Quine) We can implement a flat ordered pair in this way iff we can implement* $\mathbb{N}$*.*

Proof:

left to right

If π is a bijection between x and some proper subset y of x we can obtain an implementation of $\mathbb{N}$ by taking 0 to be an arbitrary member of $x - y$; S to be π, and $\mathbb{N}$ itself to be $\bigcap\{z : 0 \in z \wedge \pi``z \subseteq z\}$.

right to left

Conversely suppose we have an implementation of $\mathbb{N}$. We define Quine pairs as follows. Set $A = \{x : 0 \notin x\}$ and $B = \{x : 0 \in x\}$; S is successor as usual. $\theta_1`x$ is $S``(x \cap \mathbb{N}) \cup (x - \mathbb{N})$ (i.e., add 1 to every natural number in x and leave everything else alone) and $\theta_2`x$ is $(\theta_1`x) \cup \{0\}$.

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Quine pairs have the additional feature mentioned above, that everything is an ordered pair. They also have a further feature worth mentioning.

REMARK **2.2** *Any implementation of ordered pairs which arises in this way from an implementation of disjoint union is a continuous map between the two CPO's* $\mathcal{V} = \langle V, \subseteq\rangle$ *and* $\mathcal{V} \times \mathcal{V}$ *with the product ordering.*

Proof:

In fact if everything is an ordered pair the continuous map is actually an isomorphism. It is mechanical to check that $\langle x \cap y, z\rangle = \langle x, z\rangle \cap \langle y, z\rangle$, $\langle x \cup y, z\rangle = \langle x, z\rangle \cup \langle y, z\rangle$ and so on. Some of this will be useful in what follows.

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3 Streams

Now we are in a position to consider implementing streams (of naturals, for example) as ordered pair of head and tail. The original hard case was the stream of zeroes, for which we need an object $x = \langle 0, x\rangle$. This can be disposed of particularly sweetly because the function $\lambda x.\langle 0, x\rangle$ is a continuous map from the CPO $\langle V, \subseteq\rangle$ into itself and the desired object is simply its least fixed point. The more general case requires a very slight amount

[1] In standard set-theoretic notation $f``x$ is the set of values of f for arguments in x.

of work. Suppose we wish to implement the stream whose nth element in $f`n$. (We start counting at 0). Clearly we are going to want the set

$$\theta_1``(f`0) \cup \theta_2``(\theta_1``(f`1) \cup \theta_2``(\theta_1``(f`2) \cup \theta_2``(\ldots$$

which reduces to

$$\theta_1``(f`0) \cup \theta_2 \circ \theta_1``(f`1) \cup ((\theta_2)^2 \circ \theta_1)``(f`2) \ldots$$

which is to say

$$\bigcup\{((\theta_2)^n \circ \theta_1)``(f`n) : n \in \mathbb{N}\}$$

Whether or not this exists is going to depend on what axioms of comprehension are available, but not in any way on the falsity of the axiom of foundation.

4 Moral

The long-running success of ZF and its kin in monopolising the set theory market place has bequeathed a strange legacy. The very people who are telling us that we need set theory without the axiom of foundation (for the bad reasons exposed above) nevertheless still insist on consistency proofs of those theories relative to ZF and its kin. Asking for consistency proofs relative to ZF is not unreasonable: we want to know how things fit together. But once we bear in mind that such consistency proofs consist of interpretations of set theory without foundation inside set theory with foundation, we realise that we have ended up after all implementing in set theory with foundation (*via* those interpretations) all the things for which we thought in the first place that we need set theory *without* foundation.

This circular walk is a technically illuminating exercise that does no harm: at the very least one can enjoy the view. To my mind the danger lies in thinking that anything has been achieved thereby. There is also the danger that people encountering this topic for the first time may be repelled by this circularity and procede no further. This causes alarm to people like me who think that set theory without foundation (specifically with a universal set) should be taken seriously for other reasons.

The interesting question is this: once we realise how many different ways there are of implementing ordered pairs as sets; how many different ways there are of implementing natural numbers as sets; how many different ways there are of implementing streams as pairs or God-knows what, what will be left of the propaganda for set theory without foundation? What we can be sure of is that what is left is the really interesting part.

Quine, W.v.O. [1945] On ordered pairs. *Journal of Symbolic Logic* **10** pp. 95–96.

Rosser, J. B. [1952] The axiom of infinity in Quine's New Foundations. *Journal of Symbolic Logic* **17** pp. 238–42.