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Symmetry as a criterion for comprehension motivating Quine's "New Foundations"

Abstract. A common objection to Quine's set theory "New Foundations" is that it is inadequately motivated because the restriction on comprehension which appears to avert paradox is a syntactical trick. We present a semantic criterion for determining whether a class is a set (a kind of symmetry) which motivates *NF*.

Keywords: New Foundations, symmetry, permutation methods

1. Introduction

A common criticism of Quine's set theory "New Foundations" (from the name of the paper [9] in which it was introduced, hereinafter *NF*) is that it is motivated by a mere "syntactical trick". In this paper, we present a different motivation for *NF* involving a criterion for sethood which is not syntactical in nature.

In this section, we briefly introduce *NF* in the usual way and review the known difficulties with this theory, then present the motivation for the general approach we take. In the second section, we discuss obstructions to the implementation of this criterion; in the third section we present the formalized implementation.

The first and third sections should be accessible to the reader unfamiliar with *NF*; in the second section the reader may find that we presume familiarity with prior research in *NF*. The first and second sections do not as a rule contain proofs (and may presume familiarity with some definitions), but proofs of assertions made in these sections and definitions of concepts used will often be found in the third section.

NF is a first-order theory with equality and membership as primitive relations. The axioms are as follows:

Axiom 1.1. (Axiom of Extensionality) Objects with the same elements are the same: $(\forall AB.A = B \equiv (\forall x.x \in A \equiv x \in B))$.

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Definition 1.2. A formula ϕ is said to be *stratified* iff there is a function σ from variables (considered as items of syntax) to natural numbers such that for every subformula '$x = y$' of ϕ we have $\sigma('x') = \sigma('y')$, and for every subformula '$x \in y$' of ϕ we have $\sigma('x') + 1 = \sigma('y')$.

Axiom 1.3. (Axiom Scheme of Stratified Comprehension) For each stratified formula ϕ, let A be a variable not free in ϕ: we have an axiom $(\exists A.(\forall x.x \in A \equiv \phi))$. Extensionality tells us that there is only one such object A, for which we may use the notation $\{x \mid \phi\}$.

Comments 1.4. The machinery of stratification is optional: the axiom scheme of stratified comprehension is equivalent to the conjunction of a finite collection of its instances (so *NF* is finitely axiomatizable). The standard reference for this is [6] though this is far from the best finite axiomatization. It is more convenient to use this criterion in practice, and of course this is how the theory was originally defined. It is easy to show that the criterion can be relaxed to require that the conditions on σ only apply to atomic subformulas in which both variables are bound in the set abstract (this is the criterion of "weak stratification").

The stratification criterion can be described more economically yet: a set $\{x \mid \phi\}$ is provided by stratified comprehension just in case it is possible to assign types to the variables in ϕ in such a way as to obtain a well-formed formula of a simple typed theory of sets TST which we now describe.

TST is a first-order multisorted theory with primitive relations equality and membership and sorts ("types") indexed by the nonnegative integers. Informally, 0 is a type of "individuals" and $n+1$ is the type of sets of objects of type n. This is enforced by the type conditions on equations ($x^n = y^n$) and membership statements ($x^n \in y^{n+1}$). The axioms of TST are extensionality (as in *NF* in appropriate types) and comprehension ($\{x^n \mid \phi\}^{n+1}$ exists for any formula ϕ). The axioms of *NF* are exactly the axioms of TST with all indications of type removed (as long as this creates no identifications of variables of different types).

TST is obtained by considerable streamlining of Russell's theory of types of *PM*: one first applies Ramsey's elimination of orders (obtaining an impredicative theory of sets and relations) then uses the Kuratowski pair to define relations as sets. Quine observed that TST has a considerable degree of polymorphism: for any formula ϕ, define ϕ^+ as the formula obtained by incrementing each type index in ϕ: then ϕ is a theorem if and only if ϕ^+ is a theorem, and for each object $\{x^n \mid \phi\}^{n+1}$ that we can define, there is an exactly analogous object $\{x^{n+1} \mid \phi^+\}^{n+2}$ in the next higher type, and analogous objects in each higher type. Quine suggested on the basis of these observations that the types could be identified: it has been shown since by Specker that *NF* is equivalent in consistency strength to TST plus the axiom scheme $\phi \equiv \phi^+$, and also that the consistency of *NF* is equivalent to the existence of a model of TST in which the types are isomorphic to one another in a suitably defined sense.

A discussion of the historical development of the type theory TST is found in [14].

It should be clear that this procedure is suspect. The motivation for the criterion of sethood is entirely syntactical and quite innocent of any suggestion as to what a model would be like. No one has been able to come up with a proof of consistency of *NF* in the more than sixty years since the theory was proposed. No one has come up with a proof of inconsistency, either (contrary to rumors apparently based on the inconsistency found in the first edition of Quine's *Mathematical Logic* (the second edition, [10], discusses the error and gives an apparently successful correction), which derived from an overenthusiastic attempt to extend *NF* with proper classes). A disturbing result of Specker is that *NF* disproves the axiom of choice (see [12]).

There are subtheories of *NF* which are known to be consistent. *NFU* (New Foundations with urelements) differs from *NF* simply in restricting extensionality to objects with elements (it is usual to let one of the elementless objects be the empty set $\emptyset$ and refer to the other elementless objects as urelements). Jensen defined this theory in [7] and showed that it is consistent with Infinity and Choice, and that it has ω-models. (*NF* proves Infinity, because if the universe were finite it could be well-ordered and choice would hold). *NFI*, defined and shown to be consistent by Marcel Crabbé in [2], differs from *NF* in allowing only those sets $\{x \mid \phi\}$ for stratified formulas ϕ in which no variable is assigned a type higher than that of x (this is a predicativity restriction, though *NFI* does allow certain impredicative set definitions). NF_3, defined and shown to be consistent by Grishin in [5], allows those sets whose definitions can be typed using three types. The latter two theories have full extensionality.

NFU is a serviceable foundation for mathematics, with a hierarchy of extensions analogous to that of ZFC. The other two theories are probably best regarded as curiosities, though a case could be made for the use of *NFI* (or its subset *NFP* which allows only "predicative" set definitions) as weak foundational systems, and NF_3 has a quite interesting relation to the model theory of TST.

We now present the alternative motivation for *NF* which is the subject of this paper. The observation with which we start is that any definable set in TST is invariant under permutations of type 0. Any set definable in NF is analogous to a set defined in TST at some type n (and at each type above n): this set will be invariant under permutations of its n-fold iterated elements.

