Quine's Calculus of Concepts

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1 Introduction

The calculus of concepts of Quine is described; its semantics and formal rules of inference are given.

2 The Natural Semantics of *CC*

We follow Quine's example in first presenting CC via the originally intended interpretation. It will turn out that the system has other quite natural interpretations as well, as well as some rather strange ones.

We begin by describing the language of CC.

Atomic terms of the language of CC are taken from a countable supply of free variables. Atomic constants will be introduced as well. Atomic terms of CC are terms of CC; if T and U are terms, then T^c, $T \times U$, T/U, and δT are terms. Other operations on terms may be introduced by definition. All terms are built from atomic terms by the primitive operations and operations defined in terms of the primitive operations below. Unary operations are always considered to have higher precedence than binary operations. Both product and quotient group to the right, and quotient has higher precedence than product.

We now present the intended interpretation of the notion of "concept".

Definition: If $\mathcal{D}$ is our universe of discourse and $n \in \mathcal{N}$, we define a *concept of degree* n as a pair (S, n) with $S \subseteq \mathcal{D}^n$. (The pairing with n is needed because empty concepts of different degrees need to be distinct.) We identify 1-tuples of elements of $\mathcal{D}$ with elements of $\mathcal{D}$ itself, and we postulate a unique 0-tuple, so concepts of degrees 0 and 1 make sense.

Any formula in the language of first-order logic with variables indexed by the positive integers can be associated in a natural way with a concept (actually, with a concept of each sufficiently large degree):

Definition: A concept (A, n) is said to represent a formula ϕ just in case ϕ has no free variable other than variables x_i for $i \leq n$ and $\{(x_1, \ldots, x_n) \mid \phi\} = A$. Notice that concepts of degree 0 represent sentences with no free variables (the two concepts of degree 0 represent the truth values).

Each term of the language of CC is intended to represent a concept. Note that there is no explicit indication of numerical degree in CC; the explicit indications of degree are a feature of the interpretation being given.

We now introduce the operations of CC (in their intended interpretation)

We first define $x.y$, for $x \in \mathcal{D}^m, y \in \mathcal{D}^n$, as the element of $\mathcal{D}^{m+n}$ obtained by concatenating x and y.

complement: The *complement* of a concept (A, n), written $(A, n)^c$ is

$$(\mathcal{D}^n - A, n).$$

(Quine would write $-(A, n)$).

product: The *product* of concepts (A, m) and (B, n), written $(A, m) \times (B, n)$ is

$$(\{x.y \mid x \in A \land y \in B\}, m + n).$$

Quine calls this the "Cartesian product", but this is somewhat misleading, since the underlying operation is concatenation rather than pairing; an important difference is that this operation is associative and Cartesian products properly so-called are not.

quotient: The *quotient* of a concept (A, m) by a concept (B, n), written $(A, m)/(B, n)$, is

$$(\{x \mid \exists y \in B.(x.y \in A)\}, \max(m - n, 0)\}).$$

Notice that the subtraction of degrees natural in this theory is a subtraction on natural numbers. Quine called this the "image" operation and used the double-quote notation for image to represent it. The definition of quotient when the degree of B exceeds the degree of A is a "don't care" case; under the definition given here such a concept is empty, but this will not be provable in the formalization we present.

diagonalization: The *diagonalization* of a concept (A, n), written $\delta(A, n)$ is the concept

$$(\{x.x \mid x \in A\}, 2n).$$

Quine uses I as the name of this operation.

It is quite appealing that this set of operations has the expressive power of first-order logic. The operations might be thought to have some of the same intuitive appeal as the basic operations of Boolean algebra.

3 The expressive power of CC

Theorem (Quine): The expressive power of the calculus of concepts is at least that of first-order logic with equality (if concepts are understood via the interpretation given above).

Proof: We use V^1 as the notation for the concept $(\mathcal{D}, 1)$ whose extension is the whole domain of the theory being interpreted. We can define V^n for each positive n as the product of n copies of V^1 and V^0 as V^1/V^1; V^0, the universal concept of degree 0, is an absolute notion of the calculus, unlike the V^i's with $i > 0$.

