Permutation Methods in *NF* and *NFU*

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Abstract

This paper is a survey of results obtained by using the permutation method for obtaining consistency and independence results relative to "New Foundations" and its known-to-be-consistent variant *NFU*, adapted for this purpose by Scott from the method used by Rieger and Bernays to establish the independence of Foundation from the usual set theory. It gives a complete account of known results about the classes of von Neumann numerals, Zermelo numerals, and hereditarily finite sets in *NF(U)*: these can be sets but do not have to be sets since their definitions are not stratified. The conditions under which these sets can exist are equivalent to conditions related to Rosser's Axiom of Counting. New results include a model with the Axiom of Counting in which the von Neumann numerals do not make up a set and a model of *NFU* in which the Zermelo numerals make up a set, but the set is Dedekind-finite. A general characterization of formulas which are invariant under this permutation method when set permutations are used in *NFU* + Choice is given.

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1 Preliminaries: *NF* and *NFU*

In this section, we define the set theories *NF* and *NFU*.

NF was introduced by Quine in 1937, in [18], as a "simplification" of the type theory *TST* to be described below. *NF* presents difficulties: it is not known to be consistent. It is not known to be inconsistent, either – rumors that *NF* is inconsistent continue to circulate because of an inconsistency found in the first version of Quine's system *ML*; this system adds proper classes to *NF*, and the inconsistency in *ML* was found and (as far as anyone can tell) repaired by Hao Wang; the second edition [19] is corrected (and contains a discussion of the error in the original version), and the system there is known to be consistent if *NF* is consistent. *NF* is known to disprove the Axiom of Choice (Specker, 1953, in [23]).

NFU is a slight (?) modification of *NF* due to R. B. Jensen in [14]. The modification is the weakening of extensionality to allow urelements. The effects are dramatic: the consistency of *NFU* is provable in Peano arithmetic; *NFU* does not disprove Choice, and admits a hierarchy of extensions comparable in strength to the familiar hierarchy of extensions of the standard set theory *ZFC* with strong axioms of infinity (see the second author's [10]).

The motivation for the untyped set theories *NF* and *NFU*, at least historically, lies in a type theory which we now describe. It is worth noting that either of the two set theories can also be described independently of this type theory: they admit quite natural finite axiomatizations in which type-theoretical considerations do not appear (references are given below).

TST (the "simple theory of types") is a multi-sorted first order theory with sorts (called "types") indexed by the natural numbers. The language of *TST* has equality and membership as the only primitive predicates. An equality sentence $x = y$ is well-formed iff the types of the variables x and y are the same; a membership sentence $x \in y$ is well-formed iff the type of y is the successor of the type of x. The motivation is that type 0 is a sort of otherwise unspecified "individuals" (not to be confused with urelements – urelements have no elements, while we cannot meaningfully inquire as to whether individuals have elements), while type 1 consists of sets of individuals, type 2 consists of sets of sets of individuals, and so forth.

The axioms of *TST* are extensionality (if two objects of any positive type have the same elements, then they are equal) and comprehension (for any formula ϕ of the language of *TST* and for each variable x, $\{x \mid \phi\}$ exists; the type discipline requires that its type be one higher than that of x). The comprehension axiom of *TST* does not lead to paradox because the bad instances of comprehension (such as "$\{x \mid x \notin x\}$ exists") correspond to formulas ϕ of the language of "naive" set theory (e.g., $x \notin x$) which cannot be typed. The consistency strength of *TST* with an axiom of infinity is the same as that of Zermelo set theory with comprehension restricted to formulas in which all quantifiers are bounded (Δ_0 formulas; this system is also called "Mac Lane set theory": see Adrian Mathias's paper [15] for the equiconsistency result and for much valuable discussion of this system).

TST is often referred to inaccurately as "the simple theory of types of Russell". We ourselves have been frequent offenders! While it is true that *TST* is equivalent in strength to the system of *Principia Mathematica* ([25]: hereinafter abbreviated *PM*) with the Axiom of Reducibility, or to the simplified system due to Ramsey (in which the orders are dropped, so the Axiom of Reducibility is no longer needed), either of those systems has a far more complicated type system than that of *TST*, because of the presence of relation types as well as "set" types. The additional observation needed to simplify the type system of *PM* to this very simple linear form, which was not made by Russell or Ramsey, is that relation types can be eliminated by use of the Kuratowski pair (or another coding of ordered pairs as sets) to express n-ary relations as sets of n-tuples in the usual way. Something like *TST* seems to appear first in Gödel's 1930 paper [6] (but with the addition of the predicates and axioms of Peano arithmetic on the domain of individuals). There is a discussion of the history of this streamlined type theory in Hao Wang's [24].

What Quine observed about *TST* (in the pure form in which no additional assumptions are made about the individuals) is that the types seem to be indistinguishable. Any theorem of *TST* remains a theorem if the types of all variables appearing in the theorem are raised by a uniform amount. Any object $\{x \mid \phi\}$ definable in *TST* has a precise analogue in each higher type. For example, consider Frege's definition of the natural number 3 as the set of all sets with three elements. In type theory, this is the appropriate definition to use. It is used independently at each type: there is a type 2 integer 3 which is the type 2 set of all type 1 sets of type 0 objects, and there is a type 3 integer 3 which is the type 3 set of all type 2 sets of type 1 objects, and so forth. One cannot express in the language of *TST* the assertion that these are different (or the assertion that they are the same), but the suspicion that they are really all versions of the same thing is hard to escape. This "hall of mirrors" effect applies to all theorems and all mathematical objects in *TST*. It is a very sharp form of the phenomenon already noticed in *PM* and referred to there as "systematic ambiguity", which has more recently been called "polymorphism" in computer science.

Quine suggested the types be dropped completely. The resulting theory *NF* ("New Foundations") is an unsorted first-order theory with equality and membership. Its axioms are precisely the axioms of *TST* with all indications of type dropped. When type distinctions between variables are suppressed in the extensionality axioms of *TST*, they collapse to a single extensionality axiom (objects with the same elements are the same). When type distinctions between variables are suppressed in the comprehension axioms of *TST*, we do not obtain the inconsistent comprehension scheme of naive set theory: we obtain exactly those axioms "$\{x \mid \phi\}$ exists" such that there is an assignment of types to the variables in ϕ which yields a well-formed formula of *TST*. Such formulas are said to be "stratified".

It is usual to give a definition of "stratified formula" which does not depend directly on the typed language of *TST*.

Definition 1.1. *A formula ϕ is said to be* stratified *iff there is a function* `type` *from variables to natural numbers such that for all atomic subformulas "$x = y$" of ϕ we have* `type`("x") $=$ `type`("y") *and for all atomic subformulas "$x \in y$" we have* `type`("x") $+ 1 =$ `type`("y").

Definition 1.2. NF *is the first-order theory with equality and membership* *whose axioms are extensionality and each instance of comprehension "$\{x \mid \phi\}$ exists" such that ϕ is stratified.*

The stratified comprehension scheme is equivalent to a finite set of its instances (the usual reference for this is [7], but there are much nicer finite axiomatizations, as in (for example) [12] – the axiomatization for *NFU* given there requires some adaptation for use with *NF*), so it is possible to axiomatize *NF* (or *NFU*) without reference to stratification (and so without any reference to type, absolute or relative).

It is worth noting that a formula "$\{x \mid \phi\}$ exists" in which it is possible to assign types to each *bound* variable (including x) in such a way that all atomic subformulas which contain two bound variables are typed correctly, though it may not be stratified (some parameters may be impossible to type), will follow immediately from stratified comprehension: replacing each parameter with distinct variables at each of its occurrences will result in a stratified formula of which the original formula will be a substitution instance. Such formulas are termed "weakly stratified", and we will not systematically distinguish weak stratification from stratification in what follows, since we also have comprehension for weakly stratified formulas.

Our actual working language will contain term constructions. To understand how stratification works with these, observe that in *TST* a definite description $(\iota x.\phi)$ ("the x such that ϕ") would be assigned the same type as x. Analogously, the definition of stratification can be extended: the function `type` must have values at every term (for weak stratification, at every term containing a free variable) and $\texttt{type}(x) = \texttt{type}(\iota x.\phi)$ must also hold. For any function symbol F which can appear in stratified formulas, the type of a term $F(x_1, \ldots, x_n)$ will be displaced by a fixed amount from the type of each of its arguments x_i (not necessarily the same amount for each argument). For example, a singleton $\{x\}$ has type one higher than its argument x, and a function application $f(x)$ (application of a set function to an argument) is of the same type as its argument x and three types lower than the type of the set function f (if the function is understood as a set of Kuratowski pairs; if a type-level pair is used then the displacement is one).

Jensen's theory *NFU* can be obtained by weakening the axiom of extensionality to the form "Objects with elements are equal iff they have the same elements". This allows (but does not require) the existence of many distinct objects with no elements.

Jensen's original formulation didn't allow one to distinguish the empty *set* from the urelements: it is useful to introduce a constant $\emptyset$ standing for a particular object with no elements (there is at least one such object, because $x \neq x$ is a stratified formula), then to define "x is a set" to mean "$(\exists y.y \in x) \vee x = \emptyset$". An equivalent axiomatization of *NFU* takes this sethood predicate as primitive: its axioms are an axiom of sethood, asserting that any object with an element is a set, an axiom of extensionality asserting that sets with the same elements are equal, and an axiom of stratified comprehension asserting for each stratified formula ϕ that $\{x \mid \phi\}$ exists and is a set.

Hereinafter by *NFU* we will always mean *NFU* augmented with a constant representing the empty set (or with a sethood predicate) and also extended with axioms of Infinity and Choice. The precise form of these axioms will not be an issue for us (details can be seen in sources already cited). The usual equivalences between forms of the Axiom of Choice hold in *NFU* (if they are stated in stratifiable forms).

NF and *NFU* are distinctive in allowing "big" sets like the universe $V = \{x \mid x = x\}$. It is important to note that the consistency problem for *NF* does not stem from the presence of these "big" sets: *NFU*, with the same axiom of comprehension and without the axiom of Infinity, is weaker than Peano arithmetic, and with the addition of the axiom of Infinity is exactly as strong as *TST* with Infinity (slightly weaker than Zermelo set theory). There is no evidence that the problematic *NF* is any stronger than *TST* with an axiom of infinity (*NF* proves infinity because it disproves choice: if the universe were finite, it could be well-ordered. *NFU* does not prove Infinity: it is consistent with *NFU* that the cardinality of the universe is a (necessarily nonstandard) natural number). A complete development of set theory as a foundation for mathematics using *NF* is found in Rosser's [21]; a briefer development is found in the first author's more readily available [5]; one could consult the second author's [12] for a development of elementary set theory in *NFU*.

We need a brief introduction to ordinals in *NF(U)*. A well-ordering is defined in *NFU* in the usual way. The order type of a well-ordering is the set of all well-orderings isomorphic to that well-ordering. A set is an ordinal number just in case it is the order type of some well-ordering. It is useful to be aware that we take our well-orderings to be non-strict ($\leq$ rather than $<$) or we would be unable to distinguish between the ordinals 0 and 1.

The ordinals make up a set, which is well-ordered by the natural order on ordinal numbers, which is also a set, and so there is an order type Ω of the natural well-ordering on the ordinals, and one might suspect that the Burali-Forti paradox would put an end to our whole program. Briefly, this is not the case because the argument of the Burali-Forti paradox depends on the proposition that the order type of the natural order on ordinals restricted to the ordinals less than an ordinal α is equal to α. This statement appears easy to prove by transfinite induction – but this does not work, because the condition "the order type of the natural order on the ordinals less than α belongs to α" is unstratified and does not define a set.

For any well-ordering W, we define W^ι as $\{\langle\{x\},\{y\}\rangle \mid x\,W\,y\}$. It is easy to prove that if W is a well-ordering, so is W^ι. Note that W^ι would be one type higher than W in *TST*. If α is the order type of W, we define $T(\alpha)$ as the order type of W^ι (it is straightforward to prove that the value of $T(\alpha)$ does not depend on the choice of $W \in \alpha$). $T(\alpha)$ is one type higher than α.

Now it is possible to prove by transfinite induction that the natural order on the ordinals less than α has order type $T^2(\alpha)$ (the double application of the T operation makes the formula stratified, so the argument by transfinite induction works). The Burali-Forti argument converts to an argument that Ω is the supremum of the set of all ordinals $T^2(\alpha)$, which means that $\Omega > T^2(\Omega)$, which is counterintuitive but not impossible. In fact, the analogous assertion is provable in *TST* (where the two occurrences of "Ω" have different types).

