The Urysohn Space Embeds in Banach Spaces in Just One Way

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This note will give a brief account of the research found in my master's thesis [3] of 1985 and the following paper [4] of 1992 in which I described the Urysohn universal separable metric space, which I had discovered independently but which was of course not new (see [7]), and of the unique separable Banach space which appears as the linear closure of any isometric copy containing 0 of the Urysohn space in a Banach space, which was my original contribution.

The question which I was asked in a graduate general topology class was "Is there a universal separable metric space (implicitly, up to homeomorphism)"? That is, is there a separable metric space X such that for any separable metric space at all, there is a homeomorphic embedding from X into X?

Now of course an isometry is a homeomorphism, and I perhaps foolishly asked the harder question "Is there a universal separable metric space up to isometry"?

To investigate this question, I defined the concept of a "possible combination of distances" from a metric space X. It should be noted that of course Urysohn defined the same concept in [7], but I did not become aware of this for some time.

Let (X, d) be a metric space (which we will refer to as X, as is usual, if logically dubious). The metrics on all spaces will be d as long as the intended space can be understood from context: otherwise the metric on X will be d_X.

Definition 1. Let $Y \subseteq X$. Let p be a function from Y to the non-negative reals, satisfying $p(u) - p(v) \leq d(u, v) \leq p(u) + p(v)$ for all u and v in Y. (Of course $|p(u) - p(v)| \leq d(u, v)$ then holds by symmetry). Such a function will be called a *possible combination of distances from Y (as a subspace of X).*

If the domain of p is X, we can adjoin a new point q to X, stipulating that $d(q,x) = p(x)$ for each $x \in X$, and it is straightforward to verify that $X \cup \{q\}$ is a metric space with this metric. If the domain Y of p is a proper subset of X, one can extend p to the whole of X by making its value at each $x \in X - Y$ as large as possible: let $p'(x)$ be defined as $\inf\{d(x,y) + p(y) \mid y \in X\}$. It is straightforward to verify that p' agrees with p on Y and is a possible combination of distances from the whole of X. Thus it is possible to adjoin a point to X in such a way that its distance from every point y of Y is $p(y)$. The notion "possible combination of distances from Y" exactly captures the possible combinations of distances from Y of a new point to be adjoined to the ambient metric space X.

Definition 2. Of course possible combinations of distances from a proper subset Y may be handled by points already found in $Y - X$: we say that $x \in X$ *realizes* p if $d(x,y) = p(y)$ for each $y \in Y$.

The universal separable metric space of Urysohn can be characterized using this notion. Up to isometry, $\mathbb{U}$ is the unique complete separable metric space which has the property that any possible combination of distances from a finite subset of $\mathbb{U}$ is realized in $\mathbb{U}$ (this characterization was of course given much earlier by Urysohn in [7]).

$\mathbb{U}$ can be constructed using this notion as well. Let X be a metric space and let X' be the set of all possible combinations of distances from the whole of X. We put the metric $d(p,q) = \sup\{|p(x) - q(x)| \mid x \in X\}$ on X'. X' has a canonical subspace isometric to X (consisting of the possible combinations of distances p_x which take on the value zero at some point x of X). Moreover, for any $p \in P$, we have $d(p, p_x) = p(x)$. The space X' contains new points realizing every possible combination of distances from (the natural isometric copy of) X, with any two new points as close to one another as their distances from the natural embedded copy of X permit. Unfortunately, X' is not as a rule separable; so we define a subspace X'' of X' as the completion in X' of the set of extensions to all of X (as described above) of possible combinations of distances from finite subsets of X. X'' can be shown to be separable if X is separable. Now let X_0 be a one-point space and define X_{n+1} as X''_n for each natural number n. The completion of the direct limit of the X_i's (using the natural isometric embedding of each space in the sequence into the next to construct the direct limit) is isometric to $\mathbb{U}$. This was my original construction of a universal separable metric space up to isometry when I discovered this space independently in 1983. The same construction was published by Katětov in [5], in 1988, but I did not become aware of this until 2006!

No one at SUNY Binghamton had heard of the Urysohn space, but many people there knew of the well-known theorem of Banach and Mazur that $C[0,1]$ (the space of continuous functions from $[0,1]$ to the reals with the sup metric) is a universal separable metric space up to isometry (and in fact a universal separable Banach space up to linear isometry) (see [1]). So the natural question in my mind was "how do $\mathbb{U}$ and $C[0,1]$ embed into each other?"

It was immediately clear that the spaces are different. Consider the constant functions 1,2,3. A possible combination of distances from these points which cannot be realized in $C[0,1]$ maps each of these points to 1. But 2 is the only point in $C[0,1]$ which is at distance 1 from each of 1 and 3.

The perhaps valuable original contribution of my work on the Urysohn space from 1983 to 1985 is contained in the following series of observations.