The sentence $x_i = x_j$ (for $i < j$) is represented by the concept $V^{i-1} \times (\delta V^{j-i}/V^{j-i-1}) \times V^k$ (for any k).

If A and B are concepts of the same degree, $\delta A/B$ is the intersection $A \cap B$ of A and B.

We proceed to represent a formula ϕ whose variables (free and bound) have indices $\leq n$ using a concept of degree n. We proceed by induction on the structure of ϕ.

If ϕ is an atomic sentence $R[x_{a_1} \ldots x_{a_k}]$, we let A be a concept representing the formula $R[x_1, \ldots, x_k]$. Let E be the intersection of the predicates of degree $n + k$ representing the equality assertions $x_{a_i} = x_{n+i}$ for each $i \leq k$. The sentence $R[x_{a_1} \ldots x_{a_k}]$ is represented by the degree n concept $((V^n \times A) \cap E)/V^k$ If ϕ is of the form $\sim \psi$, where ψ is represented by a concept A of degree n, ϕ is represented by A^c. If ϕ is of the form $\psi \wedge \chi$, where ψ and χ are represented by concepts A and B of degree n, ϕ is represented by $A \cap B$. All propositional logic operations can be defined in terms of negation and conjunction.

Let ϕ be of the form $(\exists x_i.\psi)$, where ψ is represented by a concept A of degree n. Let E be the intersection of the concepts of degree $2n$ representing the assertions $x_j = x_{n+j}$ for j less than or equal to n and not equal to i. ψ is represented by the concept $((V^n \times A) \cap E)/V^n$. The universal quantifier can be defined in terms of the existential quantifier.

The proof of the theorem is complete.

4 Other instantiations of the calculus of concepts

In this section, we present other interpretations of the calculus of degree, which will motivate the generality of our treatment of degree.

NOTE: some of these examples are sketchy.

In any interpretation of the calculus of concepts, concepts are sets of "argument lists" all of the same size. Degree measures the size of "argument lists". The examples in this section illustrate how the notion of "argument list" can be generalized while preserving the usefulness of the operations of the calculus.

Multi-sorted theories are handled as readily as the single-sorted theories in the natural interpretation. An argument list is a function α such that for each sort τ of the theory, $\alpha(\tau)$ is a list of objects of sort τ. The degree $|\alpha|$ of an argument list is also a function, from sorts to natural numbers: $|\alpha|(\tau)$ is the length of the list $\alpha(\tau)$. The concatenation $\alpha.\beta$ of two argument lists is defined so that $(\alpha.\beta)(\tau) = \alpha(\tau).\beta(\tau)$.

A concept is a pair $(\mathcal{A}, |\alpha|)$ of a set $\mathcal{A}$ of argument lists of degree $|\alpha|$. The operations of the calculus of concepts are defined in essentially the same way as they are in the interpretation above, with attention to the "arithmetic" of degrees: addition of degrees is defined in the obvious way (and a degree $2|\alpha|$ is read $|\alpha|+|\alpha|$), and this allows the definition of product and diagonal operations just as above. The degree of the quotient A/B, where A is of degree $|\alpha|$ and B is of degree $|\beta|$, is the degree $|\alpha| - |\beta|$ defined by $(|\alpha| - |\beta|)(\tau) = \max(|\alpha|(\tau) - |\beta|(\tau), 0)$; this subtraction operation generalizes a natural number subtraction operation.

It is straightforward to establish that the interpretation of multi-sorted first-order logic with equality in the calculus of concepts goes in essentially the same way as we showed it does for single-sorted first-order logic above. The differences are that one needs to have a different base concept V^1_τ for each sort τ (the set of all one-term argument lists of that sort); all degrees are then finite sums of these base concepts. Equality is defined (between variables of the same sort) in the same way given in the argument above. Negation and conjunction are handled in exactly the same way. The treatment of existential quantification is also essentially the same; it is important to note that the equality concepts making up the intersection E may be products of the equality concepts for two variables of a given sort with universal concepts of different sorts (they need to be padded to the degree of the concept over which we are quantifying).

This generalization is clearly useful; it will be nice to have a formalization which unified multi-sorted and single-sorted logics. However, there are further generalizations whose usefulness is much less evident. For these, we will only discuss the nature of argument lists and degrees, as we are not seriously interested in doing logic in these systems.

