There is a Forster term model of simple type theory

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Simple type theory (*TST*) is taken here to be the typed theory of sets with types indexed by the natural numbers, atomic formulas of the forms $x^n = y^n$ and $x^n \in y^{n+1}$, and with axioms of extensionality (objects of any positive type with the same elements are equal), comprehension $((\exists A^{n+1}.(\forall x^n.x^n \in A^{n+1} \leftrightarrow \phi))$ for any formula ϕ in which A is not free), and infinity (a precise formulation of the axiom of infinity will be given below).

We augment the language of *TST* with set abstracts $\{x^n \mid \phi\}$ for each formula ϕ (recursively including the new formulas with set abstracts). Of course, $\{x^n \mid \phi\}$ is taken to denote the unique A^{n+1} such that $(\forall x^n.x^n \in A^{n+1} \leftrightarrow \phi)$.

A *Forster term model* of a set theory is a model all of whose elements are of the form $\{x \mid \phi\}$, where $(\exists A.(\forall x.x \in A \leftrightarrow \phi))$ is a comprehension axiom of the theory, and ϕ is a formula in the language of the theory augmented with set abstracts which contains no free variables other than x. We call these "Forster term models" because Thomas Forster has considered this kind of model for certain weak theories, as for example in [2]. The existence of a Forster term model of a theory is not a trivial question to decide. It is reasonably easy, for example, to present a consistent set theory (a fragment of *NFU*) such that any Forster term model of the theory would be a model of *NF* (details not given here). In the case of type theory, we must modify the definition of "Forster term model" because type 0 is not inhabited by sets: we add type 0 constants a_i indexed by natural numbers to our language and require that the a_i's have distinct referents, that every type 0 term be the referent of an a_i and that all objects of positive type be reference of parameter-free set abstracts $\{x \mid \phi\}$ as above (these may contain the type 0 constants as subterms).

We show that TST has a Forster term model. Robert Solovay has already shown this (unpublished communication); our argument here is independent of his in its details though it uses much the same technology and the general approach was inspired by his arguments. We claim no originality for the argument either, but certainly any errors in it have nothing to do with Solovay's unpublished work. A structure very similar to the structure which we show to be a Forster term model was described by us (entirely independently of the conversation with Solovay) in 1998 during our visit to Thomas Forster in Cambridge. However at that time we were quite certain that this structure was *not* a Forster term model of TST: it was only the conversation with Solovay which convinced us that (when slightly(?) modified) it is one. We believe that it is the same structure which Solovay showed to be a Forster term model, though presented differently: it is the minimal model of TST with a fixed countably infinite set of type 0 objects.

The description of the structure is much easier than the proof that it is a Forster term model.

A set abstract $\{x^n \mid \phi\}$ is said to be *predicative* iff no variable bound in the abstract is of type higher than n and no free variable in the abstract is of type higher than $n+1$. Notice that the type of a set abstract $\{y^m \mid \psi\}$ appearing in a predicative $\{x^n \mid \phi\}$ can be no higher than $n+1$ since the type of the bound variable y^m is constrained to be no more than n. This notion of predicativity, originally going back to Russell, is described for this kind of type theory by Marcel Crabbé in [1], for example.

It is useful to observe that the referent of any set abstract at all is an iterated set union of the referent of some predicative set abstract: if ι is the singleton operation, $\bigcup^m\{\iota^m(x^n) \mid \phi\}$ is of course $\{x^n \mid \phi\}$, and if m is taken to be large enough, $\{\iota^m(x^n) \mid \phi\}$ (written a bit more carefully) will be a predicative set abstract. This is noted in [1].

The union of $\{x^{n+1} \mid \phi\}$ is $\{y^n \mid (\exists x^{n+1}.y^n \in x^{n+1} \wedge \phi)\}$. The singleton image of $\{x^n \mid \phi\}$ is $\{y^{n+1} \mid (\exists x^n.\phi \wedge x^n \in y^{n+1} \wedge (\forall z^n.z^n \in y^{n+1} \rightarrow z^n = x^n))\}$, in the event that anyone doubts that there are legitimate operations on set abstracts.

