Strong axioms of infinity in *NFU*

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1 Introduction

This paper discusses a sequence of extensions of *NFU*, Jensen's improvement of Quine's set theory "New Foundations" (*NF*) of [16].

The original theory *NF* of Quine continues to present difficulties. After 60 years of intermittent investigation, it is still not known to be consistent relative to any set theory in which we have confidence. Specker showed in [20] that *NF* disproves Choice (and so proves Infinity). Even if one assumes the consistency of *NF*, one is hampered by the lack of powerful methods for proofs of consistency and independence such as are available for use with *ZFC*; very clever work has been done with permutation methods, starting with [18] and [5], and exemplified more recently by [14], but permutation methods can only be applied to show the consistency or independence of unstratified sentences (see the definition of *NFU* below for a definition of stratification). For example, there is no method available to determine whether the assertion "the continuum can be well-ordered" is consistent with or independent of *NF*. There is one substantial independence result for an assertion with nontrivial stratified consequences, using metamathematical methods: this is Orey's proof of the independence of the Axiom of Counting from *NF* (see below for a statement of this axiom).

We mention these difficulties only to reassure the reader of their irrelevance to the present work. Jensen's modification of "New Foundations" (in [13]), which was to restrict extensionality to sets, allowing many non-sets (urelements) with no elements, has almost magical effects. *NFU* with the axioms of Infinity and Choice is consistent relative to a weak fragment of *ZFC* (its consistency can be proved in Zermelo set theory; it is equivalent in strength to Zermelo set theory with separation restricted to Δ_0 formulas, which is also known as Mac Lane set theory). We will use the name *NFU* in this paper to refer to the theory which Jensen called *NFU* + Infinity + Choice: we regard the consistency of Jensen's *NFU* with the assertion "the universe is finite" as of purely technical interest (a model of this theory would be externally infinite in any case), and we regard Choice as an essential part of the mathematician's toolkit. *NFU* can be extended in strength as far as one is willing to extend *ZFC*; for example, Jensen demonstrated that one can prove in *ZFC* that *NFU* has α-standard models for any ordinal α. *NFU* is a fluent set theory; mathematical work can be done in this system in a style not all that different from *ZFC*. We have argued for this thesis in our paper [10] and our book [11], an elementary set theory textbook using *NFU*.

In this paper, we will discuss a sequence of extensions of *NFU*. These extensions are natural from the standpoint of *NFU* as an autonomous foundation for mathematics; they do not result from adjoining assumptions natural to *ZFC* to *NFU*. They can also be motivated from a standpoint suggested by Hinnion in [8], in which *NFU* (or an extension) is used as a "superstructure" over a Zermelo-style set theory, in the same way that von Neumann-Gödel-Bernays or Kelley-Morse set theory (*KM*) introduces proper classes as a "superstructure" over *ZFC*. The superstructure provided by the "big" sets of *NFU* is more complex and might be expected to provide more additional power than that provided by the theories with proper classes, and this indeed turns out to be the case: each of the extensions we will consider is much stronger than superficial consideration of the axioms added would suggest.

The system *NFUM* considered at the end of the paper, the strongest extension of *NFU* with which we deal here, is the system of our book [11]; this paper provides the theoretical underpinnings of the system which would not be appropriate to present in the "elementary" format of the book. The book could be used as a supporting reference for the introductory parts of this paper, since it gives in full arguments we omit here as being elementary. We defined this system so as to get an interpretation of *ZFC* in the strongly Cantorian isomorphism classes of well-founded extensional relations "with top" (this terminology will be explained below); Robert Solovay showed us (to our surprise) that the system is much stronger than *ZFC*, and, indeed, that intermediate extensions proposed by others were stronger than had been realized. The results on the exact strength of *NFUM* itself given here are ours, but the results on the precise strength of the intermediate systems *NFUA* and *NFUB* are Solovay's. The consistency strength of *NFUM* is precisely that of Kelley-Morse set theory (*ZFC* extended with proper classes, with quantification over proper classes permitted in instances of Separation, Replacement, and class comprehension) with the addition of a predicate on proper classes which is a κ-complete nonprincipal ultrafilter on the proper class ordinal κ. The Axiom of Cantorian Sets which characterizes *NFUA* was proposed a long time ago (in [6]) by C. Ward Henson; the Axioms of Small and Large Ordinals which are used to define *NFUB* and *NFUM* were proposed by us recently in the course of our development of the elementary set theory text [11].

2 Introduction to *NFU*

NFU as we present it is a first-order theory with sethood, equality and membership as primitive predicates. The axioms below give a theory equivalent to Jensen's original formulation; to these we will add axioms of Infinity and Choice.

Sets: $x \in y \rightarrow \text{set}(y)$

Ext: $(\text{set}(y) \wedge \text{set}(z) \wedge (\forall x . x \in y \rightarrow x \in z)) \rightarrow y = z$

Comp: $(\exists A . \text{set}(A) \wedge (\forall x . x \in A \leftrightarrow \phi))$, where the variable A is not free in ϕ and ϕ is "stratified" (this notion is defined below).

Jensen's original formulation did not involve a sethood predicate; our use of such a predicate follows a suggestion of Quine in his remarks accompanying [13]. The function of the sethood predicate is to allow us to pick out the empty *set* from the other objects with no elements that may be present. Our formulation of *NFU* includes the additional axioms of Infinity and Choice introduced below: the system we call *NFU* is a slight modification of the system he called *NFU* + Infinity + Choice in [13] (our Axiom of Infinity is different). The consistency strength of this system is exactly that of Russell's theory of types, as simplified by Ramsey, with the Axiom of Infinity.

A formula is said to be "stratified" iff it makes sense in Russell's simple theory of types, as simplified by Ramsey, with a suitable assignment of types to its variables. More formally, we say that a formula ϕ in the language of *NFU* is stratified iff there is a function `type` from variables to integers such that for each atomic subformula $x = y$ of ϕ we have $\texttt{type}(x) = \texttt{type}(y)$ and for each atomic subformula $x \in y$ of ϕ we have $\texttt{type}(x) + 1 = \texttt{type}(y)$.

Stratification extends naturally to new operations. The most succinct way to summarize this is to describe the effects of the introduction of terms defined using a definite description operator (following Rosser's treatment in [17]). If terms $(\iota x.\phi)$ are admitted into our language, the function `type` needs to be extended to definite descriptions as well as variables. The conditions for stratification in the extended language are the conditions already stated, extended to apply to atomic subformulas involving definite descriptions as well as variables, and the additional condition that $\texttt{type}(\iota x.\phi)$=$\texttt{type}(x)$. Any operation we define can be interpreted as a definite description with parameters, and the effect of these conditions will be to assign to the result of any operation a certain type relative to the types of its arguments (and to enforce relationships between the types of its arguments if it has more than one argument). For example, the union $A \cup B$ of two sets (definable as $(\iota C.(\forall x.x \in C \leftrightarrow (x \in A \vee x \in B)))$ must be assigned the same type as A and B by this criterion, while the singleton $\{x\}$ $= (\iota X.(\forall y.y \in X \leftrightarrow y = x))$ must be assigned type one higher than that of x. The von Neumann successor operation $x^+ = (\iota X.(\forall y.y \in X \leftrightarrow (y = x \vee y \in x)))$ is an example of an operation which is unstratified and cannot appear in any set definition in *NFU*.