We introduce some terminology, taken from [3] (definition of j, page 8). A bijection from the universe to itself is called a permutation of the universe (or just a permutation). With any permutation π of the universe, we can associate a permutation $j(\pi)$ defined by $j(\pi)(A) = \pi``A$ for any set A. We call a set A "n-symmetric" if $j^n(\pi)(A) = A$ for "every" permutation π. We call a set "symmetric" if it is n-symmetric for some n. Now we can formalize our observation of the previous paragraph: every definable type n set of TST is n-symmetric (for each concrete natural number n), and every definable set of NF is n-symmetric for some n.

This suggests that a possible axiom to adjoin to NF is the assertion that every set is symmetric. This would be the case in a model of NF in which every set is an explicit set abstract (a "term model" in a sense proposed by Thomas Forster; this is not quite the same as a term model of NF in the usual logical sense).

An interesting assertion to consider is the converse assertion that every class which is symmetric is a set. Leaving aside difficulties with formalizing this assertion (which will be dealt with in later sections) this would imply each parameter-free instance of stratified comprehension. This is not enough for NF, because we definitely need parameters in set abstracts $\{x \mid \phi\}$. But if every parameter in an instance of stratified comprehension is supposed symmetric, it is straightforward to show that the class defined by this instance is symmetric. So the conjunction of the assertions that every set is symmetric and that every class which is symmetric is a set appears to imply the comprehension scheme of NF.

This motivates our modest proposal, the following

Criterion 1.5. (Symmetry) A class is a set iff it is symmetric (i.e., iff it is n-symmetric for some n).

2. Difficulties and Refinements

In this section, we discuss our Criterion for sethood and refine it to avoid certain difficulties. Formal definitions and proofs are left for the final section.

The Criterion, if it can be made rigorous, is (apparently) semantic in character rather than syntactical. It is analogous to the Limitation of Size criterion for sethood often considered in connection with the usual set theory:

Criterion 2.1. (Limitation of Size):] A class is a set iff it is small (i.e., iff there is no class bijection between it and the universe).

To have such a criterion for sethood which yielded *NF* would be new and interesting.

Any such Criterion, to be useful, must be accompanied by some kind of theory of classes. The Limitation of Size criterion needs nothing more than the axiom of class comprehension,

Axiom 2.2. (Axiom of Class Comprehension) For any formula ϕ, the class of all sets x such that ϕ exists. (Any element of a class is understood to be a set).

To avoid triviality, an axiom asserting the existence of at least two sets is sufficient. If there are at least two sets, then the empty set and the singleton of any set are sets by Limitation of Size. This establishes the existence of at least three distinct sets, which ensures that all unordered pairs exist. Once unordered pairs are available, Kuratowski pairs can be constructed, and any formula with two open variables corresponds to a class relation in the way one expects. The existence of at least two sets is necessary because the empty universe and the universe whose sole element is the empty set both satisfy the Limitation of Size criterion. To get infinite sets one of course needs more axioms, but we are not developing the usual theory of sets and classes here.

The situation with our Criterion is more complex because the criterion of symmetry is logically more complex than the criterion of smallness. We do need to ensure that definable permutations of the universe are classes, so the axioms of Class Comprehension and Pairing seem to be needed (Pairing turns out not to be needed as an explicit axiom in the final formalization, but it is proved as a lemma in the development).

But further we need to be able to define n-symmetry for arbitrary n. This involves assertions about all permutations $j^n(\pi)$ for a given permutation π, which might seem to require a theory of collections of permutations. In fact (as is exhibited in the formalization) it is possible to define a single object which codes all permutations $j^n(f)$.

Nonetheless, it does appear that a higher-order theory, with classes of sets and superclasses of classes, is actually needed to formalize our Criterion. The reasons for this have to do with predicativity of class comprehension axioms.

The axiom of Class Comprehension, when used in conjunction with the Limitation of Size criterion, is usually taken to be "predicative": the formula ϕ is not allowed to contain quantifiers over classes. However, the usual definitions of natural numbers are impredicative, and natural numbers are needed for the definition of symmetry (= "n-symmetry for some n").

We cannot use fully impredicative class comprehension because this would allow the definition of the class of Russell-Whitehead ordinals of genuine well-orderings, which is obviously symmetric and is known not to be a set in *NF* (this was Quine's error in the first edition of *ML*). It is possible that some kind of single class comprehension axiom which allows the definition of the natural numbers but forbids the definition of true well-orderings could be used, but we have chosen to use a development using sets, classes whose elements are sets, and superclasses whose elements are classes, with a predicative axiom of comprehension for classes and an impredicative axiom of comprehension for superclasses. As can be seen in the formalization section, this allows the needed definition of natural numbers and of the single (superclass) structure formalizing all the maps $j^n(\pi)$ for any superclass permutation π.

Further, we need to consider what permutations are quantified over in the definition of symmetry. It is known that considering just set permutations will not work: certain proper classes (such as the class of "strongly cantorian" sets) are known to be symmetric with respect to set permutations. It seems reasonable to use the class of "setlike" permutations introduced by Forster ([3], p. 8): a permutation of the universe is setlike just in case $j^n(f)$ is a permutation for each n (a permutation π can fail to have $j(\pi)$ a permutation if π"A is not a set for some set A). Note that a permutation is necessarily a permutation of the universe of *sets*, since a superclass ordered pair $\{\{a\},\{a,b\}\}$ can only exist for a and b sets. Further, there seems to be no reason to restrict ourselves to setlike class permutations: we consider all setlike superclass permutations.

The use of setlike permutations relates the symmetry criterion for sethood to Rieger-Bernays permutation methods ([11], [1]; [13] is the original reference for use of these methods in *NF*). We note the interesting result of Pétry, Henson and Forster that a sentence ϕ is invariant under Rieger-Bernays permutation methods using arbitrary setlike permutations just in case it is equivalent to a stratified formula ([3], p. 94). We will see in the formalization below that a lemma important in the study of Rieger-Bernays permutation methods is also needed here to verify that stratified comprehension follows from the symmetry criterion for sethood.

The notion "strongly cantorian" has already been seen to cause difficulties. Since the class of strongly cantorian ordinals is not a set and is definitely a class (being definable) there must be a setlike permutation which moves a strongly cantorian ordinal α to a non-strongly cantorian ordinal β. As can be seen in detail in the development in the formalization of the next section, α is not n-symmetric for any n in the most general sense: it is necessary to introduce a notion of "support" to restrict the permutations considered in relation to α so that it achieves the level of symmetry expected of any ordinal. The underlying idea is that in testing the symmetry of α, we should restrict ourselves to permutations in a subset of the world in which α is a standard object, in some sense. In any event, the notion of support introduced appears to work technically. This technical refinement of the definition of symmetry preserves the condition that all sets are symmetric with respect to set permutations of the universe.