It is easy to see that the T operation respects order (in fact, it is an endomorphism of the ordinals into an initial segment of the ordinals, respecting all standard operations on the ordinals) from which it follows that the sequence $\Omega, T^2(\Omega), T^4(\Omega), \ldots$ is a descending sequence in the ordinals. This is not a contradiction, because this sequence is not a set (its definition is not stratified).

Natural numbers are defined in *NF(U)* following Frege: 0 is defined as the set $\{\emptyset\}$ (the set of all sets with 0 elements – notice that in *NFU* this is not taken to include the urelements) and for any set A we define $A + 1$ as the set of all sets $A \cup \{x\}$ with $x \notin A$ (the set of all disjoint unions of elements of A with singletons). We say that a set I is *inductive* just in case $0 \in I$ and $(\forall A.A \in I \rightarrow A + 1 \in I\}$, and we define the set $\mathcal{N}$ of natural numbers as the intersection of all inductive sets. To see that all this reasoning is stratified, it is sufficient to observe that it all makes perfect sense in *TST*. For each concrete natural number n, n will be defined as the set of all sets with n elements. It is of course impossible to rule out the presence of nonstandard natural numbers (but Jensen did show in [14] that *NFU* has ω-models).

The cardinality $|A|$ of a set A is defined as the set of all sets B such that there is a bijection between A and B. Note that the cardinal of A would be one type higher than A in *TST*. It is easy to prove that each natural number is a cardinal number (is the cardinal of each of its elements, in fact), so "finite cardinal" means the same thing as "natural number". Finite ordinals are not the same thing as finite cardinals (natural numbers), but the natural isomorphism exists.

Definition 1.3. *For any set A, we define ι"A as $\{\{x\} \mid x \in A\}$, the image of A under the singleton operation. It is useful to note that ι"A is one type higher than A if the definition is interpreted in* TST. *For any cardinal number $|A|$, we define $T(|A|)$ as $|\iota$"$A|$ (there is an easy theorem which needs to be proved to verify that this is actually a definition). It might seem that T is a trivial operation, but this is not the case. Note that $T(|A|)$ is one type higher than $|A|$ if things are understood in terms of* TST. *(It is worth noting that appeal to* TST *makes sense for* NFU *as well as for* NF*:* NFU *has an associated type theory* TSTU *in which urelements are permitted to occur along with sets in each positive type).*

There is of course a close relationship between the T operation on cardinals defined here and the T operation on ordinals defined above.

The Cantor paradox is avoided because the Cantor theorem does not take its usual form $|A| < |\mathcal{P}(A)|$. In fact, there is no reason to expect it to take this form, because this is an unstratified assertion: the power set of A is one type higher than A in *TST*. The theorem which can be proved in *NF(U)* (and in *TST*) is $|\iota$"$A| < |\mathcal{P}(A)|$. The special case $A = V$ which leads to the Cantor paradox in naive set theory here gives us the theorem $|\iota$"$V| < |\mathcal{P}(V)|$, which tells us that the set of singletons is smaller than the universe (or than the set of all sets, in *NFU*). This is not contradictory: the natural bijection $(x \mapsto \{x\})$

between these sets has an unstratified definition, so we need not expect it to be a set (and this argument proves that it is not).

The exponential function $\exp(|A|) = 2^{|A|}$ is not defined as $|\mathcal{P}(A)|$, because $|A|$ and $|\mathcal{P}(A)|$ are not at the same relative type, and exp if defined this way could be proved not to be a set function: The correct way to define $\exp(|A|)$ is as $T^{-1}(|\mathcal{P}(A)|)$. This will not always be defined, as T^{-1} is partial. Note that $\exp(T|A|) = |\mathcal{P}(A)|$ by definition, and so $|T(|A|) < \exp(T(|A|))$ by Cantor's theorem as formulated above; it is a theorem that $|A| < \exp(|A|)$ in general as well.

Sets A such that $|\iota``A| = |A|$ satisfy the usual form of Cantor's theorem, and so are called *cantorian* sets. Sets A such that the map $(x \mapsto \{x\})\lceil A$, the restriction of the singleton map to A, is a set, are obviously cantorian and are called *strongly cantorian* sets. Cardinal numbers of cantorian sets are called cantorian cardinals. Ordinal numbers of well-orderings W such that W is isomorphic to W^ι are called cantorian ordinals (it is not the case that the order type of a well-ordering of a cantorian set is necessarily a cantorian ordinal: for example, if a natural number $n \neq T(n)$ the ordinal $\omega + n$ is a noncantorian order type of a well-ordering of a cantorian set). Ordinal numbers of well-orderings of strongly cantorian sets are called strongly cantorian ordinals; the isomorphism between W and W^ι will be witnessed by a restriction of the singleton map if the domain of W is strongly cantorian.

Variables restricted to strongly cantorian sets can have their type manipulated freely. Let A be a strongly cantorian set, let σ be the restriction of the singleton map to A (a set function) and let x be any variable restricted to A: one can replace references to x with references to the sole element of $\sigma(x)$ (one type lower than the original reference to x) or with references to $\sigma^{-1}(\{x\})$ (one type higher than x). The effect of this is that variables restricted to strongly cantorian sets may have each occurrence independently assigned any desired type for purposes of stratification.

2 Preliminaries: Permutation Methods

The history of the application of permutation methods to *NF* begins with an oversight of Quine's in the original 1937 paper [18]. Quine was aware that his choice of strong extensionality (instead of the weak extensionality adopted by Jensen for *NFU* which would allow urelements) was open to challenge. He suggested that the difference between strong and weak extensionality was inessential because any non-sets one might want to talk about could be assigned their own singleton as an extension. This is a harmless way to enforce strong extensionality in Zermelo set theory or its extensions, but it is not harmless at all in *NF*, where it is a theorem that the class of singletons is smaller in cardinality than the universe. All known models of *NFU* have the cardinality of the set of urelements equal to the cardinality of the universe, so Quine's trick cannot be used to eliminate the urelements.

For this reason, objects $x = \{x\}$ are called "Quine individuals" or "Quine atoms" in the *NF* literature (with "individual" or "atom" being used here in the same sense as our "urelement", to mean a non-set; "individual" was Scott's original terminology, but we will use "atom" to avoid confusion with the use of "individual" for type 0 objects in *TST*). Dana Scott, in [22], investigated the question of consistency and independence of the existence of Quine atoms using a permutation method which we now describe, which has been until recently almost the only method known for consistency and independence results from *NF* (the entire panoply of modern methods can be adapted to *NFU*, since its model theory is better understood; the second author has recently shown (in [11]) that forcing can be implemented in *NF*, though the inconsistency of Choice with *NF* limits the usefulness of forcing for establishing consistency and independence results with *NF*). Scott's investigation was continued by Henson in [8], and results along similar lines have been obtained by later workers.

In the rest of this section, we introduce the method of Scott and prove some of his theorems about Quine atoms as examples. The notation we use will be taken from [5]. The formal description of the adaptation of this method to *NFU* appears in the recent paper [4] of Marcel Crabbé, though its first application appeared in the earlier paper [2] of Boffa.

The idea is to redefine the membership relation using a permutation of the universe. We will only consider the use of permutations π which are sets in our ambient *NF(U)*, though the method is applicable to a more general class of "set-like" permutations which may not be sets, and interesting results can be obtained using these more general techniques as well. A more general treatment is found in [5].

Definition 2.1. *If π is a set permutation of the universe fixing all urelements we define $x \in^\pi y$ as $x \in \pi(y)$. If ϕ is a formula in the language of* NFU*, we define ϕ^π as the formula obtained by replacing all occurrences of $\in$ with $\in^\pi$ (with concomitant effects on predicates or functions defined in terms of $\in$).*

Observation 2.1. *If ϕ is stratified, so is ϕ^π. It is easy to see that extensionality$^\pi$ is true; it is only slightly harder to see that (stratified comprehension)$^\pi$ is true, as $\{x \mid \phi\}^\pi$ is conveniently defined as $\pi^{-1}(\{x \mid \phi^\pi\})$.*

Definition 2.2. *For any permutation π of the universe, define $j(\pi)$ as the map which sends each set A to π"A (and fixes any urelements). Note that $j(\pi)$ will also be a permutation, so this process can be iterated. Define a hierarchy of permutations π_n of the universe, in which π_0 is the identity and $\pi_{n+1} = j^n(\pi) \circ \pi_n$.*

Observe that any sentence $x \in y$ is equivalent in truth value to the sentence $\pi(x) \in j(\pi)(y)$ for any permutation π. Replacing π with $j^n(\pi)$, we find that $x \in y \equiv j^n(\pi)(x) \in j^{n+1}(\pi)(y)$. $x \in \pi(y)$ is equivalent to $\pi_0(x) \in \pi_1(y)$ by the definitions of π_0 and π_1 and this observation. Now consider $\pi_{n+1}(x) \in \pi_{n+2}(y)$; this is by definition $j^n(\pi)(\pi_n(x)) \in j^{n+1}(\pi)(\pi_{n+1}(y))$, which by our observation is equivalent to $\pi_n(x) \in \pi_{n+1}(y)$. So, by induction, $x \in^\pi y$ is equivalent to $\pi_n(x) \in \pi_{n+1}(y)$ for any n. We make the further observation that for any permutation π, $(\forall x.\phi) \equiv (\forall x.\phi[\pi(x)/x])$, which allows us to eliminate permutations applied to bound variables. We now have the tools we need to prove the following

Theorem 2.1. *(*NF(U)*) For any* stratified *assertion ϕ in the language of set theory, ϕ^{π} is equivalent to ϕ with each parameter a replaced with $\pi_{\mathtt{type}(a)}(a)$, where* `type` *is a stratification of ϕ (in the strict form with domain all variables and range restricted to the natural numbers).*

Proof. Replace each atomic subformula $x \in^{\pi} y$ in ϕ^{π} with the equivalent formula $\pi_n(x) \in \pi_{n+1}(y)$, and each atomic subformula $x = y$ with $\pi_n(x) = \pi_n(y)$, where $n = \mathtt{type}(x)$ in both cases. Notice that $n + 1 = \mathtt{type}(y)$ in the first kind of formula and $n = \mathtt{type}(y)$ in the second, so this procedure will cause each variable or parameter z to appear only in the context $\pi_{\mathtt{type}(z)}(z)$ (under the simplifying assumption that ϕ is presented in a form involving only equality and membership formulas). The second observation above can be used to eliminate all permutations applied to bound variables, and the resulting formula equivalent to ϕ^{π} is as the theorem requires it to be.

□

Definition 2.3. *A permutation π of the universe which is not necessarily a set function is called* setlike *if $j^n(\pi)$ is defined for all n on all sets. It is not difficult to see that the theorem holds for any setlike permutation π, even if it is not a set function itself.*

Observation 2.2. *It is tempting to redefine ϕ^{π} to stipulate that as well as replacing each occurrence of $\in$ with $\in^{\pi}$, we also replace each parameter a with $\pi^{-1}_{\mathtt{type}(a)}(a)$ (where* `type` *is a stratification of ϕ). Under this definition, we would simply have $\phi^{\pi} \equiv \phi$ for stratified ϕ. For sentences (formulas without free variables) we would have $\phi^{\pi} \equiv \phi$ under either definition. However, it seems that the reference of parameters needs to remain fixed when this notation is being used in practice to discuss permutation models.*

Definition 2.4. *We see from the above that if $\langle V, \in \rangle$ is a model of* NF(U) *and π is a permutation in this model, that $\langle V, \in^{\pi} \rangle$ is also a model of* NF(U)*. Following common abuse of notation for structures, we might refer to the first model as V and we will on occasion refer to the second as V^{π}.*

The theorem reveals a strong restriction on what sorts of consistency and independence results can be proved in this way: the truth of *stratified* sentences of the language of *NF(U)* cannot be perturbed by these permutation methods. It might seem that this would greatly limit the mathematical interest of these methods, but as it turns out there are unstratified assertions with very interesting mathematical consequences in *NF(U)*, as we will see below (some of these turn out to be invariant as well), and these permutation methods can be used to study the possibility of using constructions proper to Zermelo-style set theory which rely on (instances of) unstratified comprehension and so are not obviously appropriate in the *NFU* context.