The axiom scheme of Stratified Comprehension can be replaced by a finite list of comprehension axioms. The original reference for this is Hailperin's [4], but the axioms given there are hard to understand and use. A much more natural set of axioms is used in my elementary set theory text [11], in which Stratified Comprehension is proved as a meta-theorem rather than given as a (rather undigestible!) basic assumption.

In order to introduce the Axiom of Infinity in our favorite form, we introduce new predicates π_1 and π_2, with the same stratification requirements as the equality relation. These are intended to be the projection relations of a type level ordered pair. $x\pi_1 y$, for example, is to be read "y is the first projection of x".

Inf: $(\forall x.(\exists! z.x\pi_1 z) \wedge (\exists! w.x\pi_2 w)) \wedge (\forall z.(\forall w.(\exists! x.x\pi_1 z \wedge x\pi_2 w)))$

Our "Axiom of Infinity" simply asserts the existence of a type-level pairing operation. The usual Kuratowski pair $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ is inconvenient for this kind of set theory because a Kuratowski pair is two types higher than its projections. If the Axiom of Infinity were given in the more usual form asserting the existence of an infinite set, and the Axiom of Choice were assumed as well, it would be possible to prove the existence of a type-level ordered pair operation as a theorem; this would involve the inconvenience of developing the notions of function and relation first in terms of the Kuratowski pair (with a type differential of 3 between the function and the argument in a function application term $f(x)$) then re-developing it in terms of the type-level pair, recovering the more natural type differential of 1 between a function and its argument. We think that it is more economical to take the type-level pair as primitive from the outset; proving that there is an infinite set, given a type-level pair, is quite easy. The philosophical purity of defining the ordered pair in terms of membership can be dispensed with in a system which must have urelements in any case. (In the absence of Choice, the existence of a type-level ordered pair implies but is not equivalent to the existence of an infinite set, but *NFU* with the axiom "there is an infinite set" interprets *NFU* with a type-level pair in a straightforward manner).

We also adopt the Axiom of Choice, completing the statement of our base theory *NFU*. The various equivalences between forms of the Axiom of Choice remain valid in this kind of set theory; an economical way to state AC in *NFU* is "the universal set can be well-ordered".

We briefly review the development of basic mathematical notions in *NFU*. Relations and functions are defined exactly as in the usual set theory. Cardinal numbers (including the natural numbers) are defined as equivalence classes of sets under equinumerousness; i.e., 0 is the set $\{\{\}\}$ of all sets with 0 elements, 1 is the set of all sets with one element, $\aleph_0$ is the set of all countably infinite sets, and so forth. Ordinal numbers are defined as equivalence classes of well-orderings under similarity. The von Neumann ordinals have an unstratified definition (recall that the von Neumann successor operation cannot appear with a variable argument in the definition of any set) and are not appropriate for use in set theory with stratified comprehension. It is consistent with *NFU* that no infinite von Neumann ordinal exists; it is also possible for there to be many infinite von Neumann ordinals; these results are established by permutation methods (see [5]). Note that the identifications between ordinals and initial segments in the natural well-ordering of the ordinals and between cardinal numbers and the corresponding initial ordinals found in the usual set theory do not hold here.

We introduce a definition which will be useful below:

Definition: If κ is a cardinal number, $\text{init}(\kappa)$ is defined as the first ordinal which is the order type of a well-ordering of a set of cardinality κ.

The large objects which figure in the paradoxes of Cantor and Burali-Forti exist in *NFU*, but they do not have quite the expected properties. For example, the universal set V exists, and has a cardinality $|V|$ (the set of all sets A such that there is a bijection between A and V), which is the largest cardinal. But there is no Cantor paradox of the largest cardinal. The reason for this is that Cantor's theorem on the cardinality of power sets does not take the expected form. The "theorem" $|\mathcal{P}(A)| > |A|$ would indeed give us a paradox with $A = V$. One should be suspicious of this proposed "theorem", because the set A appears in it with two different relative types. In fact, what can be proved in *NFU* is the same theorem which can be proved in the simple theory of types: $|\mathcal{P}(A)| > |\mathcal{P}_1(A)|$, where $\mathcal{P}_1(A)$ is the set of all one-element subsets of A. Note that the two appearances of A now have the same relative type. The special case of this with $A = V$ asserts that $|V| \geq |\mathcal{P}(V)| > |\mathcal{P}_1(V)|$; the cardinality of the set of all sets is greater than the cardinality of the set of all singletons. The cardinality of the universe turns out to be provably much larger than the cardinality of the set of all sets; this is a consequence of the result of Specker that *NF* disproves AC, which translates to the result that there are a lot of atoms in *NFU* + Choice (see [1]). We have proved a stronger theorem which implies that there are many cardinals between $|V|$ and $|\mathcal{P}(V)|$ (see [3], p 67). The external bijection $x \mapsto \{x\}$ from the universe to the set of all singletons is not a set; its definition is unstratified, so there is no reason to expect it to be a set. The set of all ordinals exists, and the natural well-ordering of the set of all ordinals exists. This natural well-ordering of all the ordinals is a member of an ordinal Ω. The Burali-Forti paradox is avoided, because the order type of the initial segment of the ordinals below an ordinal α with the natural order cannot be proved equal to α; there is a difference of 2 in relative type between the ordinal α and the order type of the segment determined by α. The Burali-Forti argument proves that the order type of the segment determined by Ω is less than Ω. Details can be seen in [3] or [11].

The fundamental result in the model theory of *NF* (due to Specker in [21]) is that *NF* is consistent precisely if there are models of *TST* (the simple theory of types) with a "type-incrementing isomorphism" (that is, an isomorphism between the model of *TST* and its submodel obtained by dropping the lowest type[1]). These results transfer to *NFU*: a model of *NFU* can be obtained from a model with a type-incrementing isomorphism of *TSTU*, the simple theory of types with urelements, in which each type consists of subsets of the previous type plus urelements (objects with no elements which are permitted to be distinct from one another and the empty set).

A model of *TST* with a type-incrementing isomorphism has never so far been constructed. But a model of *TSTU* with a type-incrementing isomorphism is easily described. Any sequence of levels of the cumulative hierarchy with strictly increasing indices can be interpreted as a model of *TSTU*: if type i is represented by the stage V_{α_i} for each i, the membership $\in_i$ of type i objects in type $i+1$ objects is defined thus: "$x \in_i y \leftrightarrow x \in V_{\alpha_i} \wedge y \in V_{\alpha_i+1} \wedge x \in y$"; the elements of $V_{\alpha_{i+1}} - V_{\alpha_i+1}$ are treated as urelements. The membership relations need to be indexed because an object in $V_{\alpha_{i+1}} - V_{\alpha_i+1}$ which is an urelement with respect to $\in_i$ will have its usual extension relative to $\in_{i+1}$. We give an alternative description for those who prefer disjoint types and a single membership relation $\in_{TTU}$: type i for each $i \in \mathcal{N}$ will be $V_{\alpha_i} \times \{i\}$, and $(x,i) \in_{TTU} (y, i+1)$ will hold iff the two pairs are model elements, $y \in V_{\alpha_i+1}$, and $x \in y$.