The theory developed in the formalization is not just *NF*. The additional consequence that all sets are symmetric is known to be independent of *NF* (Rieger-Bernays permutation methods can be used to construct models of *NF* in which there are non-symmetric sets, given any model of *NF*: for example, a set which is its own singleton is not symmetric, and it is easy to get such sets using permutation methods). The fact that all sets are symmetric relative to set permutations (which holds even when the notion of "support" hinted at in the last paragraph is introduced) makes it obvious that Choice does not hold (so there is no reason to appeal to Specker's rather strange proof of this fact in *NF* proper): Choice implies the existence of a well-ordering of the universe, and a well-ordering of the universe clearly cannot be symmetric. It turns out that we can prove that all classes of Frege natural numbers are sets; this is known to strengthen *NF*, and it is a useful and appealing result. Analysis of the notion of strongly cantorian set using the symmetry criterion allows us almost to demonstrate the very useful proposition that any subclass of a strongly cantorian set is a set; we suggest an additional combinatorial axiom governing permutations which would allow us to complete this proof.

The previous considerations should allow the reader to get some idea of the motivation behind features of the formalization in the next section that might seem strange. The final question to be raised in this section is philosophical rather than technical: does the Criterion of symmetry for sethood have any philosophical appeal?

An answer to this kind of question is necessarily less likely to be convincing than an answer to a technical mathematical question. Nonetheless, we have a proposal. If a set X is to be used as the domain of a set theory, this can be done in a very general way using an injection $f : X \to \mathcal{P}(X)$ from elements of X to subcollections of X. We understand that this idea is found in the foundational work of Ennio de Giorgi, so we call such a map f a "de Giorgi map", though our use of this idea is independent of de Giorgi (see our [8]). Properties of the map f correspond to properties of the implemented set theory (for example, we have forced the set theory to be extensional by stipulating in advance that the map is an injection). $\mathcal{P}(X)$ is the domain of classes for the implemented set theory, while $f``(X)$ is the domain of sets for the implemented set theory. The membership relation for the implemented theory is $x \in_f y \equiv_{\texttt{def}} x \in f(y)$. Each element x of X implements the set $f(x) \in \mathcal{P}(X)$. Note that it also implements a set of sets (an element of $\mathcal{P}^2(X)$ defined as $f``(f(x))$. More generally, an element x of X corresponds to an element $f_n(x)$ of $\mathcal{P}^n(X)$, where $f_1(x) = f(x)$ and $f_{n+1}(x)$ is computed by first taking $f_n(x)$ (finding the element of $\mathcal{P}^n(X)$ corresponding to x), then applying f to each of its n-fold elements: $f_{n+1}(x) = j^n(f)(f_n(x))$, which gives an inductive definition.

We then observe that it seems reasonable to suppose that the details of the map f from X to $\mathcal{P}(X)$ are arbitrary: what is important for the set theory is not which element of X represents a given set, but which sets (and sets of sets, etc.) are actually represented. To implement this notion, we note that if π is a permutation of X which is "setlike" in the sense that $j^n(\pi)$ is a permutation of $\mathcal{P}^n(X)$ for each n, then replacing the notion of "membership" on X $x \in_f y$ defined by $x \in f(y)$ with $x \in_{f\circ\pi} y \equiv_{\texttt{def}} x \in f(\pi(x))$ will give an implemented set theory with the same sets, sets of sets, etc. as the set theory based on the de Giorgi map f. (It is straightforward to show that stratified sentences of the interpreted set theory have their truth values preserved under this transformation). We suggest that the de Giorgi map implements "data types" "set", "set of sets", and so forth (corresponding to the types of *TST*); the correlation between specific sets of a given type and sets of a different type is arbitrary, and properties depending on this correlation are not really properties appropriate to that data type. The actual properties of an object qua the data type "element of $\mathcal{P}^n(X)$" will be those which are invariant under permutations of X n levels down – i.e., the n-symmetric properties – and the properties of sets considered in general will be those which are properties of "type n sets" for some n, which are seen to be exactly the properties whose extensions are symmetric. We introduced this motivation for *NF* in [8], though we did not at that time suggest that all sets are symmetric.

A final remark of a philosophical character is that we are not certain that "syntax" is entirely excluded from this motivation for *NF*, since the distinction between classes and superclasses seems to have something to do with classes being "definable", and definability is a notion with a strong relationship to syntax! However, the symmetry criterion has much more to do with what a class being tested for sethood is like *qua* class than does the criterion of stratification applied to its formal definition.

The obvious gap here is that we do not claim to have any idea how to construct a model of set theory with symmetric comprehension; however, we do claim that it should be easier to think about what a model of this theory must be like than to think about what a model of *NF* needs to be like.

3. Formalization

This section presents the formalization.

The theory we present is a first order theory with equality and membership as primitive relations.

General objects of this theory are called *superclasses*. Any element is a *class*. Any element of a class is a *set*.

The comprehension axiom for superclasses asserts that for any formula ϕ there is a superclass $\{x \mid \phi\}^*$ whose elements are exactly the *classes* which satisfy ϕ.

Axiom 3.1. (Extensionality) Superclasses with the same elements are the same.

Axiom 3.2. (Superclass Comprehension Scheme) For any formula ϕ, $(\exists A.(\forall x.x \in A \equiv ((\exists y.x \in y) \wedge \phi)))$, where A is not free in ϕ.

The comprehension axiom for classes asserts that for any formula ϕ in which all quantifiers are restricted to *sets* (or, equivalently, in which each quantifier is bounded by a class) and in which any parameters are sets, there is a class $\{x \mid \phi\}$ whose elements are exactly the *sets* such that ϕ.

Axiom 3.3. (Class Comprehension Scheme) For any formula ϕ in which each quantifier is restricted to a class, $(\exists A.(\forall x.x \in A \equiv ((\exists yz.x \in y \wedge y \in z) \wedge \phi)))$, where A is not free in ϕ.

We define ordered pairs, relations, and functions as usual.

Definition 3.4. $\{x, y\} = \{z \mid z = x \vee z = y\}$. $\{x\} = \{x, x\}$. $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$. Observe that there is no commitment at this point to the existence of any unordered or ordered pairs as sets; all we can tell at this point is that if x and y are sets, the unordered pair of x and y will be a class and the ordered pair of x and y will be a superclass.