Let X be a set and let a concept be a set of functions from $[0, r)$ to X where r is a nonnegative real number. An argument list here is a function from $[0, r)$ to X. The degree of an argument list is the nonnegative real number r. We concatenate argument lists of degrees r and s to get an argument list of degree $r + s$ in the obvious way. A full development of the calculus of concepts is easy on this basis. This is an "unintended" implementation of the calculus, because "argument lists" here are "nonatomic"; we see analogues of "blocks of variables" here without being able to single out individual variables.

Even worse, let $X(r)$ be a set for each element r of $[0, 1]$ and let f be a continuous function from $[0, 1]$ to $[0, \infty)$; let a concept of degree f be a set of functions with domain $\{(r, s) \mid r \in [0, 1] \wedge f(r) < s$ sending each pair (r, s) in their domain to an element of $X(r)$. The degrees here are determined by the function f. Addition of degrees corresponds naturally to addition of these functions and it is fairly obvious how to define concatenation of "argument lists". Here we do not have the expected "granularity" of variables or even of the analogues of sorts of object! (this needs expansion).

A generalization of concepts which is arguably more natural than the ones above but which suggests that we should weaken our axioms is the following:

we return to a single domain X, but we let "argument lists" be functions from countable well-orderings to X, with degree determined by the countable ordinal which is their domain. A subtlety is that we treat earlier elements of the countable ordinal as *later* elements of the "block of variables" the argument list represents; the concatenation of a concept with domain α and a concept with domain β is a concept of domain $\beta + \alpha$. This implementation is of interest as handling an infinitary logic (it is not "unnatural" in the way the last two are); but it suggests a weakening of our axioms, because the operation of product is not commutative on degrees in this implementation. We actually do adopt commutativity of product on degrees as an axiom of the formal system presented here, because it simplifies the axiomatization of subtraction of degrees, but we need to observe that a further generalization is possible.

5 Axioms for the calculus of degree

NOTE: the indexing of the axioms in this section is mixed up! I'll fix it later.

In the formal system CC, concepts do not come with explicit indications of degree. It is necessary to verify that degree is somehow representable in the calculus and to axiomatize its properties.

We rely on the intended interpretation to develop the basic approach. In the intended interpretation, there is a canonical object in each degree which we may as well use to represent that degree: this is the universal concept $(\mathcal{D}^n, n)$. In the intended interpretation, the universal concept of the same degree as a concept A can be expressed as $(\delta A/A^c)^c$; it is straightforward to check that this works (and that it works in our other interpretations). In uninterpreted CC, we introduce the following

Definition: V^A, called the *degree of* A, is defined as $(\delta A/A^c)^c$. $\emptyset^A$ is defined as $(V^A)^c$.

The operations on degrees which are needed in the intended interpretation are addition and subtraction of natural numbers; our intention is that the degree of $A \times B$ will be the sum of the degrees of A and B, while the degree of A/B will be the difference of the degree of A and the degree of B. In the intended interpretation, we decided that the degree of A/B would be 0 when the degree of B exceeded the degree of A; this corresponds to a natural decision as to how to define subtraction as a complete operation on the natural numbers. We make the modification suggested by the last example in the previous section that commutativity does not hold for "addition" of degrees, though a special case will hold.

We adopt these axioms which reflect the effects of the operations other than quotient on degree:

D1: $V^{V^A} = V^A$

D2: $V^{A^c} = V^A$

D3: $V^{\delta A} = V^A \times V^A$

D4: $V^{A \times B} = V^A \times V^B$

We adopt an axiom expressing the associativity of product on all concepts, not just degrees.

D5: $(A \times B) \times C = A \times (B \times C)$

We now consider the problem of the quotient operation and the "subtraction" operation on degrees.

We introduce the two absolutely definable concepts.

D7: $V^{A/A} = V^{B/B}$

Axiom **D7** motivates the following

Definition: We define V^0 as $V^{A/A}$ and $\emptyset^0$ as $(V^0)^c$.

Degree 0 is the only degree which can be defined in absolute terms. In the intended interpretation, degree 0 has the two inhabitants V^0 and $\emptyset^0$, which are natural representatives of the truth values. V^0 is the identity of degree addition.