Now it is easy to describe a model of *NFU*: take a nonstandard model M of the usual set theory with an external automorphism j which moves a (nonstandard) infinite ordinal α upward ($j(\alpha) > \alpha$). Consider the model of *TSTU* determined by the sequence of levels $V^M_{j^i(\alpha)}$ of the cumulative hierarchy (as seen by M); the automorphism j itself is a type-incrementing isomorphism for this model. Although this sequence of levels is not a set in M, this does not compromise the fact that it is a model of *TSTU*, because no assertion of the language of *TSTU* can refer to more than a concrete finite number of types, and any concrete finite set of successive stages is a set in M and models all appropriate axioms of *TSTU*. Type 0 of this model of *TSTU* can be taken to be the domain of a model of *NFU*: the membership relation of the model can be defined for x and y in V_α as $x \in_{NFU} y \leftrightarrow x \in^M_0 j(y)$, where $\in^M_0$ is the membership relation of type 0 objects in type 1 objects in the model of *TSTU* extracted from M.

More briefly, a model of *NFU* is obtained if one starts with a nonstandard model M of a Zermelo-style set theory (Mac Lane set theory suffices) with an

[1] Forster in [2] calls this kind of map a "type-shifting automorphism", though he notes correctly that this is not really an automorphism unless we work with Hao Wang's theory *TNT* with all integer types.

automorphism j moving a (nonstandard) infinite ordinal α upward, takes V_α^M as the domain of one's model, then defines the membership $\in_{NFU}$ of the model thus: $x \in_{NFU} y \leftrightarrow (x \in^M j(y) \wedge j(y) \in^M V_{\alpha+1}^M)$. We presented essentially the same model in a more elaborate fashion in the previous paragraph to bring out the relationship to models of TTU with a type-incrementing isomorphism. The full strength of *ZFC* is not needed: the minimal strength needed is that of Mac Lane set theory. A proof that this is a model of *NFU* can be found in [3], p. 68; the construction, though implicit in Jensen's [13], was first given by Maurice Boffa in [2].

It is useful to note that, although it accords less well with our intuitions derived from type theory, there is an alternative formulation using a downward endomorphism of a model of a level V_α with nonstandard index: if j is such a downward endomorphism (i.e., if $j(\alpha) < \alpha$), we define $x \in_{NFU} y$ as $j(x) \in y \wedge y \in V_{j(\alpha)+1}$; an advantage of this formulation is that it can be used in a situation where we cannot refer to objects outside our nonstandard V_α.

NFU thus admits a natural interpretation in terms of the cumulative hierarchy of the usual set theory. It is interesting to observe that *NFU* supports the same interpretation of itself internally, via a natural internal representation of the cumulative hierarchy. Consider the set of isomorphism types of well-founded extensional relations with a "top" element (which can be thought of as "pictures" of the sets of a Zermelo-style set theory); one can prove in *NFU* that this structure exists as a set and has an external downard endomorphism (a type-raising operation which does not define a function), and admits an interpretation as a "model" of *NFU* along exactly the same lines outlined above (we put "model" in quotes because it is not a set model; its membership relation will be a proper class relation defined in terms of the type-raising external endomorphism). This representation of the cumulative hierarchy will be discussed in more detail later in the paper; also see [7] or our recent book [11].

2.1 Cantorian and strongly Cantorian objects and T operations

All of the extensions we will consider hinge on a pair of notions peculiar to set theories with stratified comprehension. A *Cantorian set* is a set A such that $|A| = |\mathcal{P}_1(A)|$. Note that such a set will satisfy the conclusion of the unstratified form of Cantor's theorem. A *strongly Cantorian set* is a set A such that the class map $(x \mapsto \{x\})\lceil A$, the restriction of the singleton operation to the set A, is a set. It should be clear that a strongly Cantorian set is Cantorian. A *(strongly) Cantorian cardinal* is the cardinal of a (strongly) Cantorian set. A *(strongly) Cantorian ordinal* is the order type of a (strongly) Cantorian well-ordering.

Familiar sets definable in both *NFU* and *ZFC* will be Cantorian. For example, it is straightforward to establish that the number systems $\mathcal{N}$ and $\mathcal{R}$ are Cantorian sets. The property which we really want nice sets to have is that of being strongly Cantorian. The reason for this is that we can subvert stratification restrictions on the formation of sets in the presence of strongly Cantorian sets. Let A be a strongly Cantorian set. Let K be the restriction of the singleton map to A and let K^{-1} be the inverse of this map. Any occurrence of a variable x restricted to a strongly Cantorian set can have its type freely raised or lowered by replacing references to x with references to the equivalent expressions $\bigcup K(x)$ and $x = K^{-1}(\{x\})$, in which x appears one type higher and one type lower (respectively), iterating this process as needed. Thus, stratification restrictions can be ignored for variables restricted to strongly Cantorian domains. Unfortunately, the only sets which can be proven to be strongly Cantorian in *NFU* are concrete finite sets. In a model of *NFU* + Infinity + Choice constructed by the method described above with $\alpha = \omega + n$ with n a nonstandard natural number, all strongly Cantorian sets are finite, and some nonstandard finite sets are not even Cantorian.

If $\kappa = |A|$, we define $\mathrm{T}(\kappa)$ as $|\mathcal{P}_1(A)|$. It is easy to establish that the definition of $\mathrm{T}(\kappa)$ does not depend on the choice of the set A. Similarly, if α is the order type of a well-ordering W, we define $\mathrm{T}(\alpha)$ as the order type of the well-ordering $\{(\{x\}, \{y\}) \mid x\, W\, y\}$. Each of these T operations must fail to be a function.

Each T operation will coincide with the restriction of j^{-1} to its domain in the models of *NFU* described above. To see this, consider the relation between a set A and the set $\mathcal{P}_1(A)$ of *NFU* as they are represented in one of our models. The singleton in the sense of *NFU* of an element x of A is the object $\{j^{-1}(x)\}$ in terms of the underlying nonstandard model of set theory. Thus, the cardinality of the set representing $\mathcal{P}_1(A)$ in the underlying model is the image under j^{-1} of the cardinality of the set representing A.

One can prove in *NFU* that the appropriate T operation commutes with all the standard operations on cardinals and ordinals; the T operations are proper class endomorphisms on their domains. One can prove in *NFU* that the order type of the natural well-ordering of the ordinals less than α is $\mathrm{T}^2(\alpha)$; the appearance of T corrects for the type differential between the ordinal and the order type of the associated segment in the natural order on the ordinals, and the Burali-Forti argument establishes that $\mathrm{T}^2(\Omega) < \Omega$, where Ω is the order type of the natural well-ordering on the ordinals. One sees, therefore, that there is a descending sequence $\mathrm{T}^i(\Omega)$ in the ordinals; but this sequence is not a set (this result is interesting because it establishes the existence of a countable proper class). The exponential operation $|B|^{|A|}$ on cardinals cannot be defined as $|B^A|$ if one wishes it to be a function; it is necessary to define it as $T^{-1}(|B^A|)$, which has the effect of making the exponential function partial (as one might expect in a system with a universal set). In general, T operations can be used to adjust (selected) definitions which work in Zermelo-style set theory so that they become stratified.