Definition 3.5. A relation is a superclass of ordered pairs. A function is a superclass of ordered pairs f such that if $\langle x, y\rangle \in f$ and $\langle x, z\rangle \in f$, then $y = z$. A one-to-one function is a function f such that if $\langle x, z\rangle \in f$ and $\langle y, z\rangle \in f$, then $x = y$. A permutation of the universe is a one-to-one function f such that if x is a set, there is some $\langle x, y\rangle \in f$. For any function f, we define $f(x)$ as the y such that $\langle x, y\rangle \in f$.

Note that the word "universe" always refers to the class of all *sets*.

We now need to define the notion $j^n(f)$ for a permutation of the universe f. This is somewhat tricky.

Definition 3.6. We define the superclass of *Zermelo natural numbers* as the intersection of all superclasses which contain $\emptyset$ and contain $\{x\}$ whenever they contain x and x is a set. We define a Zermelo natural number as a superclass which is either the empty set or the singleton of an element of the superclass of Zermelo natural numbers (observe that it is possible (on the basis of axioms given so far) that there is a last Zermelo natural, which is a superclass whose sole element is the last class Zermelo natural). We refer to $\emptyset$ as 0. We refer to $\{n\}$ for n a Zermelo natural as $n + 1$, and we will refer to the concrete Zermelo naturals as 0,1,2..., although these will not be the natural numbers of our set theory.

We now develop the construction for any permutation f of the universe of a set j_f which will (for setlike permutations) be a function with the property $j_f(\langle n, x\rangle) = j^n(x)$. The natural numbers n are represented as Zermelo naturals.

Definition 3.7. Let f be a permutation of the universe. We define j_f as the intersection of all superclasses C which contain $\langle\langle 0, x\rangle, y\rangle$ whenever $\langle x, y\rangle \in f$ and the pair $\langle\langle 0, x\rangle, y\rangle$ is a class, and which contain $\langle\langle n + 1, A\rangle, B\rangle$ whenever $B = \{w \mid (\exists z.z \in A \wedge \langle\langle n, z\rangle, w\rangle \in C)\}$ and $\langle\langle n + 1, A\rangle, B\rangle$ is a class.

Definition 3.8. Let f be a permutation of the universe. We say that f is *setlike* just in case the superclasses $j^n(f) = \{\langle x, y\rangle \mid y = j_f(n, x)\}$ for each Zermelo natural n are permutations of the universe. Note that since we have defined $j^n(f)$ we have also defined $j^n(f)(x)$ for a set x (note that $j^n(f)$ may be a partial function in some cases). We further define $j^n(f)(A)$ for any class (or superclass) A as $j^{n-1}(f)[A]$ (note that this will agree with the definition of $j^n(f)(A)$ if f is setlike and A is a set). Observe that the mere existence of any setlike permutation implies that every Zermelo natural number is a set, and so that there is no last Zermelo natural. This is rather artificial, but will turn out to be all right.

Observation 3.9. The definition of j_f and the induced definition of the functions $j^n(f)$ makes sense even if f is not a permutation.

Definition 3.10. We define the superclass of *finite classes* as the intersection of all superclasses C which contain $\emptyset$ and contain $\{x \mid x \in a \vee x = y\}$ for every class $a \in C$ and set y.

Definition 3.11. Let x be a class. We say that x is n-symmetric with support (m, a), where m is a Zermelo natural and a is a finite class such that $j^i(f)(a) = a$ for all set permutations f and $i \geq m$, if $j^n(f)(x) = x$ for all setlike permutations f such that $j^m(f)(a) = a$ (note that this will include all set permutations f which happen to be setlike). Note that if a is the empty set this amounts to $j^n(f)(x) = x$ for all setlike f. We say that x is symmetric if x is n-symmetric with support (m, a) for some n, m, and a.

It might seem that some restriction like $m \leq n$ is needed to prevent arguing that any class x such that $j^n(f)(x) = x$ for all set x is n-symmetric with support $(n + 1, \{x\})$. But notice that this requires $\{x\}$ to be a finite class, which already requires x to be a set. An attempt to bound m with respect to n appears to break the proof that unions of sets are sets.

Definition 3.12. The "*support condition* (m, a)" (m a Zermelo numeral, a a finite class) is defined as the condition "$j^m(f)(a) = a$" on setlike permutations f. We are only interested in support conditions if they satisfy the *side condition* "$j^i(f)(a) = a$ for all set permutations f and $i \geq m$". Note that this is the same as requiring that all $j^k(f)$ for f a set permutation and k a Zermelo natural satisfy the support condition: we are only interested in support conditions satisfied by all set permutations and their iterated images under j.

We are ready to state the symmetric comprehension axiom for sets.

Axiom 3.13. (Symmetric Comprehension) A class x is a set iff it is symmetric.

Theorem 3.14. *There is a nontrivial setlike permutation. Otherwise every definable class would be vacuously symmetric and thus a set, including the Russell class.*

Theorem 3.15. *$\emptyset$ is a set.*

Proof. This is obviously 1-symmetric. ∎

Theorem 3.16. *If x is a set, then $\iota(x) = \{x\}$ is a set.*

Proof. If x is n-symmetric with support (m, a), then $\iota(x)$ is $(n+1)$-symmetric with the same support. ■

Definition 3.17. Let m and n be Zermelo naturals. Define $m < n$ as holding when n belongs to any superclass which contains $m+1$ and is closed under singleton (i.e., under successor).

Definition 3.18. If A is an unordered pair of sets (which will be a class and which may be a singleton) and $a \in A$, we define the element which occurs in A with a as the unique b such that $\{a, b\} = A$.

Definition 3.19. If x is a set and n is a Zermelo natural, let F be the intersection of all superclasses which contain $\{0, x\}$ and are closed under the operation which takes any $\{a, b\}$ to $\{\iota(a), \iota(b)\}$ (this is a nonce definition). We would like to define $\iota^n(x)$ as the unique y which is the element which occurs with n in some element of F; however, there is a bad case. If x is a Zermelo natural less than n, there will be two such y's, both Zermelo naturals, and in this case we choose the larger of the two; in all other cases there is a unique such y. Since we have not yet shown that unordered pairs of sets are sets, we cannot yet use ordered pairs to define this sensibly, but we will shortly be able to do this.