The first author has suggested that a kind of modal logic of permutation models has value:

Definition 2.5. *For any sentence ϕ, we define $\Diamond\phi$ as $(\exists\pi.\phi^\pi)$ and $\Box\phi$ as $(\forall\pi.\phi^\pi)$. Since the permutations π are sets, the "modal" propositions introduced make sense. It is not to be expected that the modal assertions will be stratified, and in any event the modal operators are of no interest when ϕ itself is stratified.*

We refine our notational conventions.

Convention 2.1. *We have defined the notations ϕ^π and $\{x \mid \phi\}^\pi$ for any formula ϕ and permutation π above. By extension, for any term T definable in terms of our set notation we have implicitly defined the notation T^π. We will sometimes need to use terms defined in terms of the old interpretation as parameters in propositions or terms to be understood in terms of the new interpretation. A term $U_{\texttt{old}}$ appearing as a parameter in a context ϕ^π and $\{x \mid \phi\}$ is to be understood in terms of the original interpretation. For example, $\{\mathcal{N}_{\texttt{old}}\}^\pi$ is the singleton in terms of the permutation interpretation of the set of natural numbers of the old interpretation, or $\pi^{-1}(\{\mathcal{N}\})$, which is not as a rule the same object as $\pi^{-1}(\{\mathcal{N}^\pi\}) = \{\mathcal{N}\}^\pi$, the object understood to be the singleton of the set of natural numbers in the permutation interpretation. The* `old` *suffix may be used from time to time to remind us how symbols are to be understood, even in contexts where its use is formally unnecessary.*

Now we prove some theorems of Scott about Quine atoms (and a theorem of Henson which rounds out Scott's results), giving the promised examples of the method. The proofs are not necessarily the same in detail as those in the original sources.

We introduce some notation for permutations.

Definition 2.6. *The symbol $(a\ b)$ denotes the permutation which sends a to b, b to a, and fixes every other object (a transposition). If f and g are one-to-one maps with the same domain S and disjoint ranges, $\Pi_{i\in S}(f(i)\ g(i))$ denotes the map which sends each $f(i)$ to $g(i)$, each $g(i)$ to $f(i)$, and fixes each other object.*

Theorem 2.2. *(Scott) It is consistent with* NF *(or* NFU*) that there are Quine atoms.*

Proof. Let π be the permutation $(\emptyset\ \{\emptyset\})$ which interchanges $\emptyset$ with $\{\emptyset\}$ and fixes every other object (note that this means that all urelements are fixed, if there are any).

Under $\in^\pi$, the original $\emptyset$ has the original $\emptyset$ as its own sole member, thus providing the requisite Quine atom. Note that the original $\{\emptyset\}$ plays the role of $\emptyset^\pi$. The theorems about permutation methods already stated assure us that we still have a model of *NF(U)*. □

Theorem 2.3. *(Scott) It is consistent with* NF *(or* NFU*) that there are no Quine atoms.*

Proof. The feature which made the π of Theorem 2.2 work was that it moved some object onto its own singleton. For the current theorem, we want a π which interchanges each singleton with something which *cannot* be its sole member. In fact, it suffices to confine our attention to singletons of singletons, because only these can be Quine atoms. Define π as swapping each $\{\{x\}\}$ with $\{\{x\}, \emptyset\}$, and fixing everything else: we have the notation

$$\prod_{x \in V} (\{\{x\}\}\ \{\{x\}, \emptyset\})$$

for this. Suppose that for some y, $y = \{y\}^\pi$. For this to be true, $\{y\} = \pi(y)$ has to have been true in the original model (and so does $y = \pi(\{y\})$). If y was fixed by π, then $y = \{y\}$ is the singleton of a singleton, and would be moved to a non-singleton set by π, which is a contradiction. If y is moved by π, it is either of the form $\{\{x\}, \emptyset\}$, in which case $\{y\}$ would be fixed by π and so could not be equal to $\pi(y)$, or it is of the form $\{\{x\}\}$, in which case $\{y\}$ would be sent by π to $\{\{\{x\}\}, \emptyset\} \neq y$. So it cannot be true in any case, and the new interpretation has no Quine atoms. □

Theorem 2.4. *(Pétry, [17]) For any strongly cantorian set A it is consistent that the Quine atoms make up a set the same size as A.*

Proof. This is a two-step proof. The first step is to apply the permutation of the previous proof which eliminates all Quine atoms. Now work inside this permutation interpretation. Take a strongly cantorian set B the same size as A containing no singletons: $B = \{\{\{a\}, \emptyset\} \mid a \in A\}$ will serve (note that this set is the same size as $\iota^2``A$ for general A; it is only because A is cantorian that B is the same size as A). Take a new permutation $\pi = \prod_{x \in B}(x\ \{x\})$, swapping each member of B with its singleton and fixing everything else. It is straightforward to verify that each element of B is a Quine atom in V^π, and there are no other Quine atoms.

It is obvious that if the Quine atoms make up a set, it must be a strongly Cantorian set: the identity map restricted to any set of Quine atoms is a set and is also the restriction of the singleton map to that set.

□

Theorem 2.5. *(Henson) It is consistent for the Quine atoms to make up a proper class.*

Proof. Using the discussion of properties of ordinal numbers above, we can define the permutation π needed for the result. First apply the permutation which kills all Quine atoms. Then apply the permutation $\prod_{\alpha \in \mathtt{Ord}}(T(\alpha)\ \{\alpha\})$ swapping each ordinal $T(\alpha)$ with the singleton $\{\alpha\}$. This permutation is a set because the two occurrences of α in its definition have the same type (we could *not* define a set permutation which swapped each ordinal with its own singleton). This permutation tranforms each ordinal which is a fixed point of T (i.e., each cantorian ordinal) to a Quine atom, and it is an easy exercise to verify that no other Quine atoms are created. The Quine atoms of the new interpretation make up a proper class because the cantorian ordinals of the old interpretation make up a proper class: it is easy to prove that the successor of any cantorian ordinal is cantorian and the supremum of any set of cantorian ordinals is cantorian, and so if the cantorian ordinals made up a set they would constitute all of the ordinals, which we have already shown not to be the case (Ω is not cantorian). (It is interesting to note that every strongly cantorian well-ordering is similar to the initial segment in the natural order on the ordinals consisting of all the (strongly cantorian) ordinals less than its order type, so every strongly cantorian well-ordering is similar to a well-ordering on Quine atoms in this permutation interpretation.)

□

3 The von Neumann numerals and the Axiom of Counting

An axiom commonly adjoined to *NF(U)* (first proposed by Rosser in [21]) is

Axiom 3.1. *(Axiom of Counting) For each natural number* n*,* $|\{m \mid m < n\}| = n$*.*

This statement is "obviously true" but in fact it cannot be proved in *NF* or *NFU*. It is known to strengthen both of these theories essentially (in the case of *NF*, on the assumption that *NF* is consistent). It might seem that it could be proved by mathematical induction, but the condition $|\{m \mid m < n\}| = n$ is unstratified: the two occurrences of n differ in type by 2 if the formula is interpreted as a formula of *TST*, and the set of natural numbers is defined as the intersection of all inductive *sets*, which only warrants induction on stratified conditions. Once the model theory of *NFU* is understood, it is easy to construct models of *NFU* + Infinity + the negation of Counting (in fact, it is easier than constructing a model in which Counting is satisfied), and in [16] Steven Orey showed that *NF* + Counting proves the consistency of *NF*, which is one of the few nontrivial independence results known for *NF*.

There is another way to state this axiom which is more instructive and will introduce concepts important for our later development.

Theorem 3.1. *(proof omitted)* T *is a (possibly external) bijection on the natural numbers: for any natural number* n*,* $T(n)$ *is a natural number and there is a unique natural number* $T^{-1}(n)$*. (*T *is only a set function if it is the identity).*

The omitted proof is an easy induction argument.

Theorem 3.2. *The assertion "for all* n*,* $T(n) = n$*" is equivalent to the Axiom of Counting.*

Proof. (sketched) $|\{m \mid m < n\}| = T^2(n)$ is easily proved by induction (note that the applications of T bring the types of the occurrences of n into alignment). Then it is easy to show that $T^2(n) = n$ for all n just in case $T(n) = n$ for all n, using the fact that T is strictly monotone. □

We set out to prove the following result of the first author:

Claim 3.1. *(Forster) It is "possible" (in the sense of Definition 2.5) for the set of von Neumann numerals to exist iff the Axiom of Counting holds.*

We first need to develop the definition of the class of von Neumann numerals.

We recall the familiar construction of the natural numbers in Zermelo-style set theory. 0 is defined as $\emptyset$. We define the "successor" x^+ of an arbitrary set x as $x \cup \{x\}$. Then we define the set of natural numbers as the intersection of all inductive sets, as we did above (but with different notions of zero and successor).

From the standpoint of *NFU*, this seems quite impossible, since the operation x^+ is unstratified and so cannot appear in any stratified formula. In *TST*, each von Neumann numeral n appears first in type $n+1$, and n^+ is one type higher than n.

We can define the *class* of von Neumann numerals vN as the intersection of all *sets* which contain the von Neumann zero and are closed under the von Neumann successor operation. The definition of this class is unstratified, so there is no reason to believe that it is a set.

Definition 3.1. *For any set x, define x^+ (its* von Neumann successor*) as $x \cup \{x\}$. We say that a set ν is a* von Neumann numeral *just in case $(\forall A.\emptyset \in A \wedge (\forall x.x \in A \rightarrow x^+ \in A) \rightarrow \nu \in A)$. If there is a set whose elements are exactly the von Neumann numerals, we call it vN.*

Theorem 3.3. *(Forster) The Axiom of Counting implies $\Diamond$(vN is a set).*

Proof. The needed permutation π is

$$\prod_{n \in \mathcal{N}} (T(n)\ \{m \mid m < n\})$$

(note that $T(n)$ and $\{m \mid m < n\}$ have the same relative type, one higher than that of n). Because the Axiom of Counting holds, $T(n) = n$, so the effect is to assign to each natural number n of the original interpretation the extension $\{m \mid m < n\}$. The original 0 is the von Neumann 0 of the new interpretation: the new extension of 0 is $\pi(0) = \{m \mid m < 0\}$, which is empty. In symbols, $0 = (\emptyset)^\pi$. For each natural number n, the new extension of n will be $\{m \mid m < n\}$ and the new extension of $n+1$ will be $\{m \mid m < n+1\} = \{m \mid m < n\} \cup \{n\}$: the natural number $n+1$ of the original interpretation will be the von Neumann successor in the new interpretation of the natural number n of the original interpretation. In symbols we would like to write $n + 1 = (n^+)^\pi$, but this could be understood as saying that the old $n+1$ is the von Neumann successor in the new interpretation of the *new* natural number n. To avoid this, we write $n + 1 = ((n_{\texttt{old}})^+)^\pi$, where the convention is that the symbol $n_{\texttt{old}}$ is interpreted in terms of the old interpretation. From this it is clear that each concrete natural number of the old interpretation becomes the corresponding von Neumann numeral in the new interpretation. We claim that the original set $\mathcal{N}$ is the set vN in the new interpretation: $\mathcal{N} = (vN)^\pi$. This requires some verification. Certainly the old set $\mathcal{N}$, considered as a set of the new interpretation, is inductive using the new von Neumann 0 and von Neumann successor (which coincide here with the old zero and successor). Moreover, any von Neumann inductive set I of the new interpretation will contain the von Neumann 0 and be closed under von Neumann successor and so will contain all the natural numbers of the old interpretation: the intersection of I with the old set of natural numbers will be a set containing the old 0 and closed under the old successor operation, and so will contain all old natural numbers. Thus vN, the intersection of all von Neumann inductive sets containing 0, in the new interpretation is the old $\mathcal{N}$, and so is a set. □

Though this result was first claimed explicitly by Forster, it appears at first blush to be an immediate corollary of the more general theorem 2.4 of Henson's [8], and the permutation used in the proof is similar. However, this is less obvious that it appears. Theorem 2.4 deals with general von Neumann ordinals rather than with von Neumann numerals, and the definition of von Neumann ordinal which Henson uses is not the same. Henson defines a "von Neumann well-ordering" as a well-ordering $\leq$ with the property that for each x in the range of $\leq$, $x = \{y|y < x\}$, (where $<$ is the corresponding strict well-ordering) and his theorem asserts that it is consistent with *NF* (the proof adapts easily to *NFU*) that there is a von Neumann well-ordering of the same length as each strongly cantorian well-ordering. The union of the domain and range of a von Neumann well-ordering is the corresponding von Neumann ordinal (if the strict well-ordering were used, we could not distinguish 0 and 1). The permutation which arranges this is

$$\prod_{\alpha \in \mathtt{Ord}} (T(\alpha)\ \{\beta \mid \beta < \alpha\}),$$

which converts each strongly cantorian ordinal to the corresponding von Neumann ordinal. If we assume that ω is strongly cantorian (this is equivalent to assuming the Axiom of Counting), it follows that ω will become the von Neumann ω, which is clearly the set of all von Neumann numerals (equiv. von Neumann finite ordinals). Forster defines the collection of von Neumann ordinals as the (proper class) intersection of all sets which contain the empty set, are closed under von Neumann successor, and are closed under union of their subsets. This definition has the same flavor as Forster's definition of vN. The Henson definition could be called an "internal" definition (looking at features of each von Neumann ordinal) while the Forster definition could be called an "external" definition (working from closure properties of the entire class of von Neumann ordinals). These definitions are equivalent, but it is not easy to prove this (it is much trickier to prove them equivalent in *NF(U)* than it is in Zermelo-style set theory). For other similar notions, such as the Zermelo numerals, "internal" and "external" definitions apparently do not coincide.