D8: $A \times V^0 = V^0 \times A = A/V^0 = A$

We tackle the general problem of axiomatizing subtraction of degrees. Our approach is to view the degrees as forming a lattice. The intention is that the difference $\delta - \epsilon$ of two degrees can be understood to be actually equal to $\delta - \min(\epsilon, \delta)$ (which is true in the natural number case).

In the natural number case, one of $\delta - \epsilon$ and $\epsilon - \delta$ will be 0; this is not the case in the multi-sorted interpretations given above. What is true in these interpretations is that $\delta - \epsilon$ and $\epsilon - \delta$ will be "disjoint" in the sense that they will not share variables of any given sort.

We will not need to have max or min operations as primitives in our calculus, because max and min are naturally definable in terms of subtraction. $\min(\delta, \epsilon) = \delta - (\delta - \epsilon) = \epsilon - (\epsilon - \delta)$ is easily verified as a property of natural number subtraction. Similarly, $\max(\epsilon, \delta) = \delta + (\epsilon - \delta) = \epsilon + (\delta - \epsilon)$.

We discuss the justification for this in our more general context. The defining property of $\delta - \epsilon$ is that $\min(\delta, \epsilon) + (\delta - \epsilon) = \delta$. The defining property of $\delta - (\delta - \epsilon)$ is then that $\min(\delta, \delta - \epsilon) + (\delta - (\delta - \epsilon)) = \delta$. Since we expect $\min(\delta, \delta - \epsilon)$ to be $\delta - \epsilon$ (if there is any justice), we see that this definition depends on commutativity. For this reason, we do adopt the axiom

D6 $V^A \times V^B = V^B \times V^A$

in spite of the fact that it restricts the generality of the calculus. It appears that an implementation general enough to cover the infinitary implementation discussed in the last section would require a primitive notion of minimum on degrees; we prefer to work in an context where all notions can be defined in terms of Quine's basic operations.

We note that the minimum of degrees V^A and V^B is $V^A/V^A/V^B$; this motivates the right grouping of the quotient operator.

D10: $V^A/B = V^{V^A/V^B}$

This axiom expresses the idea that the degree of a quotient is determined by the degrees of the concepts involved. We do not adopt the axiom $V^A/B = V^A/V^B$, because this makes commitments about bad quotients (those where the degree of B is not less than or equal to the degree of A).

D11: $(V^A \times V^B)/V^B = V^A$

D12: $(V^A/V^{A/B}) \times V^{A/B} = V^A$

These axioms capture the defining properties of subtraction.

D13: $V^A/V^A/V^B = V^B/V^B/V^A$

This axiom expresses the commutativity of the degree minimum operation.

D14: $A/(B \times C) = (A/C)/B$

D15: $(A \times B)/V^{B/C} = A \times (B/V^{B/C})$

These axioms capture technical points about the relationship between product and quotient. D14 is the only axiom which says anything about "bad quotients"; it is compatible with the notion that all bad quotients are empty.

D17: $\delta A/V^A = A$

This might not be regarded as a degree calculation, but it is another rule of calculation which will be needed in the following section.

NOTE: some argument for the adequacy of these axioms is needed, though this may turn out to be implicit in the completeness proof.

6 Rules for sequent calculus presentation of CC

Sequents are of the form $\Gamma \models \Delta$, where Γ and Δ are sets of concepts and all concepts in $\Gamma \cup \Delta$ are of the same degree.

Degree calculations need to be allowed. We also need to recognize the special category of "equality concepts" $V^A \times ((\delta V^B)/V^{B/C})$; these represent assertions of equality between blocks of variables. The fact that $(\delta V^A)/V^{A \times B} = V^A/V^B$ needs to be taken into account; this is subsumed under degree calculations.

Axioms: $\Gamma, A \models A, \Delta$.

exercise: $\models V^A$ is provable.