Definition 3.20. For any finite class x, we can find a Zermelo natural $|x|$, the unique Zermelo natural which belongs to an element also containing x (and must be 1 if x is a nonzero Zermelo natural itself) of the intersection of all superclasses which contain $(0, \emptyset)$, contain each unordered pair $\{1, \{x\}\}$, contain $\{n+1, x \cup \{y\}\}$ whenever they contain $\{n, x\}$, where n is a Zermelo natural, which must be 1 if x is also a nonzero Zermelo natural, and $y \notin x$ is a set. Using this formalization of cardinality of sets, it is possible to define standard arithmetic on the Zermelo naturals with some work. It is also possible to define $m + n$ as $\iota^m(n)$. The arithmetic notions needed below have to do with addition, subtraction, and order.

Theorem 3.21. *For any class A and Zermelo natural n, A is a set iff $\iota^n[A] = \{\iota^n(x) \mid x \in A\}$ is a set.*

Proof. A class A is m-symmetric with a given support iff $\iota[A]$ is $(m+1)$-symmetric with the same support. Induction on the Zermelo naturals allows us to extend this result to $\iota^n[A]$. ■

Theorem 3.22. *If a set is n-symmetric with support (m, a), it is also $(n+1)$-symmetric with support $(m+1, a)$.*

Proof. If f satisfies $j^{m+1}(f)(a) = a$, then $j(f)$ satisfies $j^m(j(f))(a) = a$, and if x is n-symmetric with support (m, a), we will have $j^{n+1}(f)(x) = j^n(j(f))(x) = x$, so x is $(n+1)$-symmetric with support $(m+1, a)$ as well. The side condition that $j^i(f)(a) = a$ for set permutations f and $i \geq m+1$ follows immediately from the original side condition that $j^i(f)(a) = a$ for set permutations f and $i \geq m$. ■

Theorem 3.23. *If x is a set, then $\{\{x\}, \emptyset\}$ is a set.*

Proof. If x is n-symmetric with support (m, a) this means that for any set-like f with $j^m(f)(a) = a$, we have $j^n(f)(x) = x$, so we also have $j^{n+2}(f)(\{\{x\}, \emptyset\}) = \{\{x\}, \emptyset\}$, so $\{\{x\}, \emptyset\}$ is $(n+2)$-symmetric with support (m, a). ■

Definition 3.24. If A is a class, define $A^0 = \{\{\{x\}, \emptyset\} \mid x \in A\}$. (This operation is defined for the nonce – its scope includes only the next theorem.) Note that A^0 is a set iff A is a set, as A^0 will be $(m+2)$-symmetric with a given support iff A is m-symmetric with the same support.

Theorem 3.25. *The conjunction of any two support conditions (m_1, a_1) and (m_2, a_2) is equivalent to a single support condition.*

Proof. Suppose w.l.o.g. that $m_2 \geq m_1 \geq 1$.

Observe that a_2 is fixed by $j^{m_2}(f)$ (f any permutation or indeed any function (if the definition of j is extended appropriately)) iff a_2^0 (as defined just previously) is fixed by $j^{m_2+2}(f)$. Further, a_1 is fixed by $j^{m_1}(f)$ iff $\iota^{m_2-m_1+2}[a_1]$ is fixed by $j^{m_2+2}(f)$.

The desired support condition will be $(m_2+2, a_2^0 \cup \iota^{m_2-m_1+2}[a_1])$: the second component will be fixed by $j^{m_2+2}(f)$ iff a_2 is fixed by $j^{m_2}(f)$ and a_1 is fixed by $j^{m_1}(f)$.

It is straightforward to prove that the elementwise image under ι (or ι^n) of a finite class is a finite class and that the union of two finite classes is a finite class, so the needed finite classes here exist. The reason for use of the A^0 construction is that there are no double singletons in a_2^0, whereas every element of $\iota^{m_2-m_1+2}[a_1]$) is a double singleton (at least), which ensures that no permutation (or other function) $j^2(f)$ will move items between the two classes, which is essential to showing that the single condition implies the two parts of the conjunction.

One needs to verify further that for any $i \geq m_2+2$ and set permutation f, we have $j^i(f)$ fixing $a_2^0 \cup \iota^{m_2-m_1+2}[a_1]$ (the side condition on the new support condition). Let $k = i - (m_2+2)$. $j^i(f) = j^{m_2+2}(j^k(f))$. This fixes $a_2^0 \cup \iota^{m_2-m_1+2}[a_1]$ iff $j^{m_2}(j^k(f))$ fixes a_2 and $j^{m_1}(j^k(f))$ fixes a_i: both of these conditions hold by the side conditions on the original two support conditions. It is important to note that we do not need to assume here that $j^n(f)$ is necessarily a permutation for set permutations f (we do not yet know that set permutations are setlike) though we will be able to show that this is the case below.

$\blacksquare$

THEOREM 3.26. *If x and y are sets, $\{x, y\}$ is a set.*

PROOF. Suppose that x is n_1-symmetric with support (m_1, a_1) and y is n_2-symmetric with support (m_2, a_2) and (w.l.o.g) that $n_2 \geq n_1$. Then $\{x, y\}$ is $(n_2 + 1)$-symmetric with support equivalent to the conjunction of the conditions $(m_1 + n_2 - n_1, a_1)$ and (m_2, a_2).

$\blacksquare$

COROLLARY 3.27. *Ordered pairs of sets are sets. Class relations and functions on the universe of sets are freely definable.*

THEOREM 3.28. *The image of a set under a set map is a set.*

PROOF. Setlike permutations of sufficient depth (i.e., images under j^n for large enough n) and support will fix both the set and the set map. Examination of the effect of such permutations on the set map reveals that the image of the set under the map must also be fixed: if permutations with enough depth and support fix both the set and the set map, they must also fix the restriction of the map to the set, and so fix the image as well. This allows us to show that the image is symmetric.

We give the detailed calculation. Let F be a set function and A be a set. For any set b, $b \in F[A]$ iff there is $\{\{a\}, \{a, b\}\} \in F$ with $a \in A$. For n large enough so that F is n-symmetric and A is $(n-2)$-symmetric and any setlike f satisfying appropriate support conditions we have that $\{\{j^{n-3}(f)(a)\}, \{j^{n-3}(f)(a), j^{n-3}(f)(b)\}\}$ belongs to F as well, with $j^{n-3}(f)(a) \in A$, from which it follows that $j^{n-3}(f)(b) \in F[A]$, from which it follows that $F[A]$ is $(n-2)$-symmetric with suitable support.