Theorem 3.4. *(Forster) The Axiom of Counting holds if vN is a set.*

Proof. (Holmes) We need to show a series of facts about vN (we suppose throughout the argument that vN is a set).

Each element of vN is finite (the set of finite sets is von Neumann inductive).

Each nonempty von Neumann numeral x is the von Neumann successor of some von Neumann numeral $y \neq x$: otherwise $vN - \{x\}$ would be von Neumann inductive and $x \notin vN$.

No von Neumann ordinal can be its own successor. $x = x^+$ holds for a set x iff $x \in x$. For any self-membered set x, the set $\{y \mid x \notin y\}$ exists by stratified comprehension and is easily seen to be von Neumann inductive: it follows that x cannot be a von Neumann numeral.

There is at least one element of vN of each finite size: there is an element of size 0, and given an element x of size n we see that x^+ is of size $n + 1$, since x cannot be self-membered. Note that this induction argument only works because vN is a set, so the class of cardinalities of elements of vN is a set.

We claim that the class of von Neumann numerals which are unique in their size is a set (obvious if vN is a set) and is von Neumann inductive. Clearly the von Neumann 0 is the only von Neumann numeral of size 0. Suppose x is the only von Neumann numeral of size n. Let y be any von Neumann numeral of size $n + 1$. We know that y is the von Neumann successor of some $z \neq y$. This z is of size n and so must be x, so y must equal x^+, and thus x^+ is the only von Neumann numeral of its size $n + 1$.

Now consider the class A of von Neumann numerals ν such that for some natural number n the elements of ν are the unique von Neumann numerals of each size less than n. The definition of A is (weakly) stratifiable, so A is a set (it is worth noting that vN appears at two different relative types in the definition of this set, but this is not a problem, as vN is free in the formula defining A: the defining formula is weakly stratified rather than stratified, but we have comprehension for weakly stratified formulas as well, as noted above).

The von Neumann 0 is certainly in A. If x is in A, it contains as elements a set of size 0, a set of size 1, and so on up to a set of size $n - 1$. The size of x itself will be $T(n)$ (the elements of x are one type higher than the elements of the elements of x). So x^+ will belong to A exactly if the new element x added is of size n, so exactly if $n = T(n)$.

Suppose that n is the size of the smallest von Neumann numeral not in A. It is easy to see that $n > 1$, so we can see that the von Neumann numerals x and x^+ of sizes $n-1$ and $n-2$ are in A, from which it follows that $T(n-2) = n-2$ by the discussion of the previous paragraph, from which it follows that $T(n-1) = n-1$ (it is easy to see that $T(m+1) = T(m)+1$ for all m), from which it follows that x^{++} is in A, which contradicts the putative choice of n. It follows that every von Neumann numeral is in A.

Since every von Neumann numeral is in A, we have $T(n) = n$ for every natural number n which is the size of a von Neumann numeral. We have already seen that every natural number is the size of some von Neumann numeral, so we have $T(n) = n$ for all n, an assertion equivalent to the Axiom of Counting.

This completes the proof. Henson showed that von Neumann ordinals are strongly Cantorian, so that the existence of the von Neumann ω (which is vN)

would imply the Axiom of Counting. But, as we have noted above, he was working from a different definition of von Neumann ordinals and so of von Neumann numerals and vN. □

Theorem 3.5. *(Henson) (Axiom of Counting)$^\pi$ is equivalent to the Axiom of Counting, for any permutation π.*

Proof. AxCount is equivalent to the assertion that for all $n \in \mathcal{N}$, if $A \in n$ then $\iota``A \in n$. We need to show that this is equivalent to (AxCount)$^\pi$, which means "for all $n \in \mathcal{N}$, if $A \in^\pi n^\pi$, then $(\iota``A)^\pi \in^\pi n^{\pi}$".

It is straightforward to determine that $n^\pi = \pi^{-1}(\pi^{-1}``n)$; it is somewhat less straightforward to determine what $(\iota``A)^\pi$ denotes. Clearly $\{a\}^\pi = \pi^{-1}(\{a\})$. We require $a \in^\pi A \equiv \{a\}^\pi \in (\iota``A)^\pi$, which is equivalent to requiring $a \in \pi(A) \equiv \pi^{-1}(\{a\}) \in \pi((\iota``A)^\pi)$, whence $(\iota``A)^\pi = \pi^{-1}(\pi^{-1}``\iota``(\pi(A)))$.

Now we see that "if $A \in^\pi n^\pi$, then $(\iota``A)^\pi \in^\pi n^{\pi}$" is equivalent to "if $A \in \pi(\pi^{-1}(\pi^{-1}``n)) = \pi^{-1}``n$, then $\pi^{-1}(\pi^{-1}``\iota``(\pi(A))) \in \pi(\pi^{-1}(\pi^{-1}``n)) = \pi^{-1}``n$, which is in turn equivalent to "if $\pi(A) \in n$ then $\pi^{-1}``\iota``(\pi(A)) \in n$", which is equivalent to the instance of the Axiom of Counting with which we started: $|\pi(A)| = |A| = |\iota``A| = |\pi^{-1}``\iota``(\pi(A))|$, where the middle equation is the instance of Counting and the outer equations $|\pi(A)| = |A|$ and $|\iota``A| = |\pi^{-1}``\iota``(\pi(A))|$ are consequences of the fact that π and its inverse are set maps. □

This allows us to complete the proof of Claim 3.1. If the Axiom of Counting holds, it is "possible" for the set of von Neumann numerals to exist (Theorem 3.3). Conversely, if it is "possible" for the set of von Neumann numerals to exist, it is "possible" for the Axiom of Counting to hold (since the Axiom of Counting actually holds if the von Neumann numerals actually exist, by Theorem 3.4). But, by Theorem 3.5, if it is "possible" for the Axiom of Counting to hold, then the Axiom of Counting actually does hold.

So the Axiom of Counting is equivalent to the assertion that the set of von Neumann numerals exists in the permutation interpretation V^π for some permutation π, which is what we set out to prove.

Theorem 3.6. *(Forster) The Axiom of Counting is equivalent to the assertion that there is a permutation π such that the universe V^π contains inductive closures of $\{\emptyset\}$ under F for every class (not necessarily set) map F defined by $F(x) = \{z \in \mathcal{P}^i(x) \mid \phi\}$, where ϕ is a parameter-free formula, not necessarily stratified, which is $\Delta_0^{\mathcal{P}}$ (i.e., ϕ is a formula in equality and membership with every quantifier in ϕ bounded by a variable or a concrete iterated power set of a variable, the variable being free in the scope of the quantifier). Note that because ϕ is not stratified we will need to show not only the existence of the closure but the existence of values of $F(x)$.*

Proof. One direction of this follows from the result already shown that the existence of vN (which can be expressed as an inductive closure of the kind indicated: let $F(x) = \{y \in \mathcal{P}(x) \mid y \in x \vee y = x\}$ and vN is the closure of $\{\emptyset\}$ under F) in any permutation model implies the Axiom of Counting.

Now assume the Axiom of Counting.

The permutation π which does the trick is the permutation which interchanges each natural number n with $\{m \mid$ "the mth binary digit of n is 1"$\}$ (the definition of this permutation is not stratified, but it is a set under the assumption of the Axiom of Counting: it is then equivalent to the permutation which interchanges each natural number n with $\{m \mid$ "the mth binary digit of $T^{-1}(n)$ is 1"$\}$, which definition is stratified; it is possible to define "the mth binary digit of n is 1" in such a way that the two variables have the same type). (We call this the Ackermann permutation because we believe that Ackermann was the first to note that the binary expansion can be used to code sets as natural numbers.) In the resulting permutation interpretation V^π, each old natural number becomes a set of smaller natural numbers, and each finite set of old natural numbers is coded by an old natural number. In V^π, the old $\mathcal{N}$ becomes V_ω, the class of hereditarily finite sets, defined here as the (possibly proper class) intersection of all sets which contain all of their finite subsets (there is more on V_ω below):

Definition 3.2. *We say that a set A is* hereditarily finite *iff it belongs to every set B such that all finite subsets of B are elements of B. If there is a set whose elements are exactly the hereditarily finite sets, we call this set V_ω. (It is worth noting that we are actually defining the class of* well-founded *hereditarily finite sets here, but we have little occasion to consider non-well-founded hereditarily finite sets in this paper).*

We consider the set $\mathcal{N}_{\mathtt{old}}$ (the set of natural numbers of the old interpretation) as a set of V^π. Each element of $\mathcal{N}_{\mathtt{old}}$ is assigned an extension consisting of smaller elements of $\mathcal{N}_{\mathtt{old}}$, by examination of the permutation. Further, each finite subset of $\mathcal{N}_{\mathtt{old}}$ is the extension assigned to some element of $\mathcal{N}_{\mathtt{old}}$ in V^π (again, by direct examination of the permutation). Thus we have $(\mathcal{N}_{\mathtt{old}} = \mathcal{P}^{\mathtt{fin}}(\mathcal{N}_{\mathtt{old}}))^\pi$: in the new interpretation, the set $\mathcal{N}_{\mathtt{old}}$ coincides with the set of finite subsets of $\mathcal{N}_{\mathtt{old}}$.

Let A be any set for which we have $(\mathcal{P}^{\mathtt{fin}}(A) \subseteq A)^\pi$. Let $<_{\mathtt{old}}$ be the usual order relation on $\mathcal{N}_{\mathtt{old}}$ inherited from the original model. Consider the $<_{\mathtt{old}}$-least element n of $\mathcal{N}_{\mathtt{old}}$ which does not belong to A in V^π (if there is one). In V^π, n is a finite collection of $<_{\mathtt{old}}$-smaller elements of $\mathcal{N}_{\mathtt{old}}$, all of which by choice of n belong to A, which is absurd: this makes n a finite subset of A not belonging to A, contradicting the choice of A. So every set A for which $(A = \mathcal{P}^{\mathtt{fin}}(A))^\pi$ contains all elements of $\mathcal{N}_{\mathtt{old}}$ in the new interpretation V^π, from which it follows that $\mathcal{N}_{\mathtt{old}}$, itself a set which contains all of its finite subsets in V^π, is in fact the intersection of all such sets, or V_ω (in the sense of V^π). Briefly, $\mathcal{N} = (V_\omega)^\pi$.

V_ω contains all elements and subsets of its elements and all power sets of its elements. All these sets are finite and so are strongly Cantorian by the Axiom of Counting, as is the countably infinite set V_ω itself. Now observe that every quantified variable in the definition of $F(x)$ is bounded by an iterated power set of some variable free in the scope of the quantifier, which is in turn either x or a quantified variable bounded in an iterated power set of a variable free in its scope, and so on until we get to x (the only free variable in the definition of $F(x)$). Closure properties of V_ω show that every variable can be supposed to be bounded in V_ω if x is in V_ω. From this we can conclude that $F(x)$ exists as a set for every $x \in V_\omega$, and that the restriction of F to V_ω is a set, because every bound variable in the definitions of these classes is bounded in a strongly cantorian set, and so can have the type of each of its occurrences independently adjusted for stratification purposes. From this it follows that the desired inductive closure is a subset of V_ω. It should be clear that the result extends to closures under operations with more than one argument or with parameters taken from V_ω allowed in the definition of F.