A set A is Cantorian exactly if $\mathrm{T}(|A|) = |A|$; similarly, Cantorian cardinals and ordinals are exactly the fixed points of the relevant T operators. An ordinal is strongly Cantorian iff all smaller ordinals are strongly Cantorian. This last result can be demonstrated by considering the function f which sends each ordinal $\mathrm{T}(\alpha)$ to the singleton set $\{\alpha\}$. An ordinal is Cantorian exactly if f sends it to its own singleton. All the ordinals smaller than a given ordinal are Cantorian iff f witnesses a bijection between the set of smaller ordinals and their singletons, making the segment and so the given ordinal itself strongly

Cantorian. In terms of our model construction for *NFU*, Cantorian cardinals and ordinals are exactly the fixed points of the automorphism j (because they are exactly the fixed points of T, which corresponds to j^{-1}).

More detailed proofs of theorems of *NFU* discussed in this section can be found in [11].

3 The Axiom of Counting

We now begin to introduce the extensions of *NFU* which are the object of our investigations. We use some nomenclature introduced by Solovay for specific extensions of *NFU*.

We are interested only in extensions of our base theory *NFU* (recall that *NFU* includes Infinity and Choice for us) which are natural in terms of notions proper to *NFU*; we are not interested in extensions which ape *ZFC*, such as *NFU* + "there is an inaccessible cardinal".

The first axiom we propose to extend *NFU* is the extremely natural axiom:

Axiom of Counting: All finite sets are strongly Cantorian.

which is due to Rosser in [17] (but not in this form: his form of the axiom is "$\{1 \ldots n\}$ has n elements"). Although we refer to the extension of *NFU* with Counting simply as *NFU* + Counting here, the name *NFUR* (in honor of Rosser) has been coined for it by Marcel Crabbé.

In spite of appearances, this is not an axiom about arithmetic. It must have some arithmetical consequences for meta-mathematical reasons of consistency strength, but its natural consequences are in set theory. *NFU* (with Infinity) proves the existence of $\beth_n$ for each concrete natural number n, but cannot prove the existence of $\beth_\omega$. *NFU* + Counting proves the existence of $\beth_{\texttt{init}(\beth_n)}$ for each concrete n; it proves that each $\beth_n$ is strongly Cantorian (assuming that all finite sets are strongly Cantorian implies that far larger sets are strongly Cantorian as well). The Axiom of Counting can hold in a model (as we will show below) in which $\beth_{\texttt{init}(\beth_\omega)}$ does not exist. Orey showed in [15] that the Axiom of Counting is independent of *NF* (if *NF* is consistent); he used metamathematical techniques to prove this. The results and techniques apply in *NFU*; *NFU* + Counting is strictly stronger than *NFU* (with Infinity). Jensen showed how to build ω-standard models of *NFU*, in [13]; an ω-standard model of *NFU* will certainly satisfy Counting (and will satisfy stronger assumptions as well, such as Mathematical Induction for unstratified sentences).

We outline proofs of these results. They represent the first indication that natural-seeming assertions that certain sets are strongly Cantorian may be stronger than one might naively expect.

Definition: $\exp(\kappa)$ is defined as 2^κ, for any cardinal κ; it is useful to recall from the definition of exponentiation of cardinals given earlier that $2^{|A|} = T^{-1}|\mathcal{P}(A)|$ rather than $|\mathcal{P}(A)|$.

Definition: Let B be the intersection of all sets A of cardinals such that $\aleph_0 \in A$, A is closed under the function exp[2], and A contains the supremum of each subset of A. This set is well-ordered by the natural order on cardinals; we define $\beth_\alpha$ as the cardinal κ (if any) in B with the property that the restriction of the natural order on cardinals to the elements of B less than κ has order type α (i.e., is an element of α). Note in particular that $\beth_0 = \aleph_0$ and $\beth_{n+1} = \exp(\beth_n)$ for each natural number n.

Theorem (*NFU* + Axiom of Counting): For each n, $\beth_n$ exists.

Proof: Certainly $\beth_0 = \aleph_0$ exists. Suppose that $\beth_n$ is defined. Let A be a set with $|A| = \beth_n$. We would like to assert that $|\mathcal{P}(A)| = \beth_{n+1}$, but the correct theorem of *NFU* is $|\mathcal{P}(A)| = \beth_{T(n)+1}$; the application of the T operation is needed to preserve relative type (the power set operation is type-raising). In mere *NFU* the proof would break down at this point, since it might be possible for $T(n) + 1 < n$ to hold; there are models of *NFU* in which the cardinality of the universe is a nonstandard $\beth_n$ (let $\alpha = \omega + n$ in our model construction above). But the Axiom of Counting is equivalent to the assertion $(\forall n \in \mathcal{N})(T(n) = n)$, which allows us to conclude that $|\mathcal{P}(A)| = \beth_{n+1}$ as we hoped originally. The proof is complete.

Corollary: $\beth_\omega$ exists.

Proof: Choose one set of each cardinality $\beth_n$ and take their union.

Theorem (*ZFC*): There is a model of *NFU* + Axiom of Counting + "$\beth_{\texttt{init}(\beth_\omega)}$ does not exist".

The model construction involves an unusual adaptation of the ultrapower construction. For anyone who feels that our account of this "ultrapower construction" needs to be supplemented, see the very precise description of a similar construction in Solovay's electronically available preprint [19]. We developed this construction for the purpose of constructing models of *NFU* in our Ph.D. thesis [9]; Solovay comments in [19] that it is similar to standard constructions used in the theory of large cardinals.

We define the "full language of A" as the first-order language in which there is a predicate symbol for every subset of A and an n-ary relation symbol for every subset of the Cartesian power A^n. The "full theory" of a set A is defined as the set of true sentences expressible in the full language of A with the range of all quantifiers taken to be the set A.

The usual ultrapower construction involves the use of a nonprincipal ultrafilter U on a set X; the ultrapower A^U, a nonstandard model of the full theory of A, is obtained by considering the equivalence classes of the set $[X \to A]$ of functions from X to A under the equivalence relation

[2]This means "if $a \in A$ *and* $\exp(a)$ *exists*, then $\exp(a) \in A$".

defined as follows: $f \sim_U g$ iff $\{x \in X \mid f(x) = g(x)\} \in U$. Any relation $R(a_1, \ldots, a_n)$ of the full language of A is interpreted in A^U by the relation $R^U([f_1], \ldots, [f_n])$, which holds for each sequence of $[f_i]$'s in A^U (represented as equivalence classes under $\sim_U$ of functions $f_i : X \to A$) iff $\{x \in X \mid R(f_1(x), \ldots, f_n(x))\} \in U$.

