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The structure of the ordinals and the interpretation of *ZF* in double extension set theory

Abstract. Andrzej Kisielewicz has proposed three systems of "double extension set theory", of which we have shown two to be inconsistent in an earlier paper. Kisielewicz presented an argument that the remaining system interprets *ZF*, which is defective: it actually shows that the surviving possibly consistent system of double extension set theory interprets *ZF* with Separation and Comprehension restricted to Δ_0 formulas. We show that this system does interpret *ZF*, using an analysis of the structure of the ordinals.

Keywords: double extension set theory, universal set, ordinals

1. Introduction

Various systems of double extension set theory have been proposed by Andrzej Kisielewicz in the papers [3] and [4]. Of these, all but one was shown to be inconsistent by the author of this paper in [2].

The purpose of this paper is to examine the claim of Kisielewicz that the surviving possibly consistent system of double extension set theory interprets *ZF*. For technical reasons, the account he gives in [4] is unsatisfactory: it only shows that double extension set theory interprets bounded *ZF* (with comprehension and replacement restricted to Δ_0 formulas). We show that nonetheless double extension set theory does interpret *ZF*; we do this by examining the structure of the ordinals in double extension set theory.

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2. Definition of the Theory

Double extension set theory (hereinafter *DEST*) is a first-order theory with equality and two primitive binary predicates $\in$ and ϵ which are to be thought of as two different flavors of membership. Objects of our theory are called

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"sets"; but notice that each "set" has two different extensions. The interesting sets, called "regular" sets, will have the same extension in both senses.

DEFINITION 2.1. We say that a set A is *regular* iff $(\forall x.x \in A \equiv x \epsilon A)$. We abbreviate this as $\mathtt{reg}(A)$. Sets which are not regular are said to be *irregular*.

We have the following

AXIOM 2.2. (Axiom of Mixed Extensionality)

$$(\forall AB.(\forall x.x \in A \equiv x \epsilon B) \rightarrow A = B)$$

Observe that a set shown to be equal to another set by this axiom of extensionality will necessarily be regular. The axiom also implies that any set with the same extension as a regular set in the sense of either membership relation is actually equal to that regular set.

In [2] it is shown that in the full system of *DEST* it is provable that there are distinct objects with the same extension in terms of one or the other membership relation. It is not known whether it is consistent to assume that objects with both extensions the same are equal.

We introduce an axiom which ensures that regular sets are somewhat well-behaved.

DEFINITION 2.3. We say that a set A *has regular elements* just in case $(\forall x.(x \in A \vee x \epsilon A) \rightarrow \mathtt{reg}(x))$ (i.e., when all its elements in either sense are regular).

DEFINITION 2.4. We say that a set A is *partially contained* in a set B when $(\forall x.x \in A \rightarrow x \in B) \vee (\forall x.x \epsilon A \rightarrow x \epsilon B)$ (i.e., when A is a subset of B in one sense or the other).

AXIOM 2.5. (Regularity Axiom) A set which is partially contained in a set with regular elements is regular.

Now we introduce the basic idea that drives double extension set theory.

DEFINITION 2.6. A formula is said to be *uniform* if it contains no occurrence of ϵ.

DEFINITION 2.7. For any formula ϕ, we define ϕ^*, the *dual* of ϕ, as the formula which results when every occurrence of $\in$ in ϕ is replaced with ϵ and vice versa.

Axiom 2.8. (Axiom Scheme of Comprehension) For each uniform formula ϕ with free variables $x, x_1, \ldots, x_n$, there is an object $\{x \mid \phi\}$ such that

$$(\forall x_1 \ldots x_n. \bigwedge_{i=1}^{n} \texttt{reg}(x_i) \rightarrow (\forall x.(x \in \{x \mid \phi\} \equiv \phi^*) \wedge (x \,\epsilon\, \{x \mid \phi\} \equiv \phi)))$$

.

This completes the statement of the axioms of *DEST*.

It is instructive to observe how the theory avoids Russell's paradox. $x \notin x$ is a uniform formula with no free variables other than x. By the comprehension scheme, there is then an object $\{x \mid x \notin x\}$ (the Russell class), which we may abbreviate R, with the property that $x \in R \equiv x \not\epsilon\, x \wedge x \,\epsilon\, R \equiv x \notin x$, for all x. In particular, letting $x = R$, we find that $R \in R \equiv R \not\epsilon\, R$, so instead of discovering a paradox we find that R is an irregular set. Later in the paper we will have a close encounter with the Burali-Forti paradox as well.