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^c \vdash \Delta} \, \mathbf{NEG-L}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^c, \Delta} \, \mathbf{NEG-R}$$

These rules for negation are exactly what we expect. An additional calculation allowed with complement is the fact that complement commutes with left or right padding: $V^A \times B^c = (V^A \times B)^c$ and $A^c \times V^B = (A \times V^B)^c$.

$$\frac{\Gamma, V^A \times B, A \times V^B \vdash \Delta}{\Gamma, A \times B \vdash \Delta}\,\mathbf{PROD-L}$$

$$\frac{\Gamma \vdash V^A \times B, \Delta \quad \Gamma \vdash A \times V^B, \Delta}{\Gamma \vdash A \times B, \Delta}\,\mathbf{PROD-R}$$

These rules for product are exactly what we should expect. See the additional rule **COPY** below, which eliminates right padding; this is also a kind of product rule.

$$\frac{\Gamma^A, A, V \times B \vdash \Delta}{\Gamma, A/B \vdash \Delta}\,\mathbf{QUOT-L}$$

$$\frac{\Gamma \vdash C/V^B \quad \Gamma^A, C \vdash A, \Delta \quad \Gamma^A, C \vdash V \times B, \Delta}{\Gamma \vdash A/B, \Delta}\,\mathbf{QUOT-R}$$

(where C is any concept and V is the appropriate universal concept; the trick is that C can always be taken to be a conjunction of equalities, and the leftmost hypotheses in that case is proved by calculation)

(Γ^A, Δ^A are suitably right padded versions of Γ, Δ)

An important fact about these rules is that they only support "quantification" over final "blocks of variables"; the limitation is circumvented by the **COPY** rule.

$$\frac{\Gamma_1^{\Gamma_1 \times \Gamma_2}, V^{\Gamma_1} \times \Gamma_2, (\delta V^{\Gamma_1 \times \Gamma_2})/V^A \vdash \Delta_1^{\Gamma_1 \times \Gamma_2}, V^{\Gamma_1} \times \Delta_2}{\Gamma_1, \Gamma_2 \times V^A \vdash \Delta_1, \Delta_2 \times V^A}\,\mathbf{COPY}$$

(the notation here is tricky: V^{Γ_1} is the universal concept of the common degree of Γ_1, and degrees of sets of concepts are used in several places in ways we have above used degrees of concepts. An operation $V^A \times \Gamma$ yields a set of concepts padded on the left with V^A.)

(**COPY** is a rule for elimination of right padded concepts; the application is to bring quotients (including generalized equality concepts) and concepts to which generalized equality concepts are to be applied to the position where the quotient or diagonal rules can be applied).

The notation in the **COPY** rule is strained because we need economical ways to describe a rule which allows manipulation of a set of concepts of arbitrary size on each side.

$$\frac{\Gamma, V^A \times D \times D, V^A \times C^D \times B^D \vdash V^A \times F^D \times E^D, \Delta}{\Gamma, V^A \times B^D \times C^D, V^A \times \delta D \vdash V^A \times E^D \times F^D, \Delta}\,\mathbf{EQ-L}$$

$$\frac{\Gamma \vdash V^A \times B \times B, \Delta \quad \Gamma, V^A \times C^B \times D^B \vdash V^A \times D^B \times C^B, \Delta}{\Gamma \vdash V^A \times \delta B, \Delta}\,\mathbf{EQ-R}$$

(in **EQ-R**, the concepts C^B and D^B are fresh concepts not mentioned in the conclusion and of the same degree as B)

The following rules handle the problem which we call "block splicing": an equality concept asserts the equality of two blocks of variables; we need to be able to decompose it into the conjunction of equations between corresponding sub-blocks.

$$\frac{\Gamma, V^A \times ((\delta B^{C\times D})/V^D) \times V^D, V^{A\times C} \times (\delta V^{D\times C})/V^C \vdash \Delta}{\Gamma, V^A \times \delta B^{C\times D} \vdash \Delta}\ \mathbf{SPLICE-L}$$

$$\frac{\Gamma \vdash V^A \times ((\delta B^{C\times D})/V^D) \times V^D, \Delta \quad \Gamma \vdash V^{A\times C} \times (\delta V^{D\times C})/V^C, \Delta}{\Gamma \vdash V^A \times \delta B^{C\times D}, \Delta}\ \mathbf{SPLICE-R}$$

7 The Completeness Proof

7.1 Objects from Concepts

The calculus of concepts is a language without nouns or pronouns; there is nothing in the language which appears to refer directly to “objects”. In this subsection, we discuss how to formally interpret talk of sorts of object and variables referring to objects from a theory in the calculus of concepts.