□

4 The Zermelo numerals

Here we discuss the status of a different unstratified class of sets which has also been used to implement the natural numbers in Zermelo-style set theory, namely the *Zermelo numerals* $\emptyset, \{\emptyset\}, \{\{\emptyset\}\} \ldots$.

Definition 4.1. *We say that a set x is a* Zermelo numeral *if $x \in A$ for every set A such that $\emptyset \in A$ and $(\forall y.y \in A \rightarrow \{y\} \in A)$. We refer to the class of Zermelo numerals as Zn. We say that a set A is* Zermelo-inductive *if it has $\emptyset$ as a member and has $\{x\}$ as a member whenever it has x as a member.*

Lemma 4.1. *If Zn is a set, we have $|Zn| = 1 + T(|Zn|)$.*

Proof. This holds because Zn is the union of the disjoint sets $\{\emptyset\}$ and $\iota``Zn$. If $Zn$ is a set, $\{\emptyset\} \cup \iota``Zn$ is a set and is "Zermelo-inductive", so $Zn \subseteq \emptyset \cup \iota``Zn$. Now suppose that we have $z \in (\{\emptyset\} \cup \iota``Zn) - Zn$: z is clearly not the empty set, so it must be $\{w\}$ for some $w \in Zn$. But it is obvious that $w \in Zn \rightarrow \{w\} \in Zn$. □

Corollary 4.1. *If Zn is a set, it cannot be finite.*

Proof. No natural number n can satisfy $n = Tn + 1$ because n and Tn have the same parity. □

Lemma 4.2. *If $Tn < n$ for some natural number n, it is possible for Zn to be a proper subclass of a finite set.*

Proof. Consider the permutation π which swaps 0 with $\emptyset$ and each natural number $Ti + 1$ with $\{i\}$. This will clearly convert the concrete natural numbers into the concrete Zermelo numerals. Suppose that $Tn < n$ for some natural number n. The set $\{i \in \mathcal{N} \mid i < n\}_{\texttt{old}}$ contains as its elements under the new interpretation all the natural numbers $i_{\texttt{old}}$ of the old interpretation with $i < n$. Since $\{i_{\texttt{old}}\}^\pi = Ti + 1 < Tn + 1 < n$ (where we are of course alluding to old natural numbers and their original order) will belong to this set (in the new sense) if $i_{\texttt{old}}$ does, it follows that this set is Zermelo inductive in V^π. But then the class of Zermelo numerals cannot be a set in V^π, because an infinite set cannot be a subset of a finite set. □

Lemma 4.3. *If $Tn \geq n$ for all natural numbers n then it is possible for Zn to be a countably infinite set.*

Proof. Use the permutation π which exchanges each natural number $Tn+1$ with $\{n\}$ and 0 with the empty set. The application of T is required to ensure that the map is a set. In the permutation model V^π, $\mathcal{N}_{\texttt{old}}$, whose elements are the old natural numbers, will be Zermelo inductive, because for any natural number $n_{\texttt{old}}$ of the original model, $\{n_{\texttt{old}}\}^\pi = Tn + 1 \in^\pi \mathcal{N}_{\texttt{old}}$. Now suppose that in V^π some proper subset A of $\mathcal{N}_{\texttt{old}}$ is Zermelo inductive. There will be a least natural number $n+1$ such that $(n+1)_{\texttt{old}} \not\in^\pi A$ (clearly $0_{\texttt{old}} \in^\pi A$). $n+1 = \{T^{-1}n_{\texttt{old}}\}^\pi$, whence we must have $T^{-1}n \geq n+1$, whence $T^{-1}n > n$, whence $Tn < n$, contrary to assumption. This completes the proof that $\mathcal{N} = (Zn)^\pi$. Clearly $\mathcal{N}_{\texttt{old}}$ is a countably infinite set in the permutation model. □

Observation 4.1. *The assertion $(\forall n \in \mathcal{N}.n \leq Tn)$, which has an obvious relationship to conditions in the two preceding lemmas, is a weaker form of the Axiom of Counting (which we will refer to in a later section as $AxCount_{\leq}$).*

Lemma 4.4. *If Zn is a cantorian set, it contains a countably infinite set (if we assume further that $Tn \geq n$ for all n, we show in another way that Zn is possibly countably infinite).*

Proof. If Zn is cantorian, we have a bijection f sending Zn onto $\iota``Zn$. The inductive closure of the singleton of the empty set under this map will be a countably infinite subset of Zn, which we may denote as $\{f^n(\emptyset) \mid n \in \mathcal{N}\}$. (We can define the indexing so that the map sending i to $f^i(\emptyset)$ is a function.) This completes the proof of the main result. For the additional remark, we want to construct a permutation which will convert the map f into the singleton map. We would like to exchange $f^{i+1}(\emptyset)$ with $\{f^i(\emptyset)\}$, but the best we can do is to exchange $f^{Ti+1}(\emptyset)$ with $\{f^i(\emptyset)\}$ (the application of T makes the intended map stratified). Use

$$\prod_{i \in \mathcal{N}} (f^{Ti+1}(\emptyset)\ \{f^i(\emptyset)\})$$

as π. In the resulting model V^π, the set $\{f^n(\emptyset) \mid n \in \mathcal{N}\}_{\texttt{old}}$ is Zermelo-inductive: it contains the empty set (which isn't moved by the permutation) and contains the singletons in the new sense of each of its elements. Now suppose that there was a smaller Zermelo-inductive set A in V^π. We would have a least $i+1$ such that $(f^{i+1}(\emptyset))_{\texttt{old}} \not\in^\pi A$. A must not contain $f^{T^{-1}(i)}(\emptyset)$, whose singleton in the new sense is $f^{i+1}(\emptyset)$ itself. From this it follows that $T^{-1}i \geq i+1$, whence $Ti < i$, contrary to the additional assumption. This shows that the set of iterated images of the empty set under f is the class of Zermelo numerals in V^π, so Zn is "possibly countable" (under the additional condition that T moves no natural number downward). □

Observation 4.2. *We would be able to get a sharper result here if we could show that Zn is "at most countable" in the sense that if it is a set and has a countably infinite subset it would itself be a countably infinite set: it would then* *follow that if Zn is cantorian it must be countably infinite. But we do not know how to prove this (nor do we see an approach to constructing a model in which Zn is an uncountable Dedekind-infinite set).*

Lemma 4.5. *If Zn is a strongly cantorian set then Zn is countably infinite and the Axiom of Counting holds.*

Proof. The singleton map restricted to Zn is then a set, and the inductive closure of $\{\emptyset\}$ under this map will be the set of Zermelo numerals: it is a Zermelo-inductive subset of Zn, and so has to be the whole of Zn. $\square$

It would be ideal if we could show that the possible existence of the Zermelo numerals was actually equivalent to the assertion $(\forall n \in \mathcal{N}.Tn \geq n)$. Unfortunately, the Zermelo numerals seem to have too little exploitable structure for us easily to get a result of this kind.

One might expect that if Zn were a set we could prove that it was a countably infinite set, as is the case for the apparently similar class vN, but this turns out not to be so. We will explicitly describe the construction of a model of *NFU* in which the class of Zermelo numerals is an infinite but Dedekind-finite set. To be more precise, the class will be a set which has only finite and co-finite subsets; we call this an "amorphous" set.

We briefly review the easiest construction of models of *NFU* (due to Boffa in [3]). Construct a model of bounded Zermelo set theory (or full Zermelo set theory, or *ZFC*) with an external automorphism J which moves a rank V_α of the cumulative hierarchy. Suppose without loss of generality that $J(\alpha) > \alpha$. The domain of the interpreted *NFU* will be the rank V_α. The membership relation $x \in_{\texttt{NFU}} y$ is defined as $x \in J(y) \wedge J(y) \in V_\alpha$. Observe that the rank $V_{J^{-1}(\alpha)+1}$, which is externally isomorphic to the power set of the domain of our model (in the nonstandard model of bounded Zermelo) contains the sets of our purported model of *NFU*, and everything else will be urelements. The construction could also be carried out in versions of any of the above theories with a set of atoms: in this case the rank V_0 would be the set of atoms and the atoms would be urelements as well as the sets of rank $> J(\alpha) + 1$, but for a different reason.

The proof that this yields a model of *NFU* is closely related to the proof that our permutation methods work. The proof of weak extensionality in the model is easy. We demonstrate the existence of $\{x \mid \phi\}$ in the interpreted *NFU*, where ϕ is a stratified formula and `type` is a stratification of ϕ which sends x to 0. Translate ϕ into a formula ϕ^* in the language of the nonstandard model of *ZFC* with the automorphism J. It is not clear that the class $\{x \mid \phi^*\}$ is a set, because of the role of J in its definition. We show that it actually is a set. Replace each subformula $y = z$ of ϕ^* with the equivalent $J^n(y) = J^n(z)$, where $n = \texttt{type}(y) = \texttt{type}(z)$. Replace each subformula $y \in J(z)$ of ϕ^* with the equivalent $J^n(y) \in J^{n+1}(z)$, where $n = \texttt{type}(y)$ (note that $n + 1 = \texttt{type}(z)$ will also be true). Replace each subformula $J(y) \in V_\alpha$ with the equivalent $J^n(y) \in J^{n-1}(V_\alpha)$, where $n = \texttt{type}(y)$. Observe that each variable y will now appear only with exactly $\texttt{type}(y)$ applications of J. Now observe that every bound occurrence of $J^{\texttt{type}(y)}(y)$ (with quantifier restricted to V_α) can be replaced with an occurrence of y (with quantifier now restricted to $J^{\texttt{type}(y)}(V_\alpha)$) without affecting the meaning of the formula ϕ^*. This eliminates all occurrences of J in ϕ^* except in parameters without affecting its meaning, and shows that the set $\{x \mid \phi^*\}$ exists in the nonstandard model of Zermelo set theory. $J^{-1}\{x \mid \phi^*\}$ will be the set $\{x \mid \phi\}$ of the interpreted *NFU*.

It is useful to note that the isomorphism J is closely related to the *inverses* of T operations in the induced model of *NFU*. For example, if A is a set with size n in the model of *NFU*, and $J(A)$ is also a set in the model of *NFU*, $|J(A)| = T^{-1}(n)$. Note that this implies that the Axiom of Counting for the model of *NFU* is equivalent to the assertion that J fixes all natural numbers.

We describe the construction of a model of *NFU* in which the set of Zermelo numerals exists and is not countably infinite. Start with a model Z_1 of (enough of) *ZFA* (Zermelo-Frankel set theory with atoms) with a countable set of atoms and with an external automorphism J as above. From this get a model Z_2 of (enough of) *ZFA* in which the set of atoms of Z_1 becomes an amorphous set which can be indexed externally by the natural numbers. This model is obtained by the Frankel-Mostowski permutation method (a set a of Z_1 belongs to the model Z_2 iff there is a finite set of atoms s such that a is fixed by all permutations of the universe induced by permutations of the set of atoms which fix all elements of the finite set s); the external indexing of the amorphous set is derived from a map witnessing the countability of the set of atoms in the original model. Convert Z_2 to a model of *NFU* as described above using the automorphism J of Z_1 (whose restriction to Z_2 is of course an automorphism of Z_2, still with the appropriate properties). We define a setlike permutation π of the universe of Z_1: π exchanges $\emptyset$ with the first atom in the indexing and exchanges the $(n+1)$st atom with the singleton of the nth atom, and otherwise π satisfies $\pi(a) = \pi``(a)$. This is also a setlike permutation of Z_2: Z_2 is closed under π because π sends sets with finite support to sets with (different) finite support (the same is true for any $j^n(\pi)$). In the model of *NFU* it interchanges the $(n+1)$st atom with the singleton of the $J(n)$th atom (because of the role of J in the redefinition of membership). To verify that it is still a setlike permutation in the model of *NFU*, we need verify that the model of *NFU* is closed under π and that the image of a set of *NFU* under π (or any $j^n(\pi)$) is still a set (not an urelement) in the permutation interpretation: both of these facts follow from the fact that any infinite rank is closed under π or any $j^n(\pi)$ (in particular, V_α and $V_{J^{-1}(\alpha)+1}$ are so closed). Consider the set A whose members are the atoms of Z_1; it is an amorphous set in Z_2 and in the model of *NFU*. In the permutation model of *NFU* induced by π it is a set containing the empty set and closed under the singleton operator. To ensure that it is the smallest set with this property requires an additional condition. To see what it is, we develop an argument that A is the set of Zermelo numerals and see where the gap is. Suppose that there is some other set B which contains the (new) empty set and is closed under the (new) singleton operation. We can consider the intersection of B with A; this will still have the desired closure properties. Consider the first element of A in the external indexing which does not belong to B. We can suppose that it is the $n+1$-st element (it is certainly not the first element, which is the new empty set). The $n+1$-st element is the (new) singleton of the $J(n)$-th element. If $J(n) \leq n$, or equivalently $n \leq Tn$, is true for all natural numbers, which is true in suitably chosen models of NFU, this is impossible (because an non-element of the set cannot be the singleton of an element of the set) and A is seen to be the set of Zermelo numerals. A is not countable, because passage to the permutation model will not alter the fact that it has only finite and cofinite subsets (a condition expressible by a stratified formula).