We think of the ultrafilter U as a complete description of a nonstandard object to be adjoined to X; U tells us which predicates of the full language of X are satisfied by this nonstandard object. The ultrapower A^U can be thought of as the collection of all images under standard functions in $[X \to A]$ of the nonstandard object; Łoś's theorem of model theory tells us that the ultrapower is an elementary superstructure of A for the full language of A, where the embedding of A into the ultrapower sends each element $a \in A$ to the equivalence class of the constant function with value a.

In the modified construction presented here, a sequence of ultrafilters U_i is used: each U_i is a nonprincipal ultrafilter on the set $[X]^i$ of i-element subsets of X, and the U_i's satisfy additional coherence conditions. It is assumed that X is a set of ordinals with no largest element; this means that there is a natural order on X, and, moreover, that we may choose the nonprincipal ultrafilter U_1 on $[X]^1$ to include all (images under the singleton set construction of) final segments of X. The coherence conditions on the U_i's are stated below.

The motivation of the construction which follows is that U_i will serve as a complete description of (the domain of) a sequence of i nonstandard elements of X (thus nonstandard ordinals). For each $j < i$, U_j will describe (the domain of) each consecutive subsequence of length j of the objects described by U_i, so they will have limited homogeneity properties (nonconsecutive subsequences may not have the same description). In the nonstandard model of set theory which serves as precursor to the model of *NFU* to be constructed, the ultrafilter U_i will "describe" each finite set $\{j^n(\alpha), j^{n+1}(\alpha), \ldots, j^{n+i-1}(\alpha)\}$ of i consecutive iterated images of the ordinal α used in the model construction under the automorphism j.

The elements of the "ultrapower" we construct will be equivalence classes of functions, but not of functions with domain X. The set of functions we will partition is the set of functions f with domain $[\mathcal{Z} \to X]$ and range A with the additional feature that there are $m < n \in \mathcal{Z}$ such that the values of f at an element F of $[\mathcal{Z} \to X]$ depend only on the restriction of F to $[m, n]$. $[m, n]$ is said to be a *support* of f, and such functions f are said to have *finite support* in $\mathcal{Z}$.

Functions f and g with finite support will belong to the same model element iff they satisfy the equivalence relation $f \sim_U g$ defined as follows: $f \sim_U g$ holds iff whenever $[m, n] \subseteq \mathcal{Z}$ is a common support for f and g, the set $\{\text{rng}(h) \mid h : [m, n] \to X$ is strictly increasing and $f(h') = g(h')$ for all $h' : \mathcal{Z} \to X$ such that $h' \supset h\}$ belongs to U_{n-m+1}.

The coherence conditions on the sequence U of ultrafilters are precisely what is needed to make the construction work: for any element A of U_{n+1}, the collection of elements of $[X]^n$ obtained by dropping the largest element of each element of A belongs to U_n, and likewise the collection of elements of $[X]^n$ obtained by dropping the smallest element of each element of A belongs to U_n. This has the effect of ensuring that the requirements for different common supports of f and g in the definition of $f \sim_U g$ are consistent.

The motivation here is that each element of the "ultrapower" will be the image under a standard map from some $[X]^i$ to A of a finite set of objects taken from a $\mathcal{Z}$-sequence of nonstandard objects any consecutive finite subsequence of which is indiscernible from any of its translates with respect to standard properties and relations on X. A translation of these "indiscernibles" will induce the automorphism of the "ultrapower" we are constructing.

Relations of the full language of A inherited by the "ultrapower" obtained in this way can be defined in essentially the same way that relations of the full language of A are defined in the usual kind of ultrapower, with reference to a common support $[m, n]$ of the functions involved and the appropriate U_{n-m+1}: $R^U([f_1], \ldots, [f_k])$ holds iff, for some common support $[m, n]$ of the f_i's (and so for any common support of the f_i's) we have $\{x \in [X]^{n-m+1} \mid R(f'_1(x), \ldots, f'_k(x))\} \in U^{n-m+1}$, where $f'_i(x)$, for any $x \in [X]^{n-m+1}$, is defined as the result of applying f_i to any function $\mathcal{Z} \to X$ which extends the increasing function from $[m, n]$ onto x (this result is well-defined because $[m, n]$ is a support of f_i).

Łoś's theorem for conventional ultrapowers holds for "ultrapowers" constructed in this way as well: the easiest way to see this is to observe that the restriction of this "ultrapower" to functions with support $[-n, n]$ is isomorphic to an ultrapower of the usual sort, and this sequence of restricted ultrapowers has the full "ultrapower" as a direct limit (there is a natural elementary embedding of each of these restricted ultrapowers in the next).

The distinguishing feature of this kind of "ultrapower" is that there is an external automorphism j of the ultrapower, considered as a nonstandard model of the full theory of A. The automorphism sends each equivalence class $[f]$ to the equivalence class $[f^+]$, where $f^+(F) = f(F \circ \sigma)$, where σ is the successor map on $\mathcal{Z}$, for each $F \in [\mathcal{Z} \to X]$. The fact that this is an automorphism is ensured by the coherence conditions on the U_i's. The existence of the automorphism makes these models appropriate for use in constructing models of NFU.

Lemma: Let X be an infinite set of ordinals with no largest element. Then there are non-principal ultrafilters U_i satisfying the coherence conditions given above.

Proof of Lemma: Clearly there is a non-principal ultrafilter U_1 on X which contains all final segments of X. For each set A in U_1, define A^+ as the collection of all two element subsets of X such that the larger element belongs to A and A^- as the collection of all two-element sets such that the smaller element belongs to A. We claim that the set of all $A^\pm$'s with $A \in U_1$ is a filter, which can be extended to an (obviously non-principal) ultrafilter U_2.

To verify that we have a filter, it is sufficient to verify that A_1^+ meets A_2^- for $A_1, A_2 \in U_1$. We can see further that it is sufficient to verify that A^+ meets A^- for each $A \in U$ (choose A to be a subset of both A_1 and A_2). To see this it is sufficient to observe that any A must have at least two members, since U_1 is nonprincipal.

Similarly, if U_n has been defined (and $n > 1$), for each set A in U_n, define A^+ as the collection of all $(n+1)$-element subsets B of X such that the n-element set obtained by dropping the smallest element of B belongs to A and A^- as the collection of all $(n+1)$-element sets B such that the n-element set obtained by dropping the largest element of B belongs to A. We claim that the set of all $A^\pm$'s with $A \in U_n$ is a filter, which can be extended to an (obviously non-principal) ultrafilter U_{n+1}.

Just as in the basis case, and for the same reason, it is sufficient to show for any $A \in U_n$ that A^+ and A^- must intersect. We consider the set A_+ of all $n-1$ element sets which can be extended to an element of A by adding a new maximum element; this set must belong to U_{n-1}. We consider the set A_- of all $n-1$ element sets which can be extended to an element of A by adding a new minimum element; this must be an element of U_{n-1} as well. Thus the intersection of A_+ and A_- must be nonempty (since U_{n-1} is an ultrafilter). Any element of this intersection can be extended by adding a new maximum and a new minimum to an element of the intersection of A^+ and A^-, completing the proof of the claim.

The proof of the Lemma is complete.