The restriction to regular parameters (or at least some restriction on parameters) is necessary. Otherwise consider the set $A^* = \{x \mid x \in A\}$: we would have $x \in A^* \equiv x \,\epsilon\, A$, and by mixed extensionality $A = A^*$ would be regular – for any set A, including, for example R. But we have already established that R is not regular.

It is useful to observe that a formula of the form $x \,\epsilon\, \{x \mid \phi\}$, with ϕ uniform, is actually equivalent to a uniform formula and so can be allowed in uniform formulas (this will usually be useful when a lengthy set abstract has been given a name). This observation can be generalized to give a beautiful version of the theory allowing general (not necessarily regular) parameters in a very natural way – but the resulting theory turns out to be inconsistent (the theory which results is a subtheory of the theory of [3], and the proof of inconsistency is found in [2]).

We prove a basic lemma about regular sets, due to Kisielewicz in [4].

Definition 2.9. We say that x *has a singleton* (abbreviated $S(x)$) just in case $(\exists y.(\forall z.z \in y \equiv z = x))$. We denote the dual notion by $S^*(x)$. Notice that there is no implication that the singleton is unique.

Lemma 2.10. *(Singleton Lemma)* $(\forall A.\texttt{reg}(A) \equiv (S(x) \wedge S^*(x)))$. *In English, a set is regular iff it has a singleton in both senses.*

Proof. Suppose that a is regular. $\{x \mid x = a\}$ is a set. $y \in \{x \mid x = a\}$ iff $y \,\epsilon\, \{x \mid x = a\}$ iff $y = a$, so $\{x \mid x = a\}$ witnesses the truth of both $S(a)$ and $S^*(a)$.

Now suppose that $S(a)$ is true and $S^*(a)$ is true. Then there is a b whose $\in$-extension has a as its only element and a c whose ϵ-extension has a as its only element. By mixed extensionality, $b = c$ and so $b = c$ is regular. Use the notation $\{a\}$ for this object. Since $\{a\}$ is regular, we can consider the set $A^* = \{z \mid (\exists y.z \in y \wedge y \in \{a\})\}$. For any x, $x \,\epsilon\, A^*$ iff $(\exists y.x \in y \wedge y \in \{a\})$ iff $x \in A$, from which it follows that $A = A^*$ by mixed extensionality and further that A is regular.

This completes the proof of the Lemma.

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We prove a useful relationship between sentences ϕ and ϕ^*.

LEMMA 2.11. *(Duality Lemma) For any formula ϕ with free variables $x_1, \ldots, x_n$,*

$$(\forall x_1 \ldots x_n . \bigwedge_{i=1}^{n} \mathtt{reg}(x_i) \rightarrow \phi \equiv \phi^*)$$

.

PROOF. Consider the set $D = \{z \mid \phi\}$, z not free in ϕ (supposing that all free variables in ϕ are to take regular values, as is required for the axiom scheme of comprehension to apply). For any z, $z \in D \equiv \phi^*$, and $z \,\epsilon\, D \equiv \phi$, by comprehension. Note that the extension of D in either sense is either V or $\emptyset$, from which it follows that $D = V$ or $D = \emptyset$ (by mixed extensionality) and so its two extensions are the same, by regularity of V and $\emptyset$, from which it follows that $\phi \equiv \phi^*$.

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Here is a second Lemma about sets definable using a uniform formula with regular parameters.

LEMMA 2.12. *(Definability Lemma) Suppose that $\phi(x)$ is a uniform formula in which all free variables other than x are understood to have regular values, and it is the case that*

$$(\exists x.\phi(x)) \wedge (\forall xy.\phi(x) \wedge \phi(y) \rightarrow (\forall z.z \in x \equiv z \in y));$$

i.e., there are witnesses to $(\exists x.\phi(x))$ and all such witnesses have the same $\in$-extension (of course it suffices for the witness to be unique as well). Then there is one and only one x such that $\phi(x)$, it is regular, and it is also the unique x such that $\phi^*(x)$.

PROOF. The set $X = \{y \mid (\exists x.\phi(x) \wedge y \in x)\}$ exists by the comprehension scheme, and its ϵ-extension is the same as the $\in$-extension of any x such that $\phi(x)$. By our extensionality axiom, it follows that X is regular and is the unique x such that $\phi(x)$. By the Duality Lemma, there is one and only one x such that $\phi^*(x)$, and by comprehension the $\in$-extension of X is the same as the ϵ-extension of the unique x such that $\phi^*(x)$, so they are the same object.

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3. Interpreting bounded *ZF* in *DEST*

The claim of Kisielewicz in [4] is that the class of "hereditarily regular sets" supports all mathematical constructions of *ZF*. We will reproduce his development (with some modifications) and point out why we get an interpretation of bounded *ZF* rather than full *ZF*.