$\blacksquare$

THEOREM 3.29. *Every set permutation is a setlike permutation.*

PROOF. By the previous theorem, if f is a set permutation, $j(f)$ is a class permutation. Now if f is n-symmetric with a given support, $j(f)$ is $(n+1)$-symmetric with the same support, and so is a set. It follows that $j^n(f)$ is a set permutation for every n. $\blacksquare$

THEOREM 3.30. *For any set x, $\bigcup x$ is a set.*

PROOF. If x is n-symmetric with support (m, a) and $n > 1$, then x is also $(n+1)$-symmetric with support $(m+1, a)$, whence $\bigcup x$ is n-symmetric with support $(m+1, a)$. ∎

THEOREM 3.31. *If ϕ is a stratified formula with all variables (free and bound) restricted to the class of sets, the class $\{x \mid \phi\}$ is a set.*

PROOF. For any setlike permutation f, define $x \in_f y$ as $x \in f(y)$. Define $f_0(x)$ as x and $f_{n+1}(x)$ as $j^n(f)(f_n(x))$ (as in [3], p. 96): $f_{n+1} = j^n(f) \circ j^{n-1}(f) \circ \ldots \circ j(f) \circ f$. For any stratified formula ϕ, define ϕ_f as the formula which results when all occurrences of $\in$ are replaced with occurrences of $\in_f$ and all parameters a are replaced with $f^{-1}_{\sigma(a)}(a)$, where σ is a fixed stratification of ϕ (we require w.l.o.g. that all types assigned are non-negative; we treat expressions $f^{-1}_{\sigma(a)}(a)$ as if they were single variables of the same type as a in the construction which follows). It is straightforward to prove that $\phi \equiv \phi_f$ (compare Lemma 3.1.2 in [3], p. 96). First use the fact that $x \in f(y) \equiv f_n(x) \in f_{n+1}(y)$ to convert every atomic formula $x = y$ or $x \in_f y$ (that is, $x \in f(y)$) in ϕ_f to the form $f_{\sigma(x)}(x) = f_{\sigma(y)}(y)$ or $f_{\sigma(x)}(x) \in f_{\sigma(y)}(y)$ where σ is the same fixed stratification of ϕ: this is possible because of the known relations between values of σ at x and y in an atomic formula of either form. One can then eliminate applications of the permutations f_n to variables bound by quantifiers $((\forall x.\phi(x) \equiv (\forall x.\phi(\pi(x)))$ when π is a permutation of the universe), obtaining a formula which differs from ϕ only in the application of $f_{\sigma(a)}$ to each parameter a in ϕ_f, which cancels the application of $f^{-1}_{\sigma(a)}$ introduced in the definition of ϕ_f: the formula obtained in the end is ϕ itself.

We now apply this to prove symmetry of sets $\{x \mid \phi(x)\}$, ϕ a stratified formula. Observe that $\phi(x) \equiv \phi(x)_{j^n(f)}$ for any choice of f and n. This formula will be equivalent to $\phi(j^n(f)^{-1}_k(x))$, where k is a type assigned to x, if we require that n be chosen large enough and f be chosen to satisfy suitable support conditions so that all parameters in $\{x \mid \phi\}$ will be fixed by the appropriate $j^n(f)_m$, where m is the type assigned to the parameter (note that $j^n(f)_m$ is a composition of functions $j^{n+i}(f)$ for $0 \leq i < m$: choose n large enough so that all parameters are n-symmetric; we need $j^{n+i}(f)$ to fix each parameter for each value of i below the type of that parameter; if f is a set permutation, n-symmetry of the parameter will enforce this condition, and if f is setlike this defines a support condition). Now observe that we can shift assigned types down by one, so that the type of x is $k-1$ instead of k, and replace the map f with $j(f)$, and obtain equivalence of $\phi(x)$ to $\phi(j^{n+1}(f)^{-1}_{k-1}(x))$ (now requiring $(n+1)$-symmetry of each parameter and slightly different additional support conditions). Further observe that $j^n(f)^{-1} \circ j^{n+1}(f)^{-1}_{k-1} = j^n(f)^{-1}_k$. The resulting equivalence of $\phi((j^n(f)^{-1} \circ j^{n+1}(f)^{-1}_{k-1})(x))$ to $\phi(j^{n+1}(f)^{-1}_{k-1}(x))$ for any x establishes the equivalence of $\phi(x)$ to $\phi(j^n(f)(x))$ for any x and for any large enough n and f satisfying the support conditions described above, and this establishes that the original set is fixed under application of $j^{n+1}(f)$ for such n and f. This establishes that the original set $\{x \mid \phi(x)\}$ is symmetric.

■

Observation 3.32. At this point we have shown that this theory satisfies *NF* (Quine's "New Foundations") on the class of sets.

In "New Foundations", the Frege natural numbers are used (the Frege natural number n is the set of all sets with n elements; it is well-known that this definition is not circular, as the zero and successor operations for this scheme of numeration are readily defined); the collection of Zermelo natural numbers we have used so far cannot make up a set under symmetric comprehension, because it is not n-symmetric for any n, and is not obviously a class at all (incidentally, if there are ω-models of *NF*, there are models of *NF* in which the Zermelo naturals make up a set: these are constructed by Rieger-Bernays permutation methods).

Theorem 3.33. *The superclass of sets which are n-symmetric with respect to set permutations for a fixed Zermelo natural number n is a set.*

Proof. It is not obvious from the form of the definition that this is even a class, because of the role of the superclasses j_f in the definition of symmetry. It is easy to see that this collection is a set for each concrete n: this is a consequence of stratified comprehension. Let P_0 be the set of all set permutations. For any set P of permutations, the set $j[P] = \{j(f) \mid f \in P\}$ is a set (by stratified comprehension). For any set P of permutations, the collection of sets which are fixed by all permutations in P is a set (by stratified comprehension). Define a superclass map from the Zermelo naturals to sets as follows: this will send 0 to P_0 and if it sends n to P_n it will send $n + 1$ to $P_{n+1} = j[P_n]$. Induction on the Zermelo naturals establishes that P_n is the superclass of all $j^n(f)$ for f a set permutation, and we see that it is a set by induction as well. Now for any n the set of n-symmetric sets with respect to set permutations is the set of all sets fixed by all elements of the set of permutations P_n. ■

Theorem 3.34. *The Frege natural numbers of the embedded* NF *correspond precisely to the Zermelo natural numbers as defined earlier.*