We extract this information from the calculus of degrees. Each degree corresponds to a sort of object and degrees can further be used as “indices” of “variables” of each sort. In the natural interpretation of the calculus of concepts, which corresponds to single-sorted predicate logic, degree n corresponds to lists of length n of objects of the single underlying sort. In the natural interpretation, there are blocks of variables of length n beginning at each natural-number-indexed position; the concept $V^k \times B^n \times V^m$ includes those $(k+n+m)$-tuples in which the kth block of n variables has the property represented by the degree n concept B^n.

This needs to be generalized somewhat. Each degree V^A represents a sort. The “variables” of the sort associated with V^A are associated with concepts of the form $V^B \times V^A \times V^C$, in which the degree V^C serves strictly as padding; it is the degree B that determines which “variable” of the sort associated with V^A is being considered. However, an equivalence relationship on the V^B’s, determined by V^A, must be imposed. That is, different V^B’s may index the same “variable” of sort V^A. This can be seen by considering the interpretation of multi-sorted predicate logic givne above: if V^A interpreting one of the basic sorts is being considered , only the number of variables of that sort in the “block of variables” associated with the degree V^B will be relevant to which variable is being denoted. Degrees V^B and V^C are taken to be equivalent as indices for variables of sort V^A just in case $V^{A/A/B/C}$ and $V^{A/A/B/C}$ are both equal to V^0.

A “variable” is then determined by a degree V^A determining its sort and an equivalence class of degrees under the indicated equivalence relation determining its index (HOLE: prove that it is an equivalence relation!). The models which we will construct in the next subsection are sets of variables in this sense.

Variables generally represent lists of arguments of sorts taken from the underlying logic. The operation of concatenation on variables is easily definable. (HOLE:give details).

7.2 The Proof Itself

In this subsection, we assume that a sequent $\Gamma \vdash \Delta$ is unprovable and proceed to construct a model on which the calculus of concepts has an interpretation in which this sequent turns out to be invalid. We need to start with a complete knowledge of the particular calculus of degree in which we are working, in order to be able to construct "variables" in the manner described in the previous subsection: concepts in our model will be represented as sets of "variables" of the sort corresponding to their degree.

It will be assumed that the calculus of degrees is at most countably infinite, which limits the number of sorts and variables of each sort to countably infinite size.

We suppose the existence of symbols for countably many "fresh" concepts of each degree, not present in the unprovable sequent.

We construct a tree of sequents as in the usual completeness proofs. At any level, we first carry out all applications of the negation, product, and equality rules, which involve no complications and do not change degree. Applications of equality rules may need to be iterated to make sure that everything we want is done. Block splicing does not change degree, but it involves many possible decompositions of the degree of the equality concept into two degrees; we use all the decompositions whose index in a suitable enumeration is bounded by our current level in the tree (the way we handle copying will ensure that each diagonal concept will be visited enough times that all possible decompositions will be considered).

There remain to be considered the rules for quotient and the rule of copying, which change degree. The problem here is that we can't perform many rules at the same time, because of the degree change. We handle these using enumerations of all quotients and enumerations of all pairs of finite sets of concepts of uniform degree; each enumeration is to have the property that all items occur infinitely often. At a given level of the tree, we apply the quotient rule if the indexed quotient term (left padded suitably and possibly right padded as well) is present (copying to the front if necessary), and then apply copying if the indicated pair of finite sets of concepts (right padded suitably) is present to be copied. In the rule **QUOT-R**, we use all intersections of equality concepts which have the correct degree and appear in an enumeration of such at a point before the index of the current stage.

We weaken at all levels so that concepts present on either side anywhere on a branch are present everywhere above that point on that side of the branch.

Now the usual argument establishes that there is an infinite branch in this tree of sequents consisting of unprovable sequents (if there weren't we would have a proof).. We claim that the infinite branch gives a partial description of a model on which there is a sensible interpretation of the calculus of concepts in which the original unprovable sequent is invalid.