5 Invariant Sentences as Facts about Well-Founded Relation Types

In the usual set theory, a set is uniquely characterized by the relation type of the restriction of membership to its transitive closure. We define the transitive closure of x as the intersection of all transitive sets having x as an element (note that for us x belongs to its own transitive closure). This restriction of the membership relation will be a well-founded extensional relation with a top element (in a sense which will be defined formally below) and in *ZF* any such relation type is the type of membership restricted to the transitive closure of some set.

This motivates the following definitions. Most of what follows originates with Hinnion's thesis [9]; more recent sources are the second author's [12] and Mathias's [15]. The details of the formalization here follow [12].

Definition 5.1. *A relation R is* extensional *iff any two elements of the range of R with the same preimage under R are equal.*

Definition 5.2. *A relation R is* well-founded *iff for any subset S of the domain of R, there is an element of S whose preimage contains no element of S (such an element is said to be a* minimal *element of S).*

Definition 5.3. *An object t is a* top *of a well-founded relation R iff for all elements x of the domain of R there is a finite sequence x of length n such that $x_1 = x$, $x_n = t$, and $x_i \, R \, x_{i+1}$ for all appropriate indices i.*

Definition 5.4. *We use "bfext" to abbreviate the awkward phrase "well-founded extensional relation with top". (The spelling of this is derived from the French terminology of [9])*

Observation 5.1. *An empty well-founded relation has any object as its top; a nonempty well-founded relation with a top has a uniquely determined top (if it had two, we would have a cycle in the relation, whose domain would have no minimal element).*

Definition 5.5. *A* set picture *is an isomorphism class of well-founded extensional relations with top (bfexts). Isomorphism of bfexts is defined as usual. We refer to the elements of a set picture as its "representatives". When R is a bfext, we use the notation $[R]$ for the set picture of which it is a representative.*

The collection of all set pictures is a set in NF(U)*, which we call Z. For any set picture $[R]$, $[R^\iota]$ is clearly also a set picture, and we define $T([R])$ as $[R^\iota]$ (it is easy to prove that $T([R])$ does not depend on the choice of the bfext R). T is an external (proper) endomorphism of Z.*

Definition 5.6. *If x is an element of the domain of a bfext R, the "component of x relative to R" is defined as the maximal bfext which is a subset of R and has x as its top. If t is the top of R and $x\,R\,t$, the component of x with respect to R is called an immediate component of R. We define a relation E on Z: $x\,E\,y$ iff x has a representative which is an immediate component of a representative of y.*

Observation 5.2. *In the usual set theory, the immediate components of the restriction of membership to the transitive closure of a set x are the restrictions of the membership relation to the transitive closures of the elements of x: the relation E (which is a set relation) should be the precise analogue of membership on the sets of a Zermelo-like set theory pictured by the elements of Z. It is easily seen that T respects E: $T(x)\,E\,T(y) \equiv x\,E\,y$.*

Theorem 5.1. *E is a well-founded extensional relation (it does not have a top).*

The proof is omitted.

Theorem 5.2. *The isomorphism type of the component of $x \in Z$ relative to E is $T^2(x)$.*

The proof is omitted: this is analogous to the result that the order type of the initial segment in the ordinals determined by α is $T^2(\alpha)$.

Theorem 5.3. *For any subset S of Z, there is an element s of Z such that for all $t \in Z$, $t \in S \equiv T(t)\,E\,s$ and $(\forall t\,E\,s.(\exists u.T(u) = t))$.*

Proof. (sketched) Choose a representative R_x of each element x of S. R_x is a bfext: from this obtain a new bfext R'_x by replacing each node r of R_x with the pair $(\{r\}, R_x)$. R'_x is isomorphic not to R_x but to R^ι_x. Now all the R'_x's are disjoint, so we can take the union of all of the R'_x's and add a new top element, then carry out an "extensional collapse" (needed because the disjoint union will not be extensional), to obtain a bfext with immediate components belonging to each $T(x)$ with $x \in S$. The isomorphism type of this bfext will be s. □

Observation 5.3. *The previous theorems give us extensionality and a comprehension principle for the "set theory" with E as membership. Notice that any formula in E and membership defines a subset of Z (there is no analogue to a stratification requirement, because E is a set relation), but that we do not succumb to the paradoxes of naive set theory because we only have full "comprehension" for collections of objects $T(x)$ (though there are elements of Z in the domain of E which are not images under T). Note that there will be an element z which represents the entire collection $T``Z$ of elements $T(x) \in Z$. It cannot be the case that $z\,E\,z$, because E is well-founded. So $z \neq T(z)$, because $T(z)\,E\,z$*

by definition of z. Thus we also have $T^{n+1}(z)\, E\, T^n(z)$ for each concrete n, but this does not give us ill-foundedness of E because the "descending sequence" in E is not a set (note that its definition is unstratified). This is analogous to the "descending sequence" of $T^n(\Omega)$'s in the ordinals.

Observation 5.4. *It is possible to define the analogue of the hierarchy V_α of ranks in the usual set theory in the set theory of E on Z. For any subset S of Z, define $P(S)$ as the collection of all representatives in Z of subsets of S (it does not have to be the case that all subsets of S are actually represented by elements of Z). Define H as the intersection of all sets of subsets of Z which are closed under P and under unions of their subsets. The elements of H will be analogous to the ranks V_α in the "set theory" of E on Z. The union of H will be an element of H, necessarily the largest, and in fact is provably simply Z. $P(Z) = Z$ does not contain representatives of all subsets of Z. We define Z_0 as the first rank (H is well-ordered by inclusion) for which it is not the case that Z (and so $P(Z_0)$) contains representatives of all subsets of Z_0. Note that $Z_0 \neq T``Z_0$; this would be obvious if the latter were not a rank, so suppose that $T``Z_0$ is a rank (this is actually a theorem): every subset S of $T``Z_0$ is represented by an element of Z, by applying our "comprehension" theorem to the set $T^{-1}(S) \subseteq Z_0$. We know that not all subsets of Z_0 are represented by elements of Z, so these two ranks cannot be the same.*

Theorem 5.4. *Define a complete rank as an element of H all of whose subsets are represented by elements of Z. The image of a complete rank under T is a complete rank (and the inverse of the first incomplete rank Z_0 under T is also a complete rank).*

The proof is omitted; it is found in the errata sheet for [12] (it was accidentally omitted from the book).

Observation 5.5. *There are various ways to interpret fragments of the usual set theory in the set theory of E on Z; details are found elsewhere. More important to us is that there is a way to interpret* NFU *itself in the set theory of E on Z: define $x \epsilon y$ as $T(x)\, E\, y \wedge (\forall z\, E\, y.(\exists u.T(u) = z))$, and the resulting theory on Z will satisfy the axioms of* NFU*. Details are found in [12] (the domain of the interpretation described there is Z_0, but the idea is similar). The proof that this gives an interpretation of* NFU *is very similar to the verification of the model construction for* NFU *using an external automorphism of a model of set theory given above.*

The above is a review of basic properties of the set Z of set pictures and the "membership relation" E and external endomorphism T sufficient for the needs of this paper. A full exposition is found in [12].

The main result of this section is

Theorem 5.5. *(*NFU *+ Choice, due to Holmes) Any invariant sentence ϕ (i.e., any sentence ϕ such that $(\forall \pi.\phi \equiv \phi^\pi)$) is equivalent in a uniform way to a sentence about elements of Z in the language with primitive predicates*

E and T: all invariant sentences are assertions about set pictures involving the "membership relation" and the endomorphism on the collection of all set pictures induced by the singleton operation.

Proof. The strategy is to construct an interpretation of *NFU* in Z which will satisfy the same invariant sentences as the ambient *NFU*. The theorem follows because all assertions in the language of *NFU* in the interpretation of *NFU* in Z will translate into sentences involving only the predicate E and the external function T.

Let W be a well-ordering of the universe with a last element. W is a bfext; there are $T^2(|V|)$ order types of initial segments of W, so there are at least $T^2(|V|)$ elements in Z. Let H_1 be the first rank in H which is at least as large as ι^2"V. It is demonstrable that H_1 is a complete rank. The cardinality of any complete successor rank is the image of the cardinality of the preceding rank under the exponential map exp; the cardinality of the first incomplete rank is the first cardinal of a rank to have no image under exp, and it is a theorem of Holmes (in [5]) that there are at least n iterated images under exp of $T(|V|)$ (and so of $T^2(|V|)$) for each concrete natural number n in *NFU* + Choice, from which it follows by well-known facts about cardinal arithmetic with choice that $\exp(|H_1|)$ is defined. Note that if H_1 is strictly larger than ι^2"V then it must be a successor rank: in this case use H_1^- to denote the predecessor of H_1. Define H_1^* as H_1 if the cardinality of H_1 is $T^2(|V|)$, and otherwise define H_1^* as an arbitrary subset of H_1 which is a superset of H_1^- and has cardinality $T^2(|V|)$. Note that H_1^* is defined in such a way as to be downward closed under E (its relation to ranks ensures this). It is useful to note that T"$H_1^- \subseteq T$"$H_1^* \subseteq T$"H_1: the image of a complete rank under T is always a rank, and T"H_1 is a rank lower than H_1 by considerations of cardinality ($T^3(|V|) < T^2(|V|)$) and easily shown to be lower than H_1^- as well, so T"$H_1^* \subseteq H_1^*$. Choose a bijection F from H_1^* to ι^2"V, with the stipulation that $P(T$"$H_1^*)$ (which can also be proved to be contained in H_1^*) be mapped precisely onto the double singletons of sets (the cardinality of T"H_1^* is $T^3(|V|)$, so the cardinality of $P(T$"$H_1^*)$ is the image under exp of $T^3(|V|)$, which is $T^2(|\mathcal{P}(V)|)$, the cardinality of the set of double singletons of sets).

Define an interpretation of *NFU* on H_1^* as follows. $x \epsilon y$ is defined as $T(x)\, E\, y \wedge (\forall z. z\, E\, y \rightarrow z \in T$"$H_1^*)$. This yields an interpretation of *NFU* for the same reason that the interpretation described above does so (see the proof in [12]).

Now we demonstrate that the set theory of ϵ on H_1^* is equivalent to a permutation interpretation of the ambient *NFU*. We define a permutation on the universe which transforms it into the set theory of ϵ. Each double singleton $\{\{s\}\}$ is mapped by F^{-1} to an element s_2 of $P(T$"$H_1^*)$; the sole elements of elements of the images under F of the inverse images under T of the "elements" (in the E sense) of s_2 are the elements of the set $s_3 = \pi(s)$: $s_3 = \bigcup^2(F$"$(T^{-1}$"$\{x \mid x\, E\, F^{-1}(\{\{s\}\})\}))$ It is straightforward to verify that the function π thus defined is actually a set bijection, and that $x \epsilon y$ is equivalent to $\bigcup^2 F(x) \in \pi(\bigcup^2 F(y))$, which establishes that the theory of the ambient *NFU* is the same as the theory of ϵ.