Proof. Define the class of von Neumann natural numbers as the intersection of all sets which contain the empty set and are closed under the von Neumann successor operation ($\lambda x.x \cup \{x\}$) (such sets are called "von Neumann-inductive"). There is an initial segment of the von Neumann naturals which can be placed in one-to-one correspondence with the superclass of Zermelo naturals in the obvious way. Now for each Zermelo natural n we can show that the class of all objects which are either a von Neumann natural less than n or not n-symmetric is a set, contains the empty set and is closed under von Neumann successor (the von Neumann natural corresponding to the Zermelo natural n is $(n+1)$-symmetric but not n-symmetric). It follows that any von Neumann natural must belong to each of these von Neumann-inductive sets, and since it must be n-symmetric with respect to set permutations for some Zermelo natural n, it follows that it must correspond to some Zermelo natural less than that n: every von Neumann natural corresponds to a Zermelo natural. The final stage of the argument is to define the class of Frege naturals which have a von Neumann natural as an element; this class is readily seen to be symmetric, since what we have shown (in effect) is that the cardinalities of the von Neumann ordinals are precisely the cardinalities of the finite classes, and no superclass permutation will perturb the cardinality of a finite class, and this class must itself be the whole set of Frege naturals, because it is a set, contains the Frege 0 and is closed under successor. ■

Corollary 3.35. *This means that the embedded* NF *satisfies the Axiom of Counting of Rosser (one form of this axiom is the assertion that the set of (Frege) natural numbers is strongly cantorian) and strong mathematical induction. In effect, we have the theory of an ω-model of* NF*.*

Theorem 3.36. *There is a symmetric superclass of sets which is not a set.*

Proof. Consider the class of order types (equivalence classes of well-orderings under similarity, i.e., Russell-Whitehead ordinals) of *true* well-orderings (those with the property that any subsuperclass of their domain has a minimal element). This superclass is clearly symmetric (a true well-ordering cannot be changed in order type by the application of any bijection to the elements of its domain), and equally clearly cannot be a set, as we would then have the Burali-Forti paradox. ■

Corollary 3.37. *A superclass all of whose elements are sets is not necessarily a class.*

Now we turn to the analysis of the important concept of "strongly Cantorian set".

DEFINITION 3.38. A set A is strongly cantorian (s.c.) iff $\iota\lceil A$, the restriction of the singleton map to A, is a set. This is a standard concept from *NF*.

DEFINITION 3.39. An *ordinal* is an equivalence class of well-orderings under similarity. For any well-ordering W, define W^ι as $\{\langle\iota(x),\iota(y)\rangle \mid \langle x,y\rangle \in W\}$. If α is the order type of W (i.e., the ordinal which contains W), define $T(\alpha)$ as the order type of W^ι. It is easy to prove that this does not depend on the choice of W in α. An ordinal α is called a *cantorian ordinal* if $T(\alpha) = \alpha$. An ordinal α is called a *strongly cantorian ordinal (s.c. ordinal)* if the domain of each element of α is strongly cantorian: this implies that for any $W \in \alpha$, not only is W^ι similar to W, but the similarity between them is actually a restriction of the singleton map (the singleton map itself is a proper class). A strongly cantorian ordinal α has not only $T(\alpha) = \alpha$, but also $T(\beta) = \beta$ for each $\beta < \alpha$. Some ordinals (for example, the order type of the natural order on the ordinals) are provably non-cantorian (this is a standard result about *NF*).

THEOREM 3.40. *The class of strongly Cantorian sets is 2-symmetric under set permutations. (this is a standard theorem of* NF*: see [3] for this and other "standard results").*

THEOREM 3.41. *The class of all strongly Cantorian sets is not a set. Likewise, the class of all order types of strongly Cantorian well-orderings is not a set. (these are standard theorems of* NF*).*

COROLLARY 3.42. *Fix a Zermelo natural n. There is a setlike permutation $j^n(f)$ which moves a strongly Cantorian ordinal α to a non-strongly-Cantorian ordinal. For any such f and α, $j^i(f)(\alpha) \neq \alpha$ for any $i \geq n$.*

PROOF. We can deduce that the same permutation f sends some s.c. ordinal α to a non-cantorian ordinal β (if the permutation sends an s.c. ordinal to a cantorian ordinal which is not s.c., it will send some smaller (still s.c.) ordinal to a non-cantorian ordinal). The study of this α motivates the role of support in the definition of n-symmetry. For we have $j^n(f)(\alpha) = \beta$, from which it is easy to deduce that $j^{n+i}(f)(\alpha) = T^i(\beta) \neq \alpha$, by looking at the action of f on orders on sufficiently iterated singletons: if $j^{n-1}(f)$ sends W of order type α to W' of order type β, then $j^{n+i-1}(f)$ will send $W^{\iota^i} \in \alpha$ to $(W')^{\iota^i} \in T^i(\beta)$ from which we can further deduce that $j^{n+i}(f)(\alpha) = T^i(\beta)$ for every i. However, if g is any setlike permutation such that $j^3(g)(W) = W$, where $W \in \alpha$, it is easy to see that $j^4(g)(\alpha) = \alpha$, so α is 4-symmetric with support $(4, \{W\})$ (it is easy to see that $j^3(g)(W) = W$ for set g and $W \in \alpha$, as required in the definition of symmetry). The role of support is to ensure that our setlike permutation sends well-orderings of type α to well-orderings of type α: consideration of the action of f on the map witnessing similarity between W and any other element of α allows us to see this. (Setlike permutations $j^n(f)$ with n at least 4 always send ordinals to ordinals, but non-set setlike permutations may induce external automorphisms of the ordinals; support conditions control this behavior.) ∎

Next, we analyze the structure of a strongly Cantorian set A by considering the symmetry of $\iota\lceil A$.

Theorem 3.43. *If A is strongly cantorian, then there is a fixed n such that every element of A is n-symmetric (but not necessarily with uniform support). Any set all of whose elements are n-symmetric with a fixed support (m, a) is strongly Cantorian.*

Proof. If $\iota\lceil A$ is a set, it will be n-symmetric with support (m, a) for some choice of n, m, a.