First, we need to verify that the set of hereditarily regular sets is actually definable. We will further refine the definition so that our interpreted theory will satisfy Foundation.

DEFINITION 3.1. We say that a set A is *transitive* iff $(\forall x \in A.\forall y \in x.y \in A)$. We abbreviate this as $\mathtt{Trans}(A)$ and observe that it is a uniform formula.

DEFINITION 3.2. We define $A \subseteq B$ as $(\forall x.x \in A \rightarrow x \in B)$.

DEFINITION 3.3. Let A be a regular set. Define $\mathtt{TC}(A)$ (called the *transitive closure* of A) as the intersection of all transitive sets which contain A as a subset: $\mathtt{TC}(A) = \{x \mid (\forall B.A \subseteq B \wedge \mathtt{Trans}(B) \rightarrow x \in B)\}$.

DEFINITION 3.4. A set A is *hereditarily regular* iff A is regular and $\mathtt{TC}(A)$ has regular elements (i.e., all members of $\mathtt{TC}(A)$ in either sense are regular). Note that this implies immediately that A itself is regular and also that $\mathtt{TC}(A)$ itself is regular (but it is not equivalent to either of these assertions!)

It is useful to observe that not all regular sets are hereditarily regular. $V = \{x \mid x = x\}$, the universal set, has all sets as its elements in both senses, and so is regular. But V is not hereditarily regular, because $\mathtt{TC}(V) = V$ has the irregular set R as an element (notice that we have an example of a set A such that A is regular, $\mathtt{TC}(A)$ is regular, but A is not hereditarily regular).

In addition, we wish to stipulate that sets of the interpreted *ZF* should be well-founded.

Definition 3.5. Define $x < y$ as $(\forall z.(y \subseteq z \wedge \texttt{Trans}(z)) \rightarrow x \in z)$. If y happens to be regular, $x < y \equiv x \in \texttt{TC}(y)$. If y is transitive, $x < y \equiv x \in y$. It is easy to see that $<$ is a transitive relation.

Definition 3.6. We say that a transitive set A is *well-founded* iff $(\forall C.(\forall x.(x \in A \wedge x \in C \wedge (\exists y < x.y \in C)) \rightarrow (\exists z < x.z \in C \wedge (\forall w < z.z \notin C))))$. We abbreviate this $\texttt{wf}(x)$. In English, this asserts that if a set C meets A at an element x, then either x itself is a $<$-minimal element of C or some $z < x$ (which will also be an element of A because A is transitive) is a $<$-minimal element of C.

Observation 3.7. The form of the definition makes it obvious that a transitive subset of a well-founded transitive set is also well-founded transitive.

Definition 3.8. We call a general set A a *well-founded set* just in case it is a subset of a well-founded transitive set. The preceding observation ensures that there will be no conflict between the definitions of well-foundedness for transitive sets and general sets.

Lemma 3.9. *(Wellfoundedness Lemma) A hereditarily regular set all of whose elements are well-founded is well-founded.*

Proof. Let A be hereditarily regular and let all elements of A be well-founded. We aim to show that $\texttt{TC}(A)$ is well-founded transitive, which is sufficient to show that A is well-founded. Let x be an element of $\texttt{TC}(A)$ which $\in$-belongs to a set C but is not $<$-minimal in C. If x belongs to any element y of A, well-foundedness of y and so of $\texttt{TC}(y)$ implies that there is a $<$-minimal element of C in $\texttt{TC}(y)$ and so in $\texttt{TC}(A)$. If x does not belong to the transitive closure of any element of A, it must itself be an element of A (for A hereditarily regular, we can define the set consisting of all elements of A and all elements of $\texttt{TC}(y)$'s for $y \in A$ (regularity of the sets $\texttt{TC}(y)$ for $y \in A$ allows us to define this using a uniform formula): this set is clearly transitive and contains A). There is $y < x$ which belongs to C, and by well-foundedness of $\texttt{TC}(x)$ there is $z < y < x$ which is $<$-minimal in C, completing the proof of the lemma.

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The intention is that the set of well-founded hereditarily regular sets should model *ZF*.

We review the status of the axioms of *ZF*, following Kisielewicz in [4] for the most part. Our treatment of comprehension and replacement points out the technical error, and Kisielewicz's definition of the natural numbers and proof of Infinity in the interpreted *ZF* is not used because it is difficult to prove that the set as he defines it is well-founded: we prove the interpretation of Infinity as part of our development of the properties of ordinals below (we prove that there is a regular limit ordinal).