A *structure for the language of* TST is a sequence τ_i of sets and relations $\in_{i+1} \subseteq \tau_i \times \tau_{i+1}$, with the property that the preimage under $\in_{i+1}$ of an element of τ_{i+1} uniquely determines the element. The sets τ_i are candidate types, the relations $\in_{i+1}$ are candidate membership relations, and the condition ensures that the membership relation is extensional.

The notion of satisfaction of a formula of the language of TST is defined in the obvious way in any structure for the language of TST. The axioms of TST (other than extensionality) are not necessarily satisfied in an arbitrary structure for the language of TST.

In all structures for the language of TST which we will consider, the elements of the τ_i's will be terms of a language we will describe shortly and in particular τ_0 will be a fixed countably infinite set of constants $\{a_i \mid i \in \mathbb{N}\}$.

We describe a formal language L_2, parameter-free terms in which will be candidates for membership in the structures we describe. We suppose that the definition of an ordinal used is appropriate to the context (the definition of ordinal number in TST is different from the usual definition in set theory, and we will work inside TST eventually).

For any natural number i, a_i is a term of L_2, of type 0. Of course variables of type n are terms of L_2 of type n.

Any formula of the language of TST not containing set abstracts is a formula of L_2. An additional unary predicate W is added: the intended meaning of $W(x)$ is "x is a well-ordering". The addition of W to the formal language L_2 is the additional refinement not found in the construction we described in 1998; we do not know whether this addition to the language is actually necessary.

Any notation $\{x^n \mid \phi\}^\alpha$, where α is an ordinal and ϕ is a formula of L_2 in which no term of type higher than $n+1$, no bound variable of type higher than n, no free variable other than x, and no ordinal superscript greater than α appears, is a term of L_2 of type $n+1$. Note that all set abstracts of L_2 are parameter-free.

Any notation $\bigcup t$, where t is a term of L_2 of type $n \geq 2$, is a term of L_2 of type $n-1$. We use the notation $\bigcup^i t$ to abbreviate the result of applying $\bigcup$ i times.

Any formula obtained by substituting a term of L_2 for a free variable in a formula of L_2 is a formula of the new formal language.

L_2 is the smallest class of formulas and terms satisfying these conditions.

We have described all of these objects as bits of formal syntax; they can readily be coded as mathematical objects in the usual set theory, but we do not burden ourselves with the details here. We will have something to say about how to code them in the less familiar context of TST below.

We suppose that we have defined a well-ordering of the terms of L_2 under which the a_i's appear before all other terms, terms appear after their proper subterms, terms of higher ordinal index appear after terms of lower ordinal index, and set abstract terms of higher type appear after set abstract terms of lower type (note that set union terms appear after the terms of which they are unions, which are of higher type). This is clearly possible and the details are not important to our purpose.

We construct an ordinal-indexed sequence Σ of structures for the language of TST. We denote τ_i in the structure Σ_α by $\tau_{i,\alpha}$ and $\in_{i+1}$ in Σ_α by $\in_{i+1,\alpha}$.

We set up the basis of the construction. $\tau_{0,\alpha}$ is the set of a_i's in every Σ_α. $\tau_{i+1,0}$ is empty for every i.

Suppose that Σ_β has been defined for each $\beta < \alpha$, and that each Σ_β is inhabited by terms of L_2 with ordinal index $\leq \beta$.

We define a structure $\Sigma_{\alpha,1}$ which is simply the type-wise union of the Σ_β's for $\beta < \alpha$: $\tau_{n,\alpha,1}$ is defined as the union of all $\tau_{n,\beta}$ for $\beta < \alpha$, and similarly $\in_{n+1,\alpha,1}$ is the union of all $\in_{n+1,\beta,1}$ for $\beta < \alpha$.