Definition 3. We define a *possible combination of values* of a set of functions $F \subseteq C[0,1]$ with $0 \in F$ as a function p from F to the reals such that for any $f, g \in F$, we have $p(0) = 0$ and $|p(f) - p(g)| \leq d(f,g)$. Further, we say that a real r *realizes* p iff for each $f \in F$, we have $f(r) = p(f)$. It should be clear that for any $r \in [0,1]$, the function sending each $f \in F$ to $f(r)$ is a possible combination of values for F (justifying the terminology).

Suppose that F is a finite subset of an isometric copy of $\mathbb{U}$ in $C[0,1]$, $0 \in F$, and p is a possible combination of values for F. It is straightforward to verify that for large enough N (twice the diameter of F will work) the function $(f \in F \mapsto N - p(f))$ is a possible combination of distances from F, and so there is a function g in the isometric copy of $\mathbb{U}$ which realizes these distances from F. Extend p by defining $p(g) = N$ (obviously the extended p is still a possible combination of values). Now further it is straightforward to show that $(f \in F \cup \{g\} \mapsto N + p(f))$ is a possible combination of distances from $F \cup \{g\}$, so there is a function h in the isometric copy of $\mathbb{U}$ which realizes these distances from $F \cup \{g\}$. Now $d(g,h) = 2N$ by construction, so there must be a real r_p such that $|g(r_p) - h(r_p)| = 2N$. Since $0 \in F$ and $d(g,0) = d(h,0) = N$, we are forced to have either $g(r_p) = N$ and $h(r_p) = -N$ or $g(r_p) = -N$ and $h(r_p) = N$. For each $f \in F$, we have $|f(r_p) - g(r_p)| \leq N - p(f)$ and $|f(r_p) - h(r_p)| \leq N + p(f)$. So in the first case $f(r_p)$ is forced to have the value $p(f)$ for each $f \in F$ and in the second case

$f(r_p)$ is forced to have the value $-p(f)$ for each $f \in F$. So we have shown the following rather surprising

Theorem 4. *For any finite subset F of an isometric copy of $\mathbb{U}$ with $0 \in F$, and any possible combination of values p for F, either p is realized at some $r_p \in [0,1]$ or $-p$ is realized at some $r_p \in [0,1]$.*

This is very strange! It implies, for example, the following

Corollary 5. *Any element of an isometric copy of $\mathbb{U}$ in $C[0,1]$ which contains 0, other than 0 itself, is a component of something which is almost a Peano space-filling curve.*

Proof. Let f be such a function. A possible combination of distances from f and 0 is the map sending 0 to $|f|$ and $|f|$ to $2|f|$, so there is a point f_2 at distance $|f|$ from 0 and $2|f|$ from f in the isometric copy of $\mathbb{U}$. Any element of $[-|f|,|f|]^2$ is of the form $(p(f), p(f_2))$ where p is a possible combination of values for f and f_2 (and all possible p are associated with points in this way). Thus for every point (x, y) in $[-|f|,|f|]^2$ there is a real r such that either $f(r) = x$ and $f_2(r) = y$ or $f(r) = -x$ and $f_2(r) = -y$: f and f_2 are the components of a continuous curve which visits each point of a square centered at the origin or its mirror image through the origin. □

So we see that no familiar function in $C[0,1]$ except the constant 0 can be an element of such a copy of $\mathbb{U}$! What, on the face of it, does the universal separable metric space of Urysohn have to do with space-filling curves?

Further, consider the linear closure of an isometric copy of $\mathbb{U}$ in $C[0,1]$ containing 0. Consider in particular any finite linear combination $\Sigma c_i f_i$ of elements of the copy of $\mathbb{U}$. The norm of $\Sigma c_i f_i$ is the supremum of all sums $|c_i f_i(r)|$ for $r \in [0,1]$. But this means that it is the supremum of all sums $|c_i p(f_i)|$ where p is a possible combination of values for the set of f_i's, because every such possible combination of values or its uniform negative is realized at some r. This supremum depends only on the distances among the f_i's and 0, so such norms are determined entirely by the metric structure of $\mathbb{U}$ and the selection of a point to correspond to 0. This completes the proof of another surprising

Theorem 6. *The linear closure of an isometric copy of $\mathbb{U}$ in $C[0,1]$ which contains 0 is a uniquely determined separable Banach space $\overline{\mathbb{U}}$, up to linear* *isometry (and so, because of the known universality of* $C[0,1]$*, the linear closure of an isometric copy of* $\mathbb{U}$ *containing 0 in* any *Banach space is uniquely determined up to linear isometry).*

The anonymous referee advises us to emphasize the point that a formally stronger result is proved here: an isometric embedding of the Urysohn space in a Banach space determines a unique norm, in the sense that the norm of any linear combination of points of $\mathbb{U}$ is uniquely determined as soon as the point mapping to 0 is chosen. It is not clear that this property is equivalent to the property of determining a unique linear closure up to linear isometry; it might be stronger.

There are two questions about this which present themselves. One of them was ours, on which we made little progress, but we were able to answer a question of Sierpinski.

Question 7. We know that $\mathbb{U}$ is a universal separable metric space up to isometry. Is its uniquely determined linear closure $\overline{\mathbb{U}}$ a universal separable Banach space up to linear isometry?