Pairing: If a and b are well-founded hereditarily regular sets, then $\{a, b\}$ exists by comprehension and is regular. $\texttt{TC}(\{a, b\})$ is the intersection of all sets which contain $\{a, b\}$ as a $\in$-subset (i.e., contain both a and b as elements) and are transitive. It is straightforward to show that a set belongs to the transitive closure (in either sense) of the pair iff it is equal to a, equal to b, or belongs to the transitive closure (in the same sense) of one of these two sets: the collection of things with these properties is transitive and a superset of $\{a, b\}$, so contains all elements of the transitive closure of the pair, and any element of this set clearly must belong to any transitive set which contains both a and b as elements. So the pair is also hereditarily regular. The pair is well-founded because it is hereditarily regular and all its elements are well-founded.

Union: if A is a well-founded hereditarily regular set,

$$\bigcup A = \{x \mid (\exists y.x \in y \wedge y \in A)\}$$

exists by comprehension. It is regular because all elements of elements of a (in either sense) belong to its transitive closure (in that same sense) and so are regular. It is hereditarily regular, because the transitive closure of the union of A is clearly included in the transitive closure of A: if one belongs to any transitive set which contains all elements of elements of A, one clearly belongs to any transitive set which contains all elements of A. Since the transitive closure of the union is included in the transitive closure of A, all elements of the transitive closure of the union are regular. It is also the case that the transitive closure of the union is well-founded, from which it follows that the union is well-founded.

Power Set: if A is a well-founded hereditarily regular set,

$$\mathcal{P}(A) = \{x \mid (\forall y.y \in x \rightarrow y \in A)\}$$

exists by comprehension. Any element of the power set (in either sense) is partially contained in a set with regular elements, so is regular, and so the power set itself is regular. The union of the power set of A and the transitive closure of A is a transitive set which includes the power set of A as a subset, so includes (in fact is equal to) the transitive closure of the power set. This is all that is needed to see that all elements of the transitive closure of the power set are regular, so it is hereditarily regular. It is also easy to see that the transitive closure of the power set is well-founded. If a set C meets the transitive closure of the power set at an element x of the transitive closure of A, we know that there is a $<$-minimal element of C in the transitive closure of x and so in the transitive closure of the power set. If a set C meets the transitive closure of the power set at an element x of the power set of A, then either x is $<$-minimal in C itself or there is $y < x$ in C which belongs to the well-founded transitive closure of A, so that it either is or has as an element of its transitive closure a $<$-minimal element of C. Thus the power set is well-founded.

"Comprehension": For any uniform formula ϕ and well-founded hereditarily regular set A, the set $\{x \in A \mid \phi\}$ exists by comprehension. Any element of the transitive closure of this set also belongs to the transitive closure of A and so is regular (and the transitive closure of this set inherits the well-foundedness of the transitive closure of A). So this set is well-founded and hereditarily regular. But this is not really the comprehension axiom of the interpreted *ZF*: quantification over the universe of the interpreted *ZF* would involve reference to the predicate "hereditarily regular", which is not uniform and so cannot occur in ϕ. Bounded quantification (in which every quantifier is restricted to a specific hereditarily regular set) will be successfully interpreted in this way.

"Replacement": For any uniform formula ϕ and well-founded hereditarily regular set A, suppose we know that for each x in A there is exactly one well-founded hereditarily regular y such that $\phi(x, y)$. Under these conditions $\phi(x, y) \equiv \phi^*(x, y)$ for each y and for $x \in A$ by the Duality Lemma above. It follows that $\{y \mid (\exists x \in A.\phi(x, y))\}$ is hereditarily regular. It is well-founded because a hereditarily regular set all of whose elements are well-founded is well-founded. Quantification over the universe of the interpreted *ZF* in ϕ is not supported because the formula ϕ would not be uniform.

Infinity: The proof of Infinity is deferred: it falls out as a byproduct of our analysis of the ordinals below. The treatment of Kisielewicz in [4] would not be satisfactory here because there appears to be no easy way to prove that the set he defines there is well-founded.

Other Axioms: Extensionality obviously holds. Foundation holds because we consider well-founded hereditarily regular sets (this was not done in [4]).

We have shown, following Kisielewicz's argument from [4], that the restricted version of *ZF* − Infinity in which the comprehension and replacement schemes are restricted to Δ_0 formulas is interpretable in the well-founded hereditarily regular sets of *DEST*.

4. The Plan to Interpret *ZF*

Further work allows us to show that in fact double extension set theory does interpret *ZF*. The result to be proved which implies this is the following:

Claim 4.1. One of the extensions of the set of all ordinals contains exactly the regular ordinals.