We define a second intermediate structure $\Sigma_{\alpha,2}$ which is in effect the closure of $\Sigma_{\alpha,1}$ under the set union operation. The terms in $\tau_{n,\alpha,1}$ will all appear in $\tau_{n,\alpha,2}$ and their preimages under $\in_{n,\alpha,2}$ (if $n > 0$) will be the same as their preimages under $\in_{n,\alpha,1}$. The additional terms in $\tau_{n,\alpha,2}$ (for each $n > 1$) will be selected, and their preimages under $\in_{n,\alpha,2}$ will be defined, by recursion along the order on terms. Each term added to $\tau_{n,\alpha,2}$ will be of the form $\bigcup^i t$ where t is a term of $\tau_{n+i,\alpha,1}$. Each such term will be assigned a referent in any case and will be added to $\tau_{n,\alpha,2}$ (and regarded as its own referent) just in case no earlier term has been assigned the appropriate preimage under $\in_{n,\alpha,2}$ (the union of all preimages under $\in_{n,\alpha,2}$ of elements of the preimage under $\in_{n+1,\alpha,2}$ of the referent already assigned to $\bigcup^{i-1} t$); if an earlier term has already been assigned this preimage we regard the earlier term as the referent of $\bigcup^i t$ and do not add the latter term to the type; otherwise we add the term $\bigcup^i t$ to the type and assign it the appropriate preimage.

Finally, we define Σ_α as (in effect) the "predicative closure" of $\Sigma_{\alpha,1}$. The terms in $\tau_{n,\alpha,2}$ will all appear in $\tau_{n,\alpha}$ and their preimages under $\in_{n,\alpha}$ (if $n > 0$) will be the same as their preimages under $\in_{n,\alpha,2}$. The additional terms in $\tau_{n,\alpha}$ (for each $n > 1$) will be selected, and their preimages under $\in_{n,\alpha}$ will be defined, by recursion along the order on terms. Each term added to $\tau_{n+1,\alpha}$ will be a predicative set abstract of the form $\{x^n \mid \phi\}^\alpha$. Each such term will be assigned a referent in any case and will be added to $\tau_{n+1,\alpha}$ just in case no earlier term has been assigned the appropriate preimage (the set of all x in type n of Σ_α such that ϕ is satisfied for this value of x in Σ_α: this is definable without circularity because all terms of type lower than x with ordinal index $\leq \alpha$ and all terms of the same type as x which appear as parameters in ϕ

appear earlier in the order on terms than $\{x^n \mid \phi\}^\alpha$ and so have already had their referents and the preimages of their referents defined: since ϕ is predicative this means that all constants appearing in ϕ have been assigned referents and extensions and all elements of the domains of any quantifiers in ϕ have already been assigned referents and extensions); if an earlier term has already been assigned this preimage we regard the earlier term as the referent of $\{x^n \mid \phi\}^\alpha$ and do not add the latter term to the type; otherwise we add the term $\{x^n \mid \phi\}^\alpha$ to the type and assign it the appropriate preimage.

The structure Σ_α will be a model of the version of TST in which comprehension is restricted to providing the existence of predicative $\{x^n \mid \phi\}$'s. We reiterate the point that when a term $\{x^n \mid \phi\}^\alpha$ is being considered for addition to the structure, all lower type abstracts to be added to Σ_α have already been added (and so the domains of any permitted quantified variables have been fully constructed) and all parameters in ϕ have already been added, by consideration of the properties of the order on terms.

If Σ_α is closed under set unions, it will be a model of TST. The reason for this is that every set abstract of TST can be expressed as a possibly iterated set union of a predicatively defined set, so predicative comprehension combined with set union gives the full comprehension axiom of TST.

If this construction is carried out in the usual set theory ZFC, it must terminate. The cardinality of $\tau_{i,\alpha}$ is $\leq \beth_i$ for each α (because there is a natural way to associate each element of $\tau_{i,\alpha}$ with an element of the ith iterated power set of the countable set $\tau_{0,\alpha}$), so the cardinality of $\bigcup_i \tau_{i,\alpha} \leq \beth_\omega$. If $\Sigma_\alpha = \Sigma_{\alpha+1}$, then $\Sigma_\alpha = \Sigma_\beta$ for all $\beta > \alpha$. Thus there can be no more than $\beth_\omega$ ordinals α such that new elements are added to the structure at stage α, and the structure must be remain the same at two successive stages and so at all subsequent stages (and so be closed under set unions, and so be a model of TST) at some point before stage $\beth_\omega^+$. We call the ordinal at which the construction stabilizes Ω.