Assume that $j^m(f)(a) = a$ (note that this will be true if f is a set). It follows that for any $b \in A$ there is $c \in A$ such that $j^{n-1}(f)(\langle b, \{b\}\rangle) = \langle c, \{c\}\rangle$. Here we will have $j^{n-3}(f)(b) = c$ and $j^{n-3}(f)(\{b\}) = \{c\}$, from which it follows that $j^{n-3}(f)(b) = j^{n-4}(f)(b)$. If we then consider $f = j^k(g)$, we find that $j^{n-3+k}(g)(b) = j^{n-4+k}(g)(b)$ for each k for suitable g. If f is a set, we can show that $j^{n-4}(f)(b) = j^{n-3}(f)(b) = j^{n-2}(f)(b) = \ldots = b$ (because for large enough k and set f, $j^k(f)(b) = b$); all elements of A are $(n-4)$-symmetric with respect to set permutations. If we choose maps $f = j^k(g)$ which satisfy the support conditions for b, we discover further that each element b of A is actually $(n-4)$-symmetric with some support, but the support may depend on b. Finally, if all elements of a set A are n-symmetric with the same support (m, a), it is clear that $\iota\lceil A$ will be symmetric, so A will be strongly cantorian (and any class with this property will also be a strongly Cantorian set). ∎

Observation 3.44. It appears that it ought to be possible to frame an additional combinatorial assumption about setlike permutations under which we should be able to show that each element b of A is $(n-4)$-symmetric with the same support as $\iota\lceil A$, which will imply further that every subclass of a strongly cantorian set will be a strongly Cantorian set (an appealing result which strengthens *NF* in useful ways). Note that the strongly cantorian set of Frege naturals does have uniform trivial support $(1, \emptyset)$ for the 2-symmetry of all of its elements, so every subclass of the natural numbers is a set in this theory (this is a very strong additional axiom in the context of *NF*).

Axiom 3.45. (Axiom Candidate) Let m be a Zermelo natural and a a finite class satisfying $j^m(f)(a) = a$ for every set permutation f. If $j^n(f)(x) = j^{n+1}(f)(x)$ for all setlike permutations f such that $j^m(f)(a) = a$, then $j^n(f)(x) = x$ for all such setlike permutations f.

This candidate axiom would imply that all subclasses of strongly cantorian sets are sets. It is an assumption about rigidity (it excludes the existence of certain kinds of external automorphisms of sets).

We discuss the status of choice. Because we have *NF*, we know that the axiom of choice (in a version phrased in terms of sets) must be false. What is different here is that it is entirely natural that choice be false in a system with symmetric comprehension; Specker's rather strange argument to this effect in *NF* does not need to be invoked.

Theorem 3.46. *The axiom of choice is false (when stated in terms of sets).*

Proof. Consider the set permutation (xy) which sends x to y, y to x and fixes everything else, where $x \neq y$ are sets. Suppose there is a set well-ordering of the universe (this would follow from any of the usual forms of the axiom of choice; the standard equivalences between forms of the axioms of choice (in stratified forms) hold in *NF*). The pair $\langle \iota^n(x), \iota^n(y)\rangle$ would be sent to $\langle \iota^n(y), \iota^n(x)\rangle$ by the action of $j^{n+3}((xy))$, so the supposed well-ordering could not be n-symmetric with respect to set permutations for any n, and there can be no set well-ordering of the universe. ∎

Observation 3.47. Nothing appears to prevent the existence of a class well-ordering of the universe (subclasses of whose domain would have minimal elements – not necessarily subsuperclasses). The existence of such a class well-ordering combined with the assumption that all subclasses of s.c. sets are sets would lead to the agreeable result that all s.c. sets could be well-ordered. Nor does anything prevent the existence of a superclass true well-ordering of the universe, but it does not appear that one could prove anything useful about classes or sets using such a well-ordering.

Notice that, as in *NF*, the fact that we disprove Choice (easily in this case) is interesting because it shows us that Infinity is true. The result about equivalence of Frege and Zermelo natural numbers also implies infinity.

4. Relations to Other Work

Thomas Forster has pointed out that "term models" of *NF* (models in which all elements are actually set abstracts $\{x \mid \phi\}$) would have the property that all sets are symmetric (see [3]). He has not to my knowledge considered the converse assertion that all symmetric classes are sets (which is tricky to state, as this paper illustrates). Forster has in very interesting recent work (not yet published – see [4]) investigated restricted versions of the standard set theory *ZF* in which all sets are "hereditarily symmetric" in a sense related to but not identical to the sense used here (he restricts the action of permutations to V_ω).

It is possible that the superclasses could be eliminated by adding a transitive closure operation to the logical language in which classes are defined. It then appears to be possible to define n-symmetry of classes without the need to appeal to superclasses. We have not favored this line of development because the complication of the logic is considerable and we believe that it is advantageous that the setlike permutations witnessing failures of sethood do not themselves have to be classes. One should note that much of the complication of this development is due to our proving the theorem that unordered pairs of sets are sets; if Pairing were taken to be a separate axiom the development would be simpler, but less philosophically gratifying.

References

[1] Bernays, P., "A system of axiomatic set theory VII", *Journal of Symbolic Logic*, vol 19 (1954), pp. 81-96.

[2] Marcel Crabbé, "On the consistency of an impredicative subsystem of Quine's *NF*," *Journal of Symbolic Logic*, 47 (1982), pp. 131–36.

[3] Forster, T.E., *Set Theory with a Universal Set*, second edition, Clarendon Press, Oxford, 1995.

[4] Forster, T. E., "AC fails in the natural analogues of V and L that model the stratified fragment of ZF", preprint, found at `http://www.dpmms.cam.ac.uk/ tf` or from the author.

[5] V. N. Grishin, "Consistency of a fragment of Quine's *NF* system," *Sov. Math. Dokl.*, 10 (1969), pp. 1387–90.

[6] T. Hailperin, "A set of axioms for logic". *Journal of Symbolic Logic*, 9 (1944), pp. 1–19.

[7] Ronald Bjorn Jensen, "On the consistency of a slight (?) modification of Quine's 'New Foundations'," *Synthese*, 19 (1969), pp. 250–63.

[8] Holmes, M. Randall, "The set theoretical program of Quine succeeded (but nobody noticed)", *Modern Logic*, vol. 4 (1994), pp. 1-47.

[9] W. V. O. Quine, "New Foundations for Mathematical Logic," *American Mathematical Monthly*, 44 (1937), pp. 70–80.

[10] W. V. O. Quine, *Mathematical Logic*, second edition, Harvard, 1951.

[11] Rieger, L., "A contribution to Gödel's axiomatic set theory", *Czechoslovak Mathematical Journal*, vol. 7 (1957), pp. 323-357.

[12] E. P. Specker, "The axiom of choice in Quine's 'New Foundations for Mathematical Logic'," *Proceedings of the National Academy of Sciences of the U. S. A.*, 39 (1953), pp. 972–5.

[13] Dana Scott, "Quine's individuals," in *Logic, methodology and philosophy of science*, ed. E. Nagel, Stanford, 1962, pp. 111–5.

[14] Wang, H., *Logic, Computers, and Sets*, Chelsea, 1970, p. 406.