This presupposes a definition of the ordinals, which will be given explicitly in the next section, but all we need to know about this definition at this point is that the regular ordinals will coincide with the ordinals of the interpretation of bounded *ZF* − Infinity given in the previous section.

We can quantify over all ordinals because the ordinals will be defined by a uniform formula: in one sense this will amount to quantification over the regular ordinals, and in the other sense it will amount to quantification over a larger domain. The fact that the quantification is equivocal is harmless. Let ϕ be a formula involving quantification over the set of ordinals, to be used in an instance of separation or replacement. Suppose that the interpretation of ϕ involves quantification over all regular ordinals, while the interpretation of ϕ^* involves the other extension of the set of ordinals. If A is well-founded and hereditarily regular, it is sufficient to know that $\{x \in A \mid \phi\}$ merely exists to know that it is regular (and indeed well-founded and hereditarily regular) so the x in A for which ϕ is true are the same as the x in A for which ϕ^* is true. This justifies separation for conditions containing unbounded quantifiers over the ordinals.

The proof which follows of replacement for formulas which contain unbounded quantifiers over the ordinals first demonstrates a different scheme which we call "Axiom R", which we show to be equivalent to Replacement.

If $\{y \mid \phi(x, y)\}$ is well-founded and hereditarily regular for each x in a well-founded and hereditarily regular set A, that is sufficient for $\phi(x, y)$ and

$\phi^*(x, y)$ to be equivalent whenever $x \in A$, and this means that the extension of $\{y \mid (\exists x \in A \mid \phi(x, y))\}$ is unequivocally defined, and so regular. Since all of its elements are well-founded and hereditarily regular, it is well-founded and hereditarily regular as well.

This shows that the following axiom scheme holds in the interpreted set theory (to be adjoined to Zermelo, not to double extension set theory): if $\phi(x, y)$ is a formula which may contain unbounded quantifiers over the ordinals, and we have for each $x \in A$ that $\{y \mid \phi(x, y)\}$ is a set, it follows that $\{y \mid (\exists x \in A.\phi(x, y))\}$ is a set. Call this scheme "Axiom R".

Axiom R implies replacement (in the presence of the other axioms of Zermelo set theory). Suppose that for each $x \in A$ there is exactly one set y such that $\phi(x, y)$. By the axiom of pairing, $\{y \mid \phi(x, y)\}$ is a set for each $x \in A$. By axiom R, $\{y \mid (\exists x \in A.\phi(x, y)\}$ is a set.

Replacement implies axiom R (in the presence of the other axioms of Zermelo set theory). Suppose that for each $x \in A$, $\{y \mid \phi(x, y)\}$ is a set. Then for each $x \in A$ there is exactly one w such that $(\forall y.y \in w \equiv \phi(x, y))$ Then by replacement the set $\{w|(\exists x \in A.(\forall y.y \in w \equiv \phi(x, y)))\}$ exists, and the union of this set is $\{y \mid (\exists x \in A.\phi(x, y))\}$.

If we could quantify over the regular ordinals in instances of separation and replacement, we claim that we can quantify over all sets of the interpreted *ZF*, by replacing any unbounded quantifier $(\forall x.\phi)$ with $(\forall \alpha.(\forall x \in V_\alpha.\phi))$ (and similarly for existential quantifiers). This does require that we prove something: we prove below that the hereditarily regular well-founded sets are exactly those sets which belong to V_α for some regular ordinal α (Olivier Esser pointed out the need to prove this, and simplified our original proof of this result).

So it remains to prove the theorem about ordinals, prove Infinity, and prove that the sets of the interpreted *ZF* are exactly the elements of regular-indexed ranks to complete the proof that *ZF* is interpretable in the well-founded hereditarily regular sets of *DEST*.

5. The Structure of the Ordinals in *DEST*

We recall from above that we define $\texttt{Trans}(x)$ as the $\in$-formula

$$(\forall y \in x.\forall z \in y.z \in x).$$

We define $\texttt{Trichotomy}(x)$ as the $\in$-formula asserting that for any $y, z \in x$, exactly one of $y \in z$, $z \in y$, and $y = z$ is true.

We define S as

$$\{x \mid (\exists y.(\forall z.z \in y \equiv z = x))\}.$$

This is the set of all sets which have $\in$-singletons; the Singleton Lemma from above is equivalent to the assertion that a set x is regular iff $x \in S \wedge x \,\epsilon\, S$.

We recall the definition of $\texttt{wf}(A)$ as $(\forall C.(\forall x.(x \in A \wedge x \in C \wedge (\exists y < x.y \in C)) \rightarrow (\exists z < x.(z \in C \wedge \forall w < z.w \notin C))))$. Under the hypothesis that A is transitive, this expresses the notion that A is well-founded. Note that if A is transitive and all elements of A are transitive (as will be the case with ordinals) all occurrences of $<$ can be replaced with $\in$, since these relations are equivalent on transitive sets.