The Lemma and the preceding discussion show us that we can construct a model of *NFU* from any stage V_λ of the cumulative hierarchy, with λ a limit ordinal. One can use λ itself as the set X and V_λ as A. The ultrafilters U_i give complete descriptions in the full language of V_λ (in which all sets and relations on V_λ can be used as predicates) of indiscernible nonstandard ordinals just below λ; the nonstandard element corresponding to the function taking each F in $[\mathcal{Z} \to X]$ to $F(0)$ can be taken as the nonstandard ordinal α in our standard construction of models of *NFU*; it is moved by the automorphism of the ultrapower to the larger nonstandard element corresponding to the function taking each F in $[\mathcal{Z} \to X]$ to $F(1)$. The "V_α" coded in the ultrapower model will be the domain of our model.

By constructing the nonprincipal ultrafilters U_i with more care, we proceed to prove the main theorem. This construction will use the Erdös-Rado partition theorem (See [12], p. 323).

It is an immediate consequence of the Erdös-Rado theorem that any map from $[\beth_\omega]^n$ to ω is constant on $[Y]^n$ for some infinite subset Y of $\beth_\omega$.

The construction proceeds as follows. Construct an ultrapower model (of the standard variety) of $V_{\omega+\omega+\omega}$ (the subscript here is chosen large enough to ensure that all the objects we need are in this level; it is not chosen with maximum economy!) in which there is a nonstandard "finite set" F of maps from domains $[\beth_\omega]^n$ for varying values of n to ω which includes all standard maps from $[\beth_\omega]^n$ to ω for each standard n.

This ultrapower will be $V^U_{\omega+\omega+\omega}$, where U is chosen as follows: let Maps be the set $\bigcup_{n\in\omega}[[\beth_\omega]^n \to \omega]$ of all maps from $[\beth_\omega]^n$ to ω for whatever n; then U will be a nonprincipal ultrafilter on the set $[\text{Maps}]^{<\omega}$ of finite subsets of Maps with the property that the set of supersets $\{y \in [\text{Maps}]^{<\omega} \mid x \subseteq y\} \in U$ for each finite set $x \in [\text{Maps}]^{<\omega}$. The ultrafilter U serves as the "description" of the nonstandard finite set F containing all standard elements of Maps. We give this model the short name M.

For any natural number n and map f from $[\beth_\omega]^m$ to ω, with $m < n$, the map $G_n(f)$ from $[\beth_\omega]^n$ to ω is defined thus: $G_n(f)(A)$ is the result of applying f to the set consisting of the m smallest elements of A.

Let N be the maximum "natural number" in the sense of the model M such that the nonstandard finite set F seen by M contains a map with domain $[\beth_\omega]^N$. Now consideration of the values of all the maps in $G_N[F]$ induces a nonstandard partition of $[\beth_\omega]^N$ into ω pieces, which must itself have a nonstandard homogeneous set H by the Erdös-Rado theorem. Observe that H is in fact homogeneous (according to the model M) for every standard partition of a $[\beth_\omega]^n$ (the domain of a standard partition will of course have standard index n) into countably many parts.

Use i-tuples of elements of H as an oracle for the construction of U_i's with $X = \beth_\omega$: that is, let a set $B \subseteq [\beth_\omega]^i$ in the real world belong to U_i iff $M \models$ "B meets (and so contains) $[H]^i$". It is possible to build "ultrapower models" with automorphism as above using these U_i's; it is important to note that all elements of H must be seen in M to lie between the same nonstandard $\beth_n$ and $\beth_{n+1}$, which implies that all final segments of $\beth_\omega$ will actually belong to U_1 as required by the construction. It should be clear that any "ultrapower model" $\mathcal{M}$ with automorphism j built as above using this sequence of ultrafilters will have $j(x) = x$ for each element x such that $\mathcal{M} \models$ "x is a natural number".

The model of *NFU* obtained from the "ultrapower" built from $V_{\beth_\omega}$ using this sequence of U_i's will satisfy the Axiom of Counting, because all natural numbers of the "ultrapower" are fixed by the automorphism and so their analogues in the model of *NFU* are both Cantorian (because fixed) and strongly Cantorian (because all smaller natural numbers are fixed). The model fails to contain the ordinal $\beth_{\mathtt{init}(\beth_\omega)}$, because the model contains a largest $\beth_{\mathtt{init}(\beth_n)}$ (indexed by the largest nonstandard $\beth_n$ which lies below all elements of H in the model M of the construction above).

The proof of the theorem is complete.

In this model $\beth_{\mathtt{init}(\beth_n)}$ will fail to exist for some nonstandard n; this is the best possible result, for each standard $\beth_{\mathtt{init}(\beth_n)}$ can be shown to exist in *NFU* + Axiom of Counting. The reason for this (in outline) is that $\beth_\alpha$ can be shown to exist for each strongly Cantorian α (in basically the same way that we showed $\beth_\omega$ to exist given Counting). $\beth_\alpha$ can be shown to be Cantorian, but not to be strongly Cantorian. $\beth_0$ and each concrete $\beth_n$ is strongly Cantorian given Counting (because the power set of a strongly Cantorian set is strongly Cantorian), so each concrete $\beth_{\mathtt{init}(\beth_n)}$ exists, but this cannot be proved for general, possibly nonstandard n (the model constructed above shows such a proof to be impossible; an attempt to prove that each $\beth_n$ is strongly Cantorian, and so that each $\beth_{\mathtt{init}(\beth_n)}$ exists, would require mathematical induction on an unstratified condition, which is thus seen not to be supported by *NFU* with the Axiom of Counting).

It is worth observing in conclusion that the same technique used here to construct a model in which ω is strongly Cantorian but $\beth_{\mathtt{init}(\beth_\omega)}$ does not exist can be used to construct, for any given infinite ordinal β of *ZFC*, a model of *NFU* in which β has a strongly Cantorian analogue but $\beth_{\mathtt{init}(\beth_\beta)}$ does not; there is no special property of ω involved in the model construction above.

We think that part of the interest of this section lies in the fact that it is possible to construct models of *NFU* using ultrafilters. It should be noted that these constructions are not as "sharp" as Jensen's constructions using term models in [13]: Jensen showed how to build a model of *NFU* in which the order type of the analogue of an infinite ordinal β of *ZFC* is actually β (a β-standard model); the ultrapower models constructed here will include nonstandard elements below β (in fact nonstandard natural numbers) unless the ultrafilters U_i are countably complete, which requires a measurable cardinal!

4 The Axiom of Cantorian Sets; NFUA; Getting n-Mahlo Cardinals

The second axiom we propose to extend *NFU* is again a natural simplifying assumption, originally suggested by C. Ward Henson in [6] in the context of *NF* and with an additional condition which would be superfluous here. Following Solovay, we define the system *NFUA* as *NFU* + Infinity + Choice plus the additional axiom

Axiom of Cantorian Sets: All Cantorian sets (equiv. cardinals, ordinals) are strongly Cantorian.

Since $\mathcal{N}$ is provably Cantorian in *NFU*, the Axiom of Counting is provable in this system.

Robert Solovay has shown the following:

Theorem (Solovay): The consistency strength of *NFUA* is exactly that of *ZFC* + the scheme consisting of the sentences "there is an n-Mahlo cardinal" for each concrete natural number n.