The only apparent obstruction to translation of sentences in the language of ϵ into the language of E and T is the status of the special set H_1^*. The particular choice of H_1^* has no effect on the truth values of invariant sentences (if we used an alternative $H_1^{*'}$, the fact that there is a bijection between H_1^* and $H_1^{*'}$ can be exploited to get a permutation model of the structure built using $H_1^{*'}$ which has the same theory as the structure built using H_1^*, which establishes that they agree about invariant assertions), so references to a specific H_1^* can be replaced by universal quantification over all subsets of Z satisfying a certain condition. Further, we note that there is an element h of Z which represents $T``H_1^*$, so formulas $x \in H_1^*$ can be replaced by formulas $T(x)\, E\, h$, eliminating reference to subsets of Z. This technique for eliminating reference to subsets of Z can be adapted to define ranks in the language of T and E, and the cardinality of the universe can be defined as the largest possible cardinality of an "extension" (the set theory of E has an adequate internal treatment of cardinality): thus, we can translate the required conditions on the set H_1^* into conditions on an $h \in Z$ expressible in the language of E and T. Everything else in the interpretation of any sentence ϕ in the language of ϵ translates straightforwardly into the language of E and T. Each ϕ which is invariant will be demonstrably equivalent to the translation of ϕ first to ϵ then into terms of E and T. So the assertions invariant under set permutations are uniformly translatable into assertions about the "set theory with external endomorphism" on the isomorphism types of set pictures.

It should be noted further that all assertions in the language of E and T on Z are invariant (E has a stratified definition, so assertions about it are clearly invariant, while invariance of T holds for reasons analogous to those for the invariance of the T operation on natural numbers demonstrated above), so we have in some sense precisely characterized the invariant assertions. Moreover, it seems that the truth values of non-invariant sentences ϕ have preferred truth values, namely the truth values they have when translated into the language of ϵ. It could be of interest to extend the modal logic proposed by the first author to include an "actuality" operator $\circ\phi$ defined as the truth value of the translation of ϕ into the interpretation in ϵ, since we seem to have a preferred "possible world".

The proof is complete. The argument depends in several places on the availability of the Axiom of Choice, so this does not apply to *NF*.

□

6 V_ω and AxCount$_\leq$

A weaker variant of the Axiom of Counting is

Axiom 6.1. *(AxCount$_\leq$) For all natural numbers n, $n \leq T(n)$.*

Observation 6.1. *AxCount$_\leq$ is equivalent to the assertion that $|\{1, \ldots, n\}| \geq n$ for all natural numbers n.*

Adjoining AxCount$_\leq$ to bare *NFU* allows the proof of Infinity (because

$T(|V|) < |V|$ are natural numbers in the absence of infinity). In fact, like AxCount, AxCount$_{\leq}$ strengthens even *NFU* + Infinity essentially.

We repeat the definition of V_ω.

Definition 6.1. *V_ω is defined as the intersection of all sets which contain all of their finite subsets. Note that V_ω is not obviously a set, as its definition is unstratified.*

Theorem 6.1. *AxCount$_{\leq}$ implies $\Diamond$(V_ω is a set).*

Proof. Consider the class of all set pictures (isomorphism classes of well-founded extensional relations with top) whose representatives have finite domain and range: we briefly call these "finite set pictures" (they are actually pictures of *hereditarily* finite sets). The T operation on set pictures preserves finiteness, and the finite set pictures inherit the "membership relation" E. The permutation π we use will interchange each finite set picture s with $\{T^{-1}t | t\,E\,s\}$, the elementwise image under T^{-1} of the set of preimages of the set picture under the "membership" relation E. The use of T^{-1} is required to make the permutation a set. (The Ackermann permutation on natural numbers defined in the proof of Theorem 3.6 can also be used to prove this result).

It should be clear that each concrete element of V_ω will be represented in V^π by the isomorphism class of the membership relation on its transitive closure. We claim that the set of all finite set pictures of the original interpretation will be the set V_ω in the permutation interpretation.

The set of finite set pictures of the old interpretation contains all of its finite subsets in the sense of V^π. To show this, it is necessary to observe that the T operation is onto the set of finite set pictures: thus each finite set of finite set pictures is the elementwise image under T^{-1} of the preimage under E of some finite set picture, and so the extension under the new interpretation of that old finite set picture. So we can restrict our attention to collections of old finite set pictures which contain all of their own finite subsets as elements in the permutation interpretation. Our goal is to show that any such collection contains all old finite set pictures as elements in the sense of V^π.

Suppose that such a set A did not contain all old finite set pictures as elements in V^π. Then there would be an old set picture s of minimal rank which did not belong to A in the new interpretation. The elements of s in V^π would be the images under T^{-1} of the "elements" of s in the sense of the "membership relation" E in the original interpretation. The operation T^{-1} can only lower rank (by AxCount$_{\leq}$), so the finitely many images under T^{-1} of the "elements" of the supposedly missing set picture s are all elements of A (in the V^π) by minimality of the rank of s, so the supposedly missing set picture s is a finite subset of A in V^π, and so an element of A in V^π, a contradiction.

□

It is not known whether the converse of this theorem is true (whether the possible existence of V_ω implies AxCount$_{\leq}$). The first author has shown the precise equivalence of AxCount$_{\leq}$ with the existence of a related set.

Definition 6.2. *The* finite rank function $(n \mapsto V_n)$ *is defined as the intersection of all ordered pairs which contain* $\langle 0, \emptyset \rangle$ *and are closed under the operation* $\langle n, A \rangle \mapsto \langle n+1, \mathcal{P}(A) \rangle$. *Note that the finite rank function is not necessarily a set, since its definition is not stratified.*

Observation 6.2. *$AxCount_{\leq}$ implies $\Diamond$(the finite rank function is a set). This is easy: in the permutation interpretation defined in the last proof, consider the obvious function from natural numbers to "pictures" of corresponding ranks.*

Observation 6.3. *If the finite rank function is a set, then V_ω is a set. This is also easy: the union of the range of the finite rank function obviously contains all of its finite subsets (each element of any finite subset belongs to some V_n: the whole finite subset will belong to V_{m+1} where m is the largest index associated with any of its elements). A subset of the union of the range of the finite rank function which contains all of its finite subsets can be shown to contain all of the V_n's by induction and so all of their subsets, and so must be the entire union.*

Lemma 6.1. *The existence of the finite rank function implies $AxCount_{\leq}$.*

Proof. Suppose that $T(n) < n$ for some natural number n. It follows that for any set A of size n, the power set of A is of size $2^{T(n)} < n$. It follows that in the permutation interpretation using finite set pictures, the set of finite set pictures of rank $\leq n$ in the old sense is closed under the power set operation. This would imply further that the finite rank function would be contained in this finite set, which is not possible if the finite rank function is a set. □

Theorem 6.2. *$AxCount_{\leq}$ is equivalent to $\Diamond$(the finite rank function is a set).*

Proof. We have already seen the forward implication. The converse implication follows from the immediately previous Lemma combined with the observation already made that the T operation, and so the truth of $\text{AxCount}_{\leq}$, is preserved in all permutation interpretations. Thus $\Diamond$(the finite rank function is a set) implies $\Diamond(\text{AxCount}_{\leq})$ which implies $\text{AxCount}_{\leq}$.

□

Theorem 6.3. *(Forster) $AxCount_{\leq}$ is equivalent to the assertion that there is a permutation π such that the permutation interpretation using π contains inductive closures of $\{\emptyset\}$ under F for all class (not necessarily set!) functions F with definitions $F(x) = \{z \in \mathcal{P}^i(x) \mid \phi\}$, where ϕ is parameter-free and belongs to the class $\Delta_0^{\mathcal{P}}$ (formulas with all quantifiers bounded to sets, but allowing the power-set operation as a function symbol) and which have $y = F(x)$ a stratified formula (notice that this does not imply that F is a set function unless the relative types of x and y are the same).*

Proof. One direction is handled by the result already established that the existence of the finite rank function implies $\text{AxCount}_{\leq}$, since the finite rank function is the inductive closure of $\{\emptyset\}$ under the stratified but inhomogeneous power set function.

In the other direction, we once again consider the Ackermann permutation, which under AxCount$_\leq$ converts the old set of natural numbers into V_ω (we have shown above that it converts $\mathcal{N}$ to V_ω under the stronger assumption AxCount). It must be noted that it is not n but $T(n)$ that is swapped with $\{m \mid$ "the mth binary digit of n is 1"$\}$, due to stratification requirements (this wasn't an issue in the earlier proof since the Axiom of Counting was assumed). Since $T(n) \geq n$, the essential condition that each old natural number becomes a collection of smaller old natural numbers and every finite set of old natural numbers is coded by a larger old natural number is preserved: this is sufficient to see that old $\mathcal{N}$ becomes V_ω.

The function F can be converted to a set function F^*: $F \circ T^n = F^*$ for some (possibly negative) integer n (n is chosen so that x and y will have the same type in $y = F(T^n(x))$). Observe in addition that F^* will commute with T ($F^*(T(x)) = T(F^*(x))$, because F^* is definable in a nice sense and T is an automorphism of the natural numbers). Further, V_ω is closed under F and under F^* because F is defined by a $\Delta_0^{\mathcal{P}}$ formula and V_ω is transitive and contains all power sets of its elements. We claim that the inductive closure of $\{\emptyset\}$ under F^*, which exists because F^* is a set function, is the desired set. Note first that because F^* commutes with T, the inductive closure we have defined will be closed under F as well as under F^*. What remains to be proved is that this set is the smallest set containing $\emptyset$ and closed under F. There is a map G such that $G(0) = \emptyset$ and $G(n+1) = F^*(G(n))$ for all $n \in \mathcal{N}$. The domain of G is the natural numbers and the range is the inductive closure of $\{\emptyset\}$ under F^*. Now suppose that some set containing $\emptyset$ and closed under F is not a superset of this inductive closure. Then there will be a minimal i such that $G(i)$ does not belong to this set. Now $F(G(i-1))$, by hypothesis, does belong to the set: this is $F^*(G(T^{-n}(i-1))) = G(T^{-n}(i))$. Observe that this implies immediately that $i \neq T(i)$. Further, $F^{-1}(G(i)) = G(T^n(i-1))$ cannot belong to the set, so we have $T^n(i-1) > i$, so $n > 0$. But now we see that $F(G(i)) = G(T^{-n}(i+1))$ is in the set after all, because $T^{-n}(i+1)$ must be less than i. This argument can be adapted to closures under functions with more than one argument and to functions whose definitions include parameters taken from V_ω.

This completes the proof. □

We now establish that AxCount implies that the existence of the set of all hereditarily countable sets is "possible" (holds in some permutation interpretation). We do not know whether AxCount$_\leq$ is sufficient.

In [13], Jech showed (in *ZF*) that the rank of any hereditarily countable set is less than ω_2. We transfer this result into the realm of set pictures. Note that choice is not needed, so this result works in *NF* as well.

Definition 6.3. *For any set picture x, define $\rho(x)$ as the least upper bound in the ordinals of the set $\{\rho(y)+1 \mid y\, E\, x\}$. This definition works because E is well-founded.*

Definition 6.4. *Define $H_{\aleph_1}$, the set of hereditarily countable set pictures, as the set of set pictures $[R]$ such that no element of the range of R has an uncountable* *preimage.*

Theorem 6.4. *(Jech ([13], translated into terms of set pictures) The rank of each hereditarily countable set picture is below* ω_2*.*

Proof. We want to show that all set pictures in $H_{\aleph_1}$ are of rank $< \omega_2$

Let Ω be the set of all nonempty finite sequences of countable ordinals. A finite sequence is represented here as a function s whose domain is a proper initial segment of $\mathcal{N}$: s_n is notation for $s(n)$. For any sequence s, we define $s_{\texttt{last}}$ as s_n, where n is the largest element of the domain of s, and we define s^- as $s - \{\langle n, s_n\rangle\}$, the result of dropping the last element of the sequence s.

We define a function F. The domain of F is $H_{\aleph_1} \times \Omega$. For a fixed $S \in H_{\aleph_1}$, we will use the notation $F_S(s) = F(S, s)$.

We define $F(S, s)$, when s is a sequence of length 1 with $s_0 = \alpha$, as the $T(\alpha)$th element of the elementwise image under ρ of the preimage of S under E, if this exists, and 0 otherwise (the αth element of a set of ordinals A is defined as the element β of A such that the natural order on the elements of A less than β has order type $T^2(\alpha)$ (note that this is the order type of the ordinals less than α)). The use of $T(\alpha)$ ensures that the relative type of $F(S, s)$ is the same as that of its arguments S and s. Because the preimage of S is countable, and because the T operation maps countable ordinals onto countable ordinals (though it may move some of them if AxCount does not hold) it follows that F_S maps the set of sequences of length 1 onto the set of ranks of preimages of S under E.