We define $\texttt{Ord}$, the set of ordinals, as

$$\{\alpha \mid (\forall \beta \in \alpha.\beta \,\epsilon\, S) \wedge \texttt{Trans}(\alpha) \wedge (\forall \beta \in \alpha.\texttt{Trans}(\beta)) \wedge \texttt{Trichotomy}(\alpha) \wedge \texttt{wf}(\alpha)\}.$$

We refer to ϵ-elements of $\texttt{Ord}$ as $\in$-ordinals and to $\in$-elements of $\texttt{Ord}$ as ϵ-ordinals. The motivation of the definition is that we intend all elements of an ordinal to have singletons and we intend the ordinal to be transitive and strictly well-ordered by membership.

The main result of this section is the following

THEOREM 5.1. *One of the extensions of the set of ordinals consists exactly of the regular ordinals. The other extension properly includes the collection of regular ordinals.*

PROOF. It is straightforward to check that any $\in$-transitive $\in$-subset of an $\in$-ordinal is an ordinal; it follows that any $\in$-element of an $\in$-ordinal is an $\in$-ordinal as well. Further, no $\in$-ordinal can be an $\in$-element of itself (because it is *strictly* well ordered by $\in$).

We insert the check: let B be a $\in$-transitive subset of the $\in$-ordinal α. Certainly all elements of B have $\in$-singletons. B is $\in$-transitive by hypothesis. All elements of B are elements of α, and so are $\in$-transitive. Any $\in$-subset of α is $\in$-trichotomous, so B is $\in$-trichotomous. B is well-founded by the Wellfoundedness Lemma. So any $\in$-transitive $\in$-subset of an $\in$-ordinal is an ordinal, from which it follows immediately that any $\in$-element of a $\in$-ordinal is a $\in$-ordinal.

A "regular ordinal" is defined as a regular set which is a $\in$-ordinal. Observe that if α is regular and a $\in$-ordinal, then it is also a ϵ-ordinal by the Duality Lemma.

An element of a regular ordinal is a regular ordinal: a regular ordinal has all elements having singletons in both senses, since it is both a $\in$- and an

ϵ-ordinal, and this is sufficient for all its elements to be regular. In fact, this implies that a regular ordinal is a well-founded hereditarily regular set (it is well-founded transitive and all its elements are regular). This further implies that regular ordinals have all properties of ordinals provable in bounded ZF without Infinity, because they are actually the ordinals of our interpreted bounded ZF without Infinity. They are sets of the interpreted bounded ZF $-$ Infinity because they are hereditarily regular, and it is obvious that any ordinal in the sense of $DEST$ which is a set of the interpretation is also an ordinal in the sense of ZF.

Any irregular $\in$-ordinal has all regular ordinals as $\in$-elements. Suppose otherwise: suppose β irregular does not contain α regular as an $\in$-element. β must meet the complement of α, because any subset of a hereditarily regular set is regular. The complement of α is a set, because α is regular. So there must be an element γ of β which belongs to the complement of α but which contains only elements of α, by well-foundedness of β. This is only possible if $\gamma = \alpha$, by standard reasoning about ordinals (γ, being a subset of α, must be regular). This is a contradiction.

It cannot be the case that all ordinals are regular in both senses, for the Burali-Forti paradox would follow: in this case `Ord` would itself be an ordinal (this would be proved by standard methods, because the regular ordinals satisfy all familiar properties of ordinals), and so self-membered, which is impossible because ordinals are strictly well-ordered by membership.

Without loss of generality we suppose that there is an irregular $\in$-ordinal, which we call α. If α has no irregular $\in$-element, then we are done: α will have as its $\in$-elements exactly the regular ordinals. So suppose that α has an irregular $\in$-element. Observe that the elements of α which are regular are exactly those which $\in$-belong to S: every $\in$-element of α has a $\in$-singleton because α is a $\in$-ordinal: an element of α will be regular iff it also has a ϵ-singleton, which obtains exactly if it $\in$-belongs to S. α meets S^c (because it has an irregular $\in$-element). S^c is a set, so there is an element β of α which belongs to S^c and contains only elements of S, by well-foundness of α. This β must have as its $\in$-elements all the regular ordinals (because it is irregular) and only the regular ordinals (because all its elements are regular). This completes the proof of the main theorem.