The elements of the model are all the "variables" for the given calculus of degree defined as in the preceding subsection. Concepts in the model are interpreted by sets of "variables" of the appropriate degree. Variables in the model are interpreted as being equal when they belong to the same interpreted concepts. The variable of sort determined by V^A and index determined by V^B will belong to the interpreted concept A just in case some concept $V^{B'} \times A \times V^C$ with B' equivalent to B as an index appears on the left side of some sequent in the branch.

The following can be deduced: complement and product have their natural semantics, in the sense that if A^c is asserted (denied), A will be denied (asserted), while if $A \times B$ is asserted A and B will be asserted of appropriate variables interpreted as sub-blocks, and if it is denied one of these will be denied of the appropriate sub-block. If two variables are asserted to be equal, any property asserted of one will also be asserted of the other. If two variables are denied equality, there will be a concept asserted of one and denied of the other. If a block is asserted to be equal to another block, corresponding sub-blocks will be asserted to be equal; if it is denied to be equal to another block, at least one subblock in any decomposition will be denied equality with the corresponding subblock. If a quotient A/B is asserted, there will be a variable of which A is asserted of whose appropriate final sub-block B is asserted and of whose appropriate initial block A/B is asserted. If a quotient A/B is denied, each block of variables of the degree associated with A must either be denied inclusion in A or have its final segment of the degree of B denied inclusion in B. We have combinatorial completeness, in the sense that every combination of subblocks into blocks is realized somewhere; yes, this is done by the quotient rule itself (if copying actually does it, the quotient rule can be simplified).

This is enough; the model works.

HOLE: this is enough for me to see that it works, but needs to be throughly rewritten for anyone else to see this!

8 Why should we care?

Is there any respect in which the calculus of concepts is an improvement on standard first-order logic with equality? It cannot be regarded as a practical system in which to reason! The only possible advantage is conceptual economy, and here it does have certain advantages. The notion which appears essentially in a syntactical or semantic account of standard first-order logic and which does not appear in the calculus of concepts is that of *substitution*. Moreover, the notion of substitution needed for first-order logic is quite complicated, due to the appearance of variable binding operators. In the calculus of concepts, the notion of substitution is replaced by a notion of copying. The mechanics of the system become more complex, in exactly the way that imagining an account of substitution in the presence of variable binding in terms of copying would suggest. But the basic concepts and the rules associated with those basic concepts retain a certain appealing simplicity, though somewhat encumbered by "calculations of degree".

The complete disappearance of the analogues of nouns and pronouns from this system is somewhat startling.

9 Notes to myself–not to be viewed as part of paper draft

In the completeness proof, we are developing a model in which objects are "positions" in "blocks of variables" and concepts are sets of these positions, respecting equivalence relations on the positions imposed by equality hypotheses.

I believe that the forms of the rules now given are the correct ones needed for the completeness proof.

Add a description of the typed system.

Outline the completeness proof.

Think about good treatment of arbitrary situations with the degree calculus.

Extracting sorts of objects from the degree calculus:

each degree corresponds to a sort of object (the degree $V0$ to a trivial sort with just one element). A product degree $V^A \times V^B$ is inhabited by concatenations of objects of the sort associated with V^A and objects of the sort associated with V^B. "Concatenation" rather than pairing because this is an associative operation; concatenation is the operation which builds argument lists. Note that the structure of argument lists is not thought of as linear here – there is a separate linear list of arguments of each sort.

concepts of higher degree "refer" to many objects of any given sort: degrees also indicate "positions" of blocks of a given sort. The concept $V^A \times B$ picks out objects in the "position" indicated by V^A of the sort associated with V^B and which happen to fall under B. Only the part of V^A which is "commensurable" with V^B is relevant to the position specified? Is the part of A commensurable with B computable in the calculus of degrees? No – because no such object need exist in the continuous function example!

How do we tell whether the an object of the degree of C with position A is the same as the object of the degree of C with position B? Is it enough for B/A and A/B to be incommensurable with C? That seems right; it requires some verification. (it might depend on commutativity!) This seems to express the notion that two degrees have equivalent "parts commensurable with" a third; we could add equivalence classes under this relation as virtual degrees to get a kind of completion, maybe?

commutativity for incommensurables seems to make sense, but commutativity of commensurable lists does not seem sufficiently general. $V^{A/B} \times V^{B/A} = V^{B/A} \times V^{A/B}$ ought to be true.

any pair of degrees A and B can be expressed as sums of the minimum of the two plus degrees A' and B'.