When s has length greater than 1, we define $F(S, s)$ as the $T(s_{\texttt{last}})$-th element of the elementwise image under $(\lambda x.F(x, s^-))$ of the preimage of S under E. Thus, for example, the image under F_S of the set of sequences of countable ordinals of length 2 maps onto the set of ranks of elements of the preimage of S under $E \circ E$. It is straightforward to prove by induction that F_S maps the set of sequences of length n onto the preimage of S under the relation E^n (this might be $E^{T^i(n)}$ for some fixed i depending on the details of the definition of E^n (the issue is what type is assigned to the superscript); an advantage of AxCount is that such details can be ignored).

The function F_S defined above maps Ω onto the elementwise image under ρ of the set of all iterated preimages of elements of S under E (the "transitive closure" of S in E). The range of F_S will be the set of all ordinals less than the rank of S: but the domain of $F(S)$ is the image of Ω, which has cardinality $\aleph_1$, so the rank of S is less than ω_2.

□

The translated Jech result shows that the rank of any element of $H_{\aleph_1}$ is less than ω_2, and that the cardinality of any representative of a set picture in $H_{\aleph_1}$ is less than $\beth_{\omega_2}$ (admittedly not a very tight bound on cardinality!).

We define a permutation on $H_{\aleph_1}$ which will convert it into the class HC of hereditarily countable sets, defined as the intersection of all sets which contain all their countable subsets. The desired permutation π interchanges each $T(S)$ for $S \in H_{\aleph_1}$ with $\{x \mid x\,E\,S\}$. It is important to note that the action of T

on $H_{\aleph_1}$ is bijective. Thus each element of the old $H_{\aleph_1}$ becomes a countable set of elements of the old $H_{\aleph_1}$, and each countable subset of the old $H_{\aleph_1}$ is the new extension of some element of the old $H_{\aleph_1}$. The old $H_{\aleph_1}$ is thus a set which coincides with the collection of all its countable subsets in the new interpretation V^π. Further, it is the intersection of all such sets. Let A be a set which contains all of its countable subsets in V^π. Suppose that S is an element of minimal rank in the old $H_{\aleph_1}$ which is not an element of A. Let α be the rank of S. The elements of S, a countable set of the new interpretation, will be the elements of the preimage under E of $T^{-1}(S)$ which all have ranks less than the rank of $T^{-1}(S)$. Here is where we use AxCount. It is a theorem that the successor of any strongly cantorian cardinal is strongly cantorian, and of course order types of well-orderings of strongly cantorian sets are strongly cantorian. It follows from this that ω_2 is a strongly cantorian ordinal (and so are all smaller ordinals). This means that the rank of $T^{-1}(S)$ is the same as the rank of S (since it is an ordinal less than ω_2). This implies that all elements in the new sense of S belong to A, and so it is impossible that $S \notin A$, which completes the proof. We would like to be able to assert that this works for AxCount$_\leq$ as well, but it does not seem easy to prove that the image of a countable ordinal (such as the rank of S) under T is necessarily less than that ordinal under AxCount$_\leq$.

Question 6.1. *Does AxCount$_\leq$, the assertion that for all natural numbers n, $n \leq T(n)$, imply the assertion that for all countable ordinals α, $\alpha \leq T(\alpha)$?*

This is not an esoteric *NF* question: it is a question about automorphisms of nonstandard models of the countable ordinals.

7 Eliminating the set of von Neumann numerals

A question left open by the first author in his discussion in [5] of the relationship between the Axiom of Counting and the existence of the set of von Neumann numerals was the question as to whether it is possible to eliminate the set of von Neumann numerals if it actually does exist.

The second author has answered this question: in this section we describe a permutation χ such that $\neg(vN \text{ exists})^\chi$.

It seems to be a good idea to start by describing the motivation behind the construction of this permutation. The idea is define χ in such a way as to cause the von Neumann numerals to be interpreted in V^χ by a descending sequence of ordinals of the original model, which of course cannot be a set. This means that each von Neumann numeral of V^χ will be an ordinal, and the permutation will send ordinals to sets of ordinals (and presumably economically swap appropriate sets of ordinals back to ordinals).

The von Neumann ordinal 0^χ will be the largest of the ordinals Ω_i considered, and clearly a nonstandard one. It will be sent by χ to the empty set. The von Neumann numeral $1^\chi = \Omega_1$ will be the second largest of the ordinals considered, and will be sent by χ to $\{\Omega_0\}$. The von Neumann ordinal 2^χ will be sent to $\{\Omega_0, \Omega_1\}$, and so forth.

What the second author noticed about this was that finite sets needed to be preceded by "downward extensions" of the same finite sets in the order on finite sets of ordinals induced by the map χ. This caused him to consider a particular order on finite sets of ordinals.

The order $\prec$ on finite sets of ordinals which we consider is as follows: $\{0\}$ is the least element in the order. When all sets whose maximum element is less than α have been ordered, we extend the order to sets with maximum element α as follows: $\{\alpha\}$ is largest; all sets containing α are larger than all sets not containing α; where A and B have maximum less than α (and so the order between them has already been defined) $A \prec B \equiv A \cup \{\alpha\} \prec B \cup \{\alpha\}$. If the order on the sets with maximum element less than α was a well-ordering, we see that the order on sets with maximum less than or equal to α is also a well-ordering, because it is made of two isomorphic copies of the previously given order plus an additional element at the top. By transfinite induction, this process defines a well-ordering on all nonempty finite sets of ordinals; we extend it to $\emptyset$ by stipulating $A \prec \emptyset$ for all nonempty finite sets of ordinals A.

This is a modification of colex order on finite sets of ordinals, making the empty set last and putting "tails" of nonempty sets after those sets rather than before them.

There is a unique injective map F from an initial segment of the ordinals onto the finite sets of ordinals such that $\alpha < \beta \rightarrow F(\alpha) \prec F(\beta)$ for all ordinals α, β. To see that this is true, we point out that it is straightforward to show by induction that if the cardinality of the set of ordinals up to α is κ, the cardinality of the set of finite sets of ordinals with maximum ordinal α is $T(\kappa)$. Since $T(\kappa) < \kappa$ for the largest such cardinals, we will not run out of ordinals.

The permutation χ will send each ordinal α in the domain of F to $F(\alpha)$, each finite set of ordinals A to $F^{-1}(A)$, and fix all other objects.

Let Ω_0 be the ordinal corresponding to the empty set (the last set in the order on finite sets of ordinals). Clearly this will be the empty set of the permutation interpretation.

Some Ω_1 corresponds to $\{\Omega_0\}$; this will be the von Neumann numeral 1 of the permutation interpretation, and clearly $\Omega_1 < \Omega_0$. For each concrete natural number n, suppose that the von Neumann numerals for $m < n$ are coded by a decreasing sequence of ordinals Ω_m, each corresponding in the coding to the set $\{\Omega_p \mid p < m\}$. The von Neumann numeral $n + 1$ will then be coded by the ordinal Ω_{n+1} corresponding to the set $\{\Omega_m \mid m \leq n\}$, and since this is a downward extension of the finite set of ordinals $\chi(\Omega_n)$, we have $\Omega_{n+1} < \Omega_n$. By induction, the concrete von Neumann numerals correspond to a decreasing sequence in the ordinals, and so cannot make up a set.

However, we have not yet shown that vN is not a set, because the concept of "concrete von Neumann numeral" is not really captured by vN. We give the proof.

Theorem 7.1. *vN is not a set in V_χ.*

Proof. Suppose otherwise for the sake of a contradiction. Observe first that the set of old ordinals α which are moved by χ is clearly von Neumann inductive in V^χ: we have already seen that the empty set of V_χ is an old ordinal, and if α is an old ordinal, α^+ in the sense of V_χ will be $\chi(\chi(\alpha) \cup \{\alpha\})$ in the sense of the old interpretation, which is also an old ordinal. Thus the set vN in V_χ, if there is one, is a set of old ordinals. It must contain a minimal old ordinal element α, coding a finite set $A = \chi(\alpha)$ of old ordinals. Clearly α does not code the empty set (we have seen above that the old ordinal coding $\{\emptyset\}$ is smaller than the old ordinal coding $\emptyset$), so $\alpha = \beta^+$ in V^χ for some $\beta \in vN$. We know that $\beta \neq \alpha$, because no von Neumann ordinal is self-membered (we showed this earlier in this paper). It follows that $\beta > \alpha$ as an old ordinal. We have $A = \chi(\alpha) = \chi(\beta) \cup \{\beta\}$. Since $\beta > \alpha$, we must have $\chi(\alpha) = \chi(\beta) \cup \{\beta\} \prec \chi(\beta)$, from which it follows that β must be less than any element of $\chi(\beta)$ (otherwise the definition of $\prec$ would force the other order). Now consider α^+ (in the sense of V^χ: $\chi(\alpha) \cup \{\alpha\}$ in the original model): we see that α is less (as an old ordinal) than β or any element of $\chi(\beta)$, so α^+ is a downward extension of A, from which it follows that $\chi(\alpha^+) < \alpha$, which is a contradiction. Thus vN cannot be a set. □

We can show yet more.

Theorem 7.2. *The class of Zermelo naturals is not a set in V_χ.*

Proof. First note that no self-singleton $q = \{q\}$ can be a Zermelo natural, because the set of all sets which do not contain q as an element contains $\emptyset$ and is closed under the singleton operation. Now suppose that Z is a set in V_χ. It is a set of old ordinals, because the empty set of V_χ is an old ordinal and the singleton in the sense of V_χ of an old ordinal α is $\chi(\{\alpha\})$, which is also an old ordinal. There is a smallest old ordinal in Z: call it α. α is clearly not $\emptyset$, so $\alpha = \{\beta\}$ for some $\beta \in Z$. Further, β is an old ordinal itself. β cannot be equal to α because a Zermelo natural cannot be a self-singleton, so we have $\alpha < \beta$ as ordinals in the old interpretation. Now the set $\{\alpha\}$ in the sense of V_χ is also an old ordinal δ and a Zermelo natural in the new interpretation, and because $\alpha < \beta$ we have $\{\alpha\} \prec \{\beta\}$, or $\chi(\delta) \prec \chi(\alpha)$, whence $\delta < \alpha$, contrary to the choice of α. □

Theorem 7.3. *V_ω is not a set in V_χ.*

Proof. Suppose otherwise. Every finite set of old ordinals in the sense of V_χ is an old ordinal, so the set of old ordinals contains all its finite subsets as elements and so includes V_ω as a subset (all in the sense of V_χ). Thus there is a minimal old ordinal α (in the order on ordinals of the old interpretation) which belongs to V_ω in the new interpretation. Now the set $\{\alpha\}$ of the new interpretation is also an element of V_ω and so is an old ordinal β which must be greater than α by minimality of α. Thus we have (in the original interpretation) $\chi(\alpha) \prec \{\alpha\}$, from which it follows that the supremum of $\chi(\alpha)$ must be $\leq \alpha$ (and so exactly α, by minimality of α and transitivity of V_ω). But then $\chi(\alpha)$ needs to be a proper downward extension of $\{\alpha\}$, which contradicts minimality of α. □

We define the notion "well-founded set" in a way appropriate to the context of *NF(U)*, and formulate a conjecture.

Definition 7.1. *A set is said to be* well-founded *iff it belongs to every set A such that $\mathcal{P}(A) \subseteq A$. This definition supports induction on membership for stratified conditions.*

It is a question originally asked by Maurice Boffa whether there is a permutation that eliminates infinite well-founded sets. We conjecture that there are no infinite well-founded sets in the permutation model V_χ introduced here, but we do not yet see how to establish this. A first step toward proving this would be to show that there are no infinite sub*sets* of the class V_ω in V_χ, which would fairly immediately imply that there are no infinite well-founded *transitive* sets in V_χ, but we do not yet know how to show this either. So, we close with these conjectures:

Conjecture 7.1. *In the permutation model V_χ, there is no infinite set which is a subset of the (proper) class V_ω.*

Conjecture 7.2. *In the permutation model V_χ, there is no infinite well-founded set.*

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