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We conclude further that if there is an irregular $\in$-ordinal there cannot be an irregular ϵ-ordinal. If there were, we would have an ϵ-ordinal β^* whose elements are exactly the regular ordinals, by the dual of the argument of the previous paragraph. We would have $\beta = \beta^*$ by mixed extensionality, and so we would have a set consisting exactly of the regular ordinals in both senses. But the Burali-Forti paradox could then readily be derived: standard reasoning about the regular ordinals would show that β was also a regular ordinal and so was self-membered.

Further, observe that all regular ordinals (in the case considered so far in this paper all ϵ-ordinals) are the only sets with their ϵ-extensions, so, by the Duality Lemma, all $\in$-ordinals are the only sets with their $\in$-extensions. This has the amusing consequence that the $\in$-ordinal with $\in$-extension containing exactly the regular ordinals is in fact the same object as the set `Ord` whose $\in$-members are the ϵ-ordinals, so $\texttt{Ord} \, \epsilon \, \texttt{Ord}$ (`Ord` is an $\in$-ordinal) is a theorem.

We show that `Ord` is a limit $\in$-ordinal, from which it follows by duality that there is a limit ϵ-ordinal, so, equivalently, a regular limit ordinal. It follows that Infinity holds in the interpreted *ZF*. This is easy: if $\alpha^+ = \texttt{Ord}$, then $\alpha < \texttt{Ord}$ would be a regular ordinal, but of course then α^+ would also be a regular ordinal (we can define α^+ as $\{\beta \mid \beta \in \alpha \vee \beta = \alpha\}$, and this is clearly regular since α is regular), and we know that while `Ord` is an ordinal, it is not a regular ordinal.

To complete the proof that *DEST* interprets *ZF*, we need to prove one more theorem.

Theorem 5.2. *The sets of the bounded* ZF *are exactly those sets which belong to* V_α *(suitably defined) for some regular ordinal* α.

Proof. We describe an approach to definitions by transfinite recursion on the regular ordinals.

Let $\phi(x, y)$ be any uniform formula with any free variables other than x and y understood to have regular values. For any regular ordinal α, we define $F^\alpha_\phi(z)$ to be the formula which asserts that z is a function (a $\in$-extension consisting of $\in$-Kuratowski pairs with the usual properties: a notion definable entirely in terms of $\in$) with domain $\alpha+1$ and the property that $\phi(z\lceil\beta, z(\beta))$ for each $\beta \leq \alpha$.

If there is a unique z such that $F^\alpha_\phi(z)$, it is a regular set by the Definability Lemma, which is also the unique z such that $F^{\alpha*}_\phi(z)$. Moreover, $z(\beta)$ will be regular for each $\beta \leq \alpha$ (and the same object as $z(\beta)$ understood in the dual sense) for the same reason: it is easy to see that $z(\beta)$ is uniquely described by a uniform formula with regular parameters (note that α and β are regular).

Note further that if $F^\alpha_\phi(z)$ is not true, the set of $\beta \leq \alpha$ such that $\neg F^\beta_\phi(z)$ exists by comprehension, is nonempty, and so has a least element. This means that (with care) we can use this machinery to carry out proofs by transfinite induction.

Now we specify the formula $\phi(x, y)$ that we will use: this formula asserts that x is a function with domain an ordinal β and y is the union of the power sets of the elements of the range of x, and that each element of y has a singleton. ("power set of x" here means a unique object whose $\in$-extension consists of all $\in$-subsets of x; if some required power set fails to exist, the formula fails to be true.) It should be clear that in the usual set theory if z witnesses $F^{\alpha}_{\phi}(z)$, then $z(\alpha)$ will be V_α: we define V_α in this way (this definition was considerably simpified by Olivier Esser).

By considerations above V_α is regular for each ordinal α for which it exists, and if there is a regular ordinal α for which V_α does not exist, there is a least such ordinal. Further, if V_α exists, we see that each element of V_α is regular, because each element of V_α has a singleton in both senses. Suppose that β is the least regular ordinal such that V_β does not exist. If β is a successor $\gamma + 1$, we see that V_γ, a regular set with regular members, has a uniquely determined regular power set, which we can use as a value to extend the function z such that $F^{\gamma}_{\phi}(z)$ to a function z such that $F^{\beta}_{\phi}(z)$, contradicting the choice of β. If β is limit, then the union of the V_γ's for $\gamma < \beta$ will be definable, regular, and suitable to serve as the value at β of a function z witnessing $F^{\beta}_{\phi}(z)$, which it is then straightforward to construct.

We can also prove by a similar transfinite induction argument that each V_β is transitive and so hereditarily regular, and it is easy to prove that each V_β is well-founded, for quite standard reasons.