It is useful to note that the predicate W of well-orderedness is definable in terms of equality and membership in the limit model (in exactly the usual way). The effect of including this predicate is to exclude the possibility that there might be apparent well-orderings in Σ_Ω which are not actually well-orderings from an external standpoint.

A subtler approach shows that Ω must actually be *countable*. Briefly, the reason for this is that we can construct a countable model of the theory of Σ_Ω, in which the construction of Σ_α's stabilizes at a countable ordinal γ, and then absoluteness considerations show that the construction of Σ_α's really does stop at the countable stage $\gamma = \Omega$. We explain the details, leaving one major point to subsequent discussion (which does not depend on the countability of Ω, so no circularity ensues).

Augment the language L_2 with two additional notations to obtain a further language L_3. All formulas of L_2 are formulas of L_3. For any formula ϕ of L_3, admit $(\epsilon x^n.\phi)$ as a notation for the lexicographically first object in Σ_Ω such that ϕ is satisfied (or a default object if there is no such object). In addition, add ordinal variables to our notation and admit $(\mu\alpha < \beta.\phi)$ as notation for the first ordinal α such that ϕ, for any formula ϕ of L_3 and ordinal β (and 0 if there is no such ordinal). Note that the μ notation can be used to define bounded quantifiers over ordinals: $(\exists\alpha < \beta.\phi(\alpha))$ is equivalent to $\phi(\mu\alpha < \beta.\phi)$. The notation $\alpha \leq \beta$ should also be admitted. We can then construct a term model all of whose elements can be built as type 0 constants, set abstracts with ordinals defined using the μ notation, and Hilbert symbols. This structure will be countable and will have the same first-order theory in the augmented language as the true Σ_α. Its ordinals are well-ordered (being a subcollection of the true ordinals). Any set defined as a Hilbert symbol nonetheless has an ordinal index: there is a definable ordinal $\texttt{Ult}_n$ such that all type n sets must be constructed before stage $\texttt{Ult}_n$ (we will see the reasons for this below) and the index of a term T^n is the first ordinal $\alpha < \texttt{Ult}_n$ such that $T^n \in \{x^n \mid x^n = x^n\}^\alpha$. Every element defined as a Hilbert symbol can be presented in a form in which no Hilbert symbol appears except inside μ-terms: any term T is actually a term of L_2 of the form $\bigcup^j \{x \mid \phi\}^\alpha$. We have already shown how to express the rank α of T as a μ term. The difficulty which might seem to exist here is that the formula ϕ might contain ordinals which we cannot express as μ-terms. If T is of the form $\{x^n \mid \phi(\alpha_1, ..., \alpha_n)\}^\alpha$, we define an α_1 which will work as $(\mu\alpha_1 \leq \alpha.(\exists\alpha_1 \ldots \alpha_n \leq \alpha.T = \bigcup_j \phi(\alpha_1, ..., \alpha_n)))$: call this α_1^*. Similarly, when we have defined $\alpha_1^*, \ldots, \alpha_i^*$, we define α_{i+1}^* as $(\mu\alpha_{i+1}.\alpha_{i+1} \leq \alpha \wedge (\exists\alpha_{i+2} \ldots \alpha_n \leq \alpha.T = \bigcup^j \phi(\alpha_1^*, \ldots, \alpha_i^*, \alpha_{i+1}, \ldots, \alpha_n)))$. So we see that any parameter-free term can be presented in a form in which any Hilbert symbols appear inside μ-terms. Notice that there is no uniform way to do this expressible internally to the term model: we need to know externally what j and ϕ will work to find these ordinals.