I would like to prove that the equations of the calculus of degree implicit in any finite sequent can be realized in a conventional multi-sorted context; this would allow a sensible equation of the calculus of concepts with multi-sorted first order logic with equality, leaving aside the bizarre nonatomic calculi of degrees which are possible.

Can we take any degree calculus and construct a completion thereof which will behave sensibly (allowing things like commensurable parts)?

is commutativity of degree addition really needed? Think about reverse well-ordered lists of arguments; these would seem to satisfy the conditions for a calculus of degree, but the degree calculus one obtains is not commutative! This can be used as an example to illustrate why the asymmetry in the treatment of left and right ends might be appropriate – there might not be a left end!

Implementation: concepts are sets of functions from countable ordinals to a set X, with the domain of a concept determining its degree, and the "beginning" of the domain corresponding to the final part of the "argument list". (or maybe this example suggests that we should view quotients as acting on first variables and copying as proceeding toward the beginning?) product, quotient, complement and diagonal all have sensible interpretations, and product is not commutative on degrees! it seems that in the absence of commutativity it might be difficult to come up with a sensible way of handling bad quotients.

For construction of objects, suppose the calculus of degree is known completely; we have proposed a definition for equivalence of variables of each sort (sorts indexed by degrees and variables in each sort indexed by degrees up to equivalence of parts commensurable with the degree indexing the sort). The actual objects can then be identified with positions, with an equivalence relation determined by the equality concepts in the sequents one is working with.

So we should assume that we know what the calculus of degree is in the completeness proof (this seems natural anyway) and that we already know that implicit conditions on degree are satisfied in our sequent. Then we can use the calculus of degree itself as building material for the elements of our "model": each degree codes a sort, there is a natural way to relate objects of different degree as concatenations, and so forth. But we cannot do this without full commutativity: positions are not specifiable in this way in the reverse well-ordering example (think about a reverse well-ordering of type ω; all positions here are left padded by the same degree! So stick with commutativity. But can positions still be specified in relation to the sequents we work with? I think so. Noncommutativity still might work; our method of specifying position is what is incorrect! It is the relation to the end of the sequent that we should look at (and fix things in relation to earlier sequents as well; but this isnt a problem since concepts from earlier sequents are always retained and updated). The correct way to describe position is to consider concepts $V \times V^B \times V^A$, with V^B determining sort and V^A determining position (automatically adjusted in each sequent); the fact that V is not necessarily uniquely determined by V^A is only a curiousity. The equivalence between different V^A's is determined in the same way as before. it is useful for the calculus of degrees to be countable, since we need to limit the number of witnesses to be handled (the number of equality concepts that need to be used at each stage). It is natural for this calculus to be countable as well! (notice that countable calculi can still exhibit the weird incommensurability phenomena).

Would it be better to build a complete model description as in Marcel's proof? Then I would not need to worry about partial valuations in quite the same way?

Am I going to have trouble with partial valuations if the sort structure is ill-founded? No, I don't see such a problem. We are looking at concrete objects at each type level; simply accept only those equations we are forced to have, and everything should work out correctly – maybe!!! Consider phenomena that occur with equations between ill-founded pairing structures? I don't see that these are really a problem.

Just for fun, I think I should give an axiom which allows analysis of bad quotients. How hard would it be to express the idea that every bad quotient is to be identified with the appropriate empty concept?

If we accept as a logical truth the assertion that each sort contains more than one object, we can assert that A/B is empty if V^B/A contains more than one element. (we also become able to express ideas such as linearity of the type structure – but maybe that's already expressible?) The problem here is expressing the idea of the nontriviality of a sort. If we allow bad quotients to be universal rather than empty, there might be a cleaner treatment?

another idea: A/B is equivalent to A/B's conjunction with the assertion that B is logically equivalent to $V^{B/A} \times B/V^{B/A}$. This might be pretty clean!

Define left quotient $A//V^{A/B}$, the left quotient of A by $V^{A/B}$; this is a pretty easy diagramming problem.

Then $A/B = A/B//V^{B/A}$ expresses what is perhaps the nicest solution to the bad quotient problem.