It remains to prove that every hereditarily regular well-founded set belongs to some V_α for α regular. We carry out part of our argument in bounded *ZF*. Note that in bounded *ZF* as interpreted in *DEST*, each set has a transitive closure (because the transitive closure of a hereditarily regular well-founded set is also a hereditarily regular well-founded set). This means that membership restricted to the transitive closure of any element of our bounded *ZF* defines a set relation. This relation will be well-founded and will have an ordinal rank in the usual sense. In bounded *ZF*, we cannot prove that this ordinal rank can be implemented as a von Neumann ordinal, but we can prove that it can be implemented as a Scott ordinal (a Scott ordinal is the equivalence class of all well-orderings of minimal rank similar to a given well-ordering).

It remains to prove that every Scott ordinal corresponds to a von Neumann ordinal. If there is a Scott ordinal which does not correspond to a von Neumann ordinal, there is a first one. Choose a well-ordering W belonging to the first bad von Neumann ordinal. In the ambient *DEST*, construct the map sending each element of W to the von Neumann ordinal with the same type as the segment below that element in W. This map is definable as a set abstract because it is defined by a uniform formula with a regular parameter. It must be regular, because it is clear that the value of the function at each element of W must be the same in either sense. But this means that it is hereditarily regular and well-founded, because every element of the set is clearly in the interpreted bounded ZF. This would further imply that its range, the set of all regular ordinals, is regular, which is absurd.

■

This completes the proof that $DEST$ interprets ZF.

It might seem that one could show that `Ord` is inaccessible, and so by duality that there is an inaccessible regular ordinal, but this appears not to be the case. One can show `Ord` to be strong limit, but there is no obvious way to show that it is regular: there can't be a cofinal subset of `Ord` which is shorter than `Ord` and which is definable by a formula with parameters taken from `Ord` (because such a set can be shown to be regular, by using its regular length as an additional parameter, and would have as its union the irregular `Ord`) but there is no obvious obstruction to the existence of such a cofinal subset which is not so definable. This situation is closely analogous to the status of the first proper class ordinal in Ackermann set theory.

The results for the structure of the ordinals have consequences for the structure of the cumulative hierarchy. As we have seen above, there is a notion of ordinal rank definable in terms of a single membership relation. Now consider the set of all sets which belong to some ordinal rank (in the sense defined in terms of $\in$). This set will have as its $\in$-extension the collection of objects which belong to an ordinal rank in the sense of ϵ, which is exactly the universe of our interpreted ZF (the collection of well-founded hereditarily regular sets); its ϵ-extension will contain the objects which have an ordinal rank in the sense of $\in$, which will properly extend the collection of hereditarily regular sets for the same reason that the $\in$-ordinals properly extend the ϵ-ordinals. The rank $V_{\texttt{Ord}}$ (recall that `Ord` is the first irregular $\in$-ordinal) will have as its $\in$-elements exactly the elements of the interpreted ZF as well, and can be shown to be equal to the set of all objects which belong to some ordinal rank in the same way that we showed that the first irregular ordinal is the same object as `Ord` (by showing that all ordinal ranks in the sense of $\in$ are the unique objects with their $\in$-extension, since the dual statement is obviously true).

It is well-known that the following set of axioms is essentially equivalent to ZF (this is important in the study of Ackermann set theory):

Zermelo without infinity: Assume all axioms of Zermelo except infinity.

There is a special rank: A special rank V_κ of the cumulative hierarchy is specified.

The special rank is elementarily equivalent to V: For any sentence ϕ with parameters taken from V_κ, the truth value of ϕ is the same as the truth value of the relativization of ϕ to V_κ.

It is worth noting that this theory is readily interpreted in *DEST*: let V be the collection of all objects belonging to an ordinal rank in the sense of $\in$ and let $\kappa = \mathtt{Ord}$. The elementary equivalence of the interpreted V and V_κ follows from the Duality Lemma. We know that the interpreted V_κ (the universe of hereditarily regular sets) satisfies Zermelo without infinity, so we know that V satisfies Zermelo without infinity by the elementary equivalence already established. Olivier Esser encouraged us to include this observation; it was already clear to us that there was some analogy between the interpretation of *ZF* in *DEST* and the interpretation of *ZF* in Ackermann set theory (re my comments about Ackermann set theory and its relation to reflection properties of *ZF* just described, see [1] and [5]).

The results of this paper allow us to draw a distinction between the membership relations (the symmetry between them is broken). We can stipulate, for example, as we did in this paper, that $\in$ is the membership relation such that there are irregular $\in$-ordinals, so $x \in \mathtt{Ord}$ means "x is a regular ordinal", or we could adopt the opposite convention.

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