Note that the ordinals of the term model are well-ordered (being a subset of the true ordinals of the original model). We claim that the stages of the term model are actual stages and none are skipped. To verify this, it is sufficient to establish that each stage of the construction in the term model is in terms of the term model itself the predicative closure of the closure under unions of the typewise union of all earlier stages in the term model itself (not in the general model). We have just shown this: every object in the stage indexed by α in the term model (α being an ordinal represented by a μ term) is in fact representable in terms of the term model as either an ordinal-indexed set abstract with index $\leq \alpha$ or the set union of an ordinal-indexed set abstract with index $< \alpha$ or a type 0 constant.

This implies in turn that the stages of the construction in the term model are exactly the same as the stages in the true model.

Finally, we show that the term model construction, and so the true construction, must terminate at a countable ordinal stage. We defer to below the proof that there is an ordinal stage $\mathtt{Ult}_n$ definable in terms of TST by which all type n terms must be constructed. Since the ordinal $\mathtt{Ult}_n$ is an ordinal in the term model of TST, it is countable from an external standpoint (almost all of the ordinals $\mathtt{Ult}_n$ are uncountable ordinals internally to the model of TST). Since the term model construction fills type n after countably many stages, the true construction does so. Now this implies that the term model construction (and so the true construction) terminates at or before the countable ordinal limit of the countable ordinals $\mathtt{Ult}_n$ (this limit is not an ordinal in the model of TST at all; it is too large). We will call the actual ordinal at which the construction terminates Ω. Note that we cannot define Ω in the language L_3 of the term model construction, but we do not need to.

The main claim of this paper is that Σ_Ω is in fact a Forster term model of TST. Note that this is not at all obvious. Every element of Σ_Ω is defined as a set abstract with the additional decoration of an ordinal index. It is not obvious that the ordinal indexed stages Σ_α can be represented in terms internal to TST, nor is it obvious that all the ordinals used in the construction can be represented in terms internal to TST. Both of these non-obvious conjectures turn out to be true, but demonstrating them will require work; moreover, the construction of Σ_α's inside a model of TST will be rather delicate (in the end, we will be constructing Σ_α's for each α strictly less that Ω inside Σ_Ω itself).

We review the mathematical competence of TST. The pair $\langle x, y\rangle$ can be defined for objects x, y of the same type n as $\{\{x\},\{x,y\}\}$ (as in the usual set theory). This implementation of the pair is somewhat unsatisfactory because $\langle x, y\rangle$ is two types higher than x or y. In sufficiently high types (certainly above type 10, say), a type level ordered pair (having the same type as its projections) can be defined (see Quine's [4] for details). Relations and functions can then be defined using ordered pairs as usual.

Two sets A and B of the same type are said to be equinumerous if and only if there is a bijection from A to B. The cardinal number of A, written $|A|$, is the set of all sets which are equinumerous with A. Cardinal numbers are equivalence classes under equinumerousness. The sum of two cardinals κ and λ is the (uniquely determined) cardinality of the union of an element of κ and element of λ which are disjoint from each other. We define 0 as $|\emptyset|$ and 1 as $|\{x\}|$ (this does not depend on the choice of x). The Axiom of Infinity is the assertion that for every cardinal κ, $\kappa + 1$ is defined (we assume this). The finite cardinals (natural numbers) are the cardinals which belong to every set which contains 0 and is closed under addition of 1. Notice that the natural numbers are defined independently in each type $n \geq 2$.

For each x, we define $\iota(x)$ as $\{x\}$, the singleton of x, and $\iota``A$ as $\{\iota(x) \mid x \in A\}$, the singleton image of $A$. For any cardinal $\kappa$, we define $T(\kappa)$ as $|\iota``A|$ for any element A of κ. For any cardinal κ, the cardinal $T(\kappa)$ is "the same" cardinal one type higher. In particular, for any natural number n, $T(n)$ is also a natural number and can reasonably be thought of as the same natural number.

We now consider the construction of Σ_α's inside a model of TST. The definition of formal notations in TST presents no essential difficulties (this is evident since arithmetic and a general type-level ordered pair are definable in TST). The definition of a structure for the language of TST does not present difficulties either, though it should be noted that all the sets τ_i and relations $\in_{i+1}$ will inhabit the same fixed type k in terms of the ambient TST in which one is working. Note that satisfaction of formulas in a structure for the language of TST is routinely definable in TST (in a type higher than the working type in which the structure is given).