I did not make much headway on this. In [4] I got as far as demonstrating that $\overline{\mathbb{U}}$ did not have a certain homogeneity property which would have facilitated a proof of universality. This question has been answered positively by Godefroy and Kalton in [2], as a corollary of a much stronger result: if a separable Banach space embed isometrically into another Banach space, Godefroy and Kalton showed that it also embeds linearly isometrically, which neatly solves the problem at hand: any separable Banach space embeds isometrically in $\mathbb{U}$ so of course into $\overline{\mathbb{U}}$, and by the result of Godefroy and Kalton embeds linearly isometrically into $\overline{\mathbb{U}}$.

The second question, which I did answer, is difficult to phrase precisely. The usual proofs that $C[0,1]$ is a universal separable Banach space under linear isometry (at least, the ones familiar to us) involve space-filling curves. We present a version adapted to embedding metric spaces rather than Banach spaces (we believe this adaptation is from [6]). Let X be a separable metric space and fix an element of X which will be mapped to 0. Let D be a countable dense subset of X. Take the space D^* of all possible combinations of values of D (defined as above, but of course this was not their terminology) and put the pointwise convergence topology on it. This space is a connected compact metric space, so one can define a continuous map f from $[0,1]$ onto D^*. Now with each point $d \in D$ associate the function which sends each

$r \in [0,1]$ to $f(r)(d)$. Under the supremum metric, these functions will make up an isometric copy of D in $C[0,1]$ whose completion will be a copy of X. Sierpinski observed, in commenting on this proof in [6], that for most familiar spaces nothing as nasty as this construction using a Peano curve is required, and he asked specifically this

Question 8. (Sierpinski) Is there a better way to embed $\mathbb{U}$ in $C[0,1]$ than the general method of Banach and Mazur, as adapted to metric spaces?

The results above linking isometric embeddings of $\mathbb{U}$ with $C[0,1]$ strongly suggest that the answer should be No. However, it is tricky to formulate the negative answer precisely.

In [4], I formulated precise conditions under which a finite subset F of $C[0,1]$ can be extended to an isometric copy of $\mathbb{U}$ containing 0. The condition is equivalent to the statement that there is a positive constant N and a function g at distance $N + d(0, f)$ from f for each $f \in F$, such that for each possible combination of values p for $F \cup \{g\}$, either p is realized or $-p$ is realized. It follows easily from the discussion above that these conditions are necessary; additional work is required to show that these conditions are sufficient. Such a set F is called *inflatable* in [4]. The basic idea of the proof is that one can choose any possible combination of distances p from F, then use g to guide the construction of two functions, a function f' which has the desired distances from the elements of f and a function g' which has distance $N + d(f, 0)$ from each $f \in F \cup \{f'\}$. This allows the construction of a countable dense subset of a copy of $\mathbb{U}$ from an inflatable set, and taking the completion of a subset of $C[0,1]$ of course presents no difficulties. This allows an exposition of the construction of $\mathbb{U}$ entirely in terms of $C[0,1]$, which is given in detail in [4].

An easy way to answer Sierpinski's question is the following: any embedding of $\mathbb{U}$ into $C[0,1]$ is associated via the construction outlined above with a continuous curve in D^* (where D is a countable dense subset of $\mathbb{U}$) which "half-fills" D^* (visits each element of D^* or its negative). So mod the difference between "half-space-filling curves" and frankly space-filling curves, the answer to the question of Sierpinski is indeed No. A more subtle approach involves choosing D cleverly so that a universal construction of isometric embeddings of $\mathbb{U}$ in $C[0,1]$ can be presented whose only parameter is a "half-space-filling curve" in the usual Hilbert cube. This can be done in such a way that there is a one-to-one correspondence between half-space-filling curves and isometric embeddings, if D is chosen in such a way that all instances of the triangle inequality are strict (so no finite assignment of values at a given r to points of D can exactly fix the value at r of any other element of D) (it is noted that this can be done in [4] but complete details are not given for the more refined version).

I close with some acknowledgements. I am grateful to the organizers of the Workshop on the Urysohn Space for inviting me, and happy to see that this interesting area of mathematics is being actively explored (and that [4] is being read!). I am very grateful to the anonymous referee for catching some serious slips in the original version of this paper, and also for making sure that results were correctly attributed; I freely admit my ignorance of the literature and frequent re-invention of the wheel during the brief period I was working in this area. The referee asked why the publication of [4] did not occur until 1992, when I say that the work was done between 1983 and 1985. Immediately after I completed my Master's thesis in 1985, I became immersed in the entirely different research in set theory and combinatory logic which led to my Ph. D. in 1990; after that I was looking for a job, which was difficult at that time. But I felt that the work I had done on the Urysohn space and the question I had left open should be published in a more accessible format than [3], and once I was employed and settled I turned my attention to preparing and submitting [4], which contained some new ideas not found in [3], notably the attempt to present the construction of $\mathbb{U}$ entirely in terms of $C[0,1]$, but for the most part presented refinements of the results I had in 1985.

References

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