We admit having been amazed by this result, which concerns a system we were aware of but did not even consider in our attempt to find an extension of *NFU* with a "nice" representation of *ZFC*. A complete proof of this result is not given here; it is pending from Solovay himself.

We present a proof that *NFUA* implies the existence of n-Mahlo cardinals. This is a refinement of a proof of Solovay's in two senses: Solovay's original results were for L, the constructible universe, as coded in the interpretation of Zermelo-style set theory in the isomorphism classes of well-founded extensional relations. We showed that it is not necessary to work in L (but Solovay supplied a further refinement of our technique.) Further, Solovay had separate proofs for the existence of an inaccessible and for the existence of n-Mahlo cardinals; we discovered how to refine Solovay's proof of inaccessibles to give n-Mahlo cardinals as well, which is a considerable improvement on the earlier situation, because Solovay's proof of the existence of n-Mahlo cardinals was quite complex.

The proof which follows is similar to the proof for the case of inaccessibles given in our elementary text [11], which supplies information about the definition of "routine" set theoretical concepts (e.g., cofinality, ordinal indexing) in *NFU*.

Definition: A 0-Mahlo cardinal is an inaccessible. An $(n+1)$-Mahlo cardinal is a cardinal κ with the property that any closed unbounded set in the natural order on seg(init(κ)) (the set of ordinals less than init(κ)) contains the initial ordinal of an n-Mahlo cardinal.

Theorem (Solovay, refined by Holmes): *NFUA* proves the existence of n-Mahlo cardinals for each concrete natural number n (This is not the same as the assertion that it proves $(\forall n \in \mathcal{N})(\exists \kappa)(\kappa$ is n-Mahlo), which it in fact does not).

Proof: We first prove that non-Cantorian inaccessible cardinals exist; we then adapt the proof to the stronger case of n-Mahlos.

We fix a well-ordering $\leq$ on sets of cardinal numbers.

We define an operation F on pairs of cardinals. This operation is defined in the following cases; in all other cases it is undefined.

1. Suppose $\alpha \neq \beta$ are both not strong limit cardinals. We define $F(\alpha, \beta)$ as $(\alpha-, \beta-)$, where we define $\kappa-$ for any cardinal κ as the smallest cardinal λ such that $2^\lambda \geq \kappa$.
2. Suppose that $\alpha \neq \beta$ are singular strong limit cardinals with distinct cofinalities. Then $F(\alpha, \beta)$ is defined as (cf(α),cf(β)).
3. Suppose that $\alpha \neq \beta$ are singular strong limit cardinals with the same cofinality. Let A be the first set (in the sense of our well-ordering $\leq$ of sets of cardinals) of cardinals of order type cf(α) = cf(β) cofinal in the

natural order on cardinals less than α; let B be chosen in the same way from the cardinals less than β, but using the order $T(\leq)$ (that is, the order $\{(\mathrm{T}[A],\mathrm{T}[B]) \mid A \leq B\}$ where $\mathrm{T}[A]$ is the elementwise image of a set of cardinals A under the T operation on cardinals) instead of the order $\leq$ (so the definition will fail if β is not an image under T). Let these sets be indexed by ordinals in increasing order: $F(\alpha, \beta)$ is defined as (A_δ, B_δ), where δ is the smallest ordinal such that $A_\delta \neq B_\delta$.

Observe that projections of $F(\alpha, \beta)$ will always be strictly less than the corresponding projections of (α, β), when defined.

We now define a sequence (α_i, β_i) of pairs of cardinals as follows: α_0 is an arbitrary non-Cantorian cardinal, and $\beta_0 = \mathrm{T}(\alpha_0) \neq \alpha_0$. We define $(\alpha_{i+1}, \beta_{i+1})$ as $F(\alpha_i, \beta_i)$, if this is defined, and stipulate that $(\alpha_{i+1}, \beta_{i+1})$ is undefined otherwise. Clearly this sequence must be finite.

If α is a sequence of cardinals, we adopt the notation $\mathrm{T}(\alpha)$ for the sequence $\{(\mathrm{T}(n),\mathrm{T}(\kappa)) \mid (n, \kappa) \in \alpha\}$. The definition of this sequence is stratified, so it exists as a set. This is the natural T operation on sequences of cardinals, and would correspond (mod differences in the representations of cardinals as sets) to j^{-1} on sequences of cardinals in the nonstandard models of set theory with automorphism j underlying the models of *NFU* we have discussed.

We would like to show by induction that $\beta_i = \mathrm{T}(\alpha_i)$ and $\beta_i \neq \alpha_i$ whenever it is defined. This is a little suspect due to the involvement of the type-raising proper class map T, but it is a consequence of the Axiom of Cantorian Sets that $i = \mathrm{T}(i)$ for each natural number i (this is equivalent to the assertion that the Axiom of Cantorian Sets implies Counting in *NFU*): the condition is equivalent to the condition $\beta_i = (\mathrm{T}(\alpha))_{T(i)} = (\mathrm{T}(\alpha))_i$, in which reference to a T operation is confined to a constant; the condition does define a set of natural numbers, so induction will work!

1. Suppose $\alpha_i \neq \beta_i = \mathrm{T}(\alpha_i)$ are both not strong limit cardinals. We defined $(\alpha_{i+1}, \beta_{i+1}) = F(\alpha_i, \beta_i)$ as (α_i-, β_i-), where we define $\kappa-$ for any cardinal κ as the smallest cardinal λ such that $2^\lambda \geq \kappa$. Clearly $\beta_{i+1} = \mathrm{T}(\alpha_{i+1})$, since T is an endomorphism; we see further that $\beta_{i+1} \neq \alpha_{i+1}$, because if they were equal $\alpha_i \neq \beta_i = \mathrm{T}(\alpha_i)$ would both be dominated by $2^{\alpha_i-} = 2^{\beta_i-} = \mathrm{T}(2^{\alpha_i-})$, which would be Cantorian, so (by the Axiom of Cantorian Sets) α_i would have to be Cantorian and so equal to β_i.
2. Suppose that $\alpha_i \neq \beta_i = \mathrm{T}(\alpha_i)$ are singular strong limit cardinals with distinct cofinalities. Then $(\alpha_{i+1}, \beta_{i+1}) = F(\alpha_i, \beta_i)$ was defined as $(\mathrm{cf}(\alpha_i),\mathrm{cf}(\beta_i))$. It is sufficient here to observe that T commutes with cf.
3. Suppose that $\alpha_i \neq \beta_i = \mathrm{T}(\alpha_i)$ are singular strong limit cardinals with the same cofinality. Let A be the first set (in the sense of $\leq$) of

cardinals of the (Cantorian) order type $\mathrm{cf}(\alpha_i) = \mathrm{cf}(\beta_i) = \mathrm{T}(\mathrm{cf}(\alpha_i))$. cofinal in the natural order on cardinals less than α; let B be chosen in the same way from the cardinals less than β, but using the order $\mathrm{T}(\leq)$ instead of the order $\leq$ (note that β_i is by hypothesis an image under T). Let these sets be indexed by ordinals in increasing order: $(\alpha_{i+1}, \beta_{i+1}) = F(\alpha_i, \beta_i)$ is defined as (A_δ, B_δ), where δ is the smallest ordinal such that $A_\delta \neq B_\delta$. $\delta < \mathrm{cf}(\alpha_i)$, which is Cantorian, so δ is Cantorian by the Axiom of Cantorian Sets. It is clear that $\mathrm{T}[A] = B$ (it is important here that $\mathrm{T}(\leq)$ is used in place of $\leq$ in the definition of B). $\mathrm{T}(A_\delta) = (\mathrm{T}[A])_{T(\delta)} = B_\delta$ and the assumed distinctness of A_δ and B_δ complete this case.