The principal obstruction to the construction of Σ_α's in TST is that in each given type k there are fewer ordinals than there are in type $k + 2$. An ordinal of type k is an equivalence class of well-orderings of type $k - 1$ under isomorphism; we suppose here that we use the type level ordered pair to implement well-orderings, so these well-orderings act on type $k - 2$ objects). If we change our working type from k to $k + 2$, the stages up to the largest ordinal in type k will be isomorphic to the corresponding stages in type $k + 2$, but type $k + 2$ may contain further stages (because it definitely does contain further ordinals, such as the order type of the natural order on type k ordinals).

Some models of TST are large enough to contain all of Σ_Ω in a single type (and will see it to be countable internally). However, Σ_Ω itself cannot carry out its own construction fully in any single type, since it would then be possible to define truth in Σ_Ω inside Σ_Ω itself, which is impossible by Tarski's theorem (the definition of truth in the internal Σ_Ω will agree with that in the external Σ_Ω because the well-orderings and so the ordinals of Σ_Ω are true well-orderings).

We show how to associate an actual set $S_t^{i,e}$ of type i with a term $t \in \tau_{i,\alpha}$ (which is itself an object of a working type k in the ambient TST), given the additional parameter e, a finite set of assignments of values in the actual type 0 to the constant terms a_i whose domain includes all a_i's appearing in t (increasing the domain of e will not change the meaning of $S_t^{i,e}$). If $i = 0$, the term t is actually equivalent to a term a_i, and we define $S_t^{i,e}$ as $e(a_i) = e(t)$. If $i = j + 1$, we suppose that we know how to define $S_{u,f}^j$ for any type j term u and suitable environment f, and we define $S_t^{j+1,e}$ as the set of all $S_u^{j,f}$ where f extends e and $u \in_{j+1} t$. In type theoretic terms there is no uniform definition of an operation S: there is a sequence of definitions of operations S^i for each type i. Note that any $S_t^{i,e}$ is the referent of a type 0 term a_i or of a closed set abstract in the language of TST.

It follows from these considerations that any term of L_3 translates into an actual set in any model of TST. Terms of L_2 have their intended reference. Terms of L_3 have correct reference because of the fact that only bounded μ-terms are provided. We will see below that any ordinal index of a stage in the construction does appear as an ordinal in some type of Σ_Ω.

The natural numbers of the model Σ_Ω are standard. This is true because the sets of type 1 in Σ_Ω are exactly the finite and cofinite subsets, so the definable notion of being a type 1 set which embeds into its complement precisely captures "standard finite", and the notion of "$(i-1)$-fold image under the singleton operation of a standard finite set" precisely captures "standard finite set of type i". It is then obvious that the natural numbers (defined as Frege cardinals in each appropriate type) are standard. (This in turn ensures that Quine's definition of the ordered pair in [4] works correctly without any need for annoying technical adjustments) The ordinals of the model Σ_Ω are standard, because the well-orderings in Σ_Ω are true well-orderings. The semantics of the W predicate in the language L_2 ensures that any linear order which is not well-ordered is recognized internally as not well-ordered (the set of all initial segments of an externally non-well-ordered linear order which are true well-orderings will be constructed, and the domain of the linear order minus the unions of the domains of the well-ordered initial segments will be a nonempty subset of the domain with no least element in the order). This implies that the types, formulas and terms in Σ_α's constructed internally to Σ_Ω are precisely isomorphic to true types, formulas and terms, and the construction of $S_t^{i,e}$ will construct the true referent of t.

It cannot be the case that Ω_{n+3} (the ordinal at which type $n+3$ first fills up in the construction) is an ordinal in type n. If this were true we could in type $n+3$ of Σ_Ω define satisfaction for statements about type $n+3$ as represented internally to $\Sigma_{\Omega_{n+3}}$ (as constructed with type $n+1$ as the working type, in which type n ordinals can be used as notations in formulas). But this internally represented type $n+3$ would be the same structure as the type $n+3$ in which satisfaction was being defined, which would violate Tarski's theorem.