Since T is an endomorphism, each pair (α_i, β_i) will be of the same "kind" (successor, singular limit or inaccessible); the only way that the sequence can terminate is with a pair of distinct (and so non-Cantorian) inaccessible cardinals.

The proof that there are inaccessibles will now be upgraded to prove that there are non-Cantorian inaccessibles in certain clubs:

We describe a set A as *semi-natural* iff for each x which is less than or equal to an element of A, $x \in A$ iff the minimum of $\mathrm{T}(x)$ and $\mathrm{T}^{-1}(x)$ is also in A (if $\mathrm{T}^{-1}(x)$ is undefined, $\mathrm{T}(x)$ is taken as the minimum). Equivalently, if x and $T(x)$ are both bounded by elements of A, $x \in A \leftrightarrow T(x) \in A$.

If we fix a semi-natural set $\mathcal{A}$ of cardinals which has non-Cantorian elements and is closed except possibly at its upper limit, we can adapt the proof above to show that $\mathcal{A}$ contains non-Cantorian inaccessibles. Define $\kappa_{\mathcal{A}}$ as the largest element of $\mathcal{A}$ less than or equal to κ, for each cardinal κ; the fact that $\mathcal{A}$ is closed except possibly at its upper limit ensures that $\kappa_{\mathcal{A}}$ exists as long as κ is dominated by some element of $\mathcal{A}$. Define $G(\alpha, \beta)$ as $(\pi_1(F(\alpha, \beta))_{\mathcal{A}}, \pi_2(F(\alpha, \beta))_{\mathcal{A}})$; in each clause of the definition of F, we apply the additional operation $\kappa \mapsto \kappa_{\mathcal{A}}$ to each projection of the result. The fact that $\mathcal{A}$ is semi-natural ensures that $\kappa \mapsto \kappa_{\mathcal{A}}$ commutes with T as long as its argument is dominated by some element of $\mathcal{A}$; this is ensured in the cases that interest us by choosing the initial pair of cardinals (α_0, β_0) from $\mathcal{A}$.

It is important for the adaptation of the induction argument to observe that if $\kappa \neq \mathrm{T}(\kappa)$ and both cardinals are dominated by some element of $\mathcal{A}$, it will be the case that $\kappa_{\mathcal{A}} \neq \mathrm{T}(\kappa_{\mathcal{A}}) = (\mathrm{T}(\kappa))_{\mathcal{A}}$. This is needed to ensure distinctness of the projections of values of G in the cases that interest us. To see that this is true, it is sufficient to observe that the least element γ of a semi-natural set $\mathcal{A}$ greater than a given Cantorian element δ of $\mathcal{A}$ must itself be Cantorian: otherwise $\min(\mathrm{T}^{-1}(\gamma), \mathrm{T}(\gamma))$ would be a smaller element of $\mathcal{A}$ greater than δ. Thus if $\kappa_{\mathcal{A}} = \mathrm{T}(\kappa_{\mathcal{A}}) = (\mathrm{T}(\kappa))_{\mathcal{A}}$, we would have to conclude that κ was dominated by a Cantorian element of $\mathcal{A}$ and so was itself Cantorian, contrary to assumption.

The argument for inaccessibles in $\mathcal{A}$ then proceeds in exactly the same way as the argument for inaccessibles in general, using G instead of F.

To show the existence of Mahlo cardinals, we add the following new case to the definition of F (and a corresponding case to the induction proof):

4. Suppose that α and β are distinct inaccessible cardinals which are not Mahlo. Select the $\leq$-first closed unbounded set A of cardinals containing no inaccessibles in the cardinals below α_i, and the T($\leq$)-first closed set B of cardinals with no inaccessibles in the cardinals below β_i. Index these two sets with ordinals in increasing order. There must be a point at which the two sets differ (otherwise the limit of the smaller set, an inaccessible, would belong to the other set, because the sets are closed). Let δ be the first index at which the corresponding elements of the two sets differ: let (A_δ, B_δ) be taken as $F(\alpha, \beta)$.

The corresponding case of the induction proof is tricky in that the only way we can ensure that $\mathrm{T}(A_\delta) = B_\delta$ is to prove that the index δ must be fixed under T (i.e., Cantorian), which is not obvious.

If δ is non-Cantorian, then there are non-Cantorian A_α (equivalently B_α) with the property that $A_\beta = B_\beta$ for all $\beta \leq \alpha$; a suitable α would be the minimum of $\mathrm{T}(\delta)$ and $\mathrm{T}^{-1}(\delta)$.

We claim that the set of ordinals A_α such that $A_\beta = B_\beta$ for all $\beta \leq \alpha$ is semi-natural. Obviously $B = \mathrm{T}[A]$. Let $\beta = \min(\mathrm{T}(\alpha),\mathrm{T}^{-1}(\alpha))$. Then $A_\beta = B_\beta$ by definition of the indicated set. Now observe that if $\beta = \mathrm{T}(\alpha)$ we can conclude that $\mathrm{T}(A_\alpha) = \mathrm{T}[A]_{T(\alpha)} = B_\beta = A_\beta$, establishing that $\mathrm{T}(A_\alpha)$ is in the set; if $\beta = \mathrm{T}^{-1}(\alpha)$, we argue that $\mathrm{T}^{-1}(A_\alpha) = (\mathrm{T}^{-1}[A])_{T^{-1}(\alpha)} = B_\beta = A_\beta$. In either case, we see that the minimum of $\mathrm{T}(A_\alpha)$ and $\mathrm{T}^{-1}(A_\alpha)$ is in the set, supporting the claim that the set is semi-natural. There is a little more to show. It is necessary to show that if λ is a cardinal bounded by an A_α in the indicated set which is not itself a member of the indicated set (and so is not an A_γ at all), that the minimum of $T(\lambda)$ and $T^{-1}(\lambda)$ is not in the indicated set. It is sufficient to observe that if $T(\lambda)$ or $T^{-1}(\lambda)$ were equal to some A_β in the indicated set, the same calculations given above for $T(A_\beta)$ and $T^{-1}(A_\beta)$ would work to show that λ was in fact an A_γ, which would contradict the choice of λ.

Since the indicated set is semi-natural and closed except possibly at its upper limit, it must have inaccessible elements if it has any non-Cantorian elements. By definition of A and B, it can have no inaccessible elements, so it must have no non-Cantorian elements. We have seen above that if δ is non-Cantorian, the indicated set must have non