It is provable in TST that the construction, as viewed from any working type k sufficiently larger than n, fills type n at or before an definable ordinal stage found in type $n+4$ (4 here is large enough but might be too large). The argument for this is similar to the argument for the condensation lemma in the usual constructible universe (see for example [3], which contains much other useful information about the usual constructible universe L). Consider a language for the theory of TST which contains constants for each element of the first n types. This can be represented internally to TST (in type $n+2$, say). Build a term model made up of terms of this enriched language for the theory of the working type of TST within which the construction is being carried out (however much higher than n this may be taken to be). This can be carried out in say type $k+2$. Stages in the construction of the Σ_α's as represented inin this term model will be isomorphic to true stages of the construction for the usual reasons of absoluteness. Here we can see that all sets in type n ever built in the construction at the given working type k are actually built (because the term model for this language contains names for them) and are built at stages which are order types of well-orderings of type n objects (because all terms in the enriched language can be taken to be (suitably iterated singletons of) type n objects as long as n is large enough that type n supports a type level pair and contains the natural numbers). Define $\texttt{Ult}_n$ as the first ordinal which is not the order type of any well-ordering of (suitably iterated singletons of) type n objects; in types $n+2$ and above this ordinal is defined as a closed set abstract in the language of TST. By absoluteness considerations, the construction of type n terms in the given working type k must actually stop before stage $\texttt{Ult}_n$ (no matter how large k is). Note that higher working types may add more type n terms to the construction (this is not ruled out) but all working types agree that all type n terms are added by the ordinal stage $\texttt{Ult}_n$ of the construction. Note further that $\texttt{Ult}_n$ is definable by a closed set abstract (being definable purely in terms of TST): the availability of $\texttt{Ult}_n$ as a bound on ordinal stages at which type n sets can be added in the construction is the reason that the formal language L_3 does not need unbounded μ-terms.

Now each ordinal $\texttt{Ult}_n$ is a true ordinal from an external standpoint (and so is the possibly smaller Ω_n), from which fact and the considerations in the paragraph above it follows that the (external) ordinals of stages in the construction run through all (internal) ordinals of the model. Moreover, it is clear that the limit of all the (internal) ordinals of the model is the external ordinal Ω, since each type n fills up at a stage Ω_n short of Ω, from which we can see that no further sets could be added at the limit stage. We know that every set of whatever type in the model is defined externally by a term in L_3; this term is also a term internally to the model and at a high enough working type must represent the same set internally that it does externally. We know how to convert internal terms of L_3 to closed set abstracts. So the proof is complete: every element of the model is externally identified with a term t of L_3, which has an internal correlate in the model, which we know how to convert to a set $S_t^{i,e}$ (letting e assign referents to all type 0 objects which are actually involved in t) which is the referent of a closed set abstract (or an a_i if it is of type 0) in the language of TST and which is actually the same set (or type 0 object) we started with: so every set is the referent of a closed set abstract (or an a_i if of type 0) as desired.

A feature of this development not found in the original presentation of this paper at the conference in Cambridge is the addition of the predicate W of true well-ordering to L_2. We do not know whether this is necessary for a proof of the result, but we do not know how to prove that all internal well-orderings of Σ_Ω (as defined without use of the predicate W) are well-orderings from the external standpoint. If this could be proved, then our original 1998 construction would also be shown to produce a Forster term model.

We have two questions. One is indicated in the previous paragraph: can W be eliminated from the construction, and does the original 1998 construction give a Forster term model? The second question has to do with the theory TNT defined by Hao Wang in [5], which differs from TST in having types indexed by all integers, and is easily seen to be consistent by a compactness argument. Does *TNT* have a Forster term model (note that if *TNT* does not, then *NF* certainly does not)? Superficial examination indicates that this is a *much* harder question than the question about *TST* answered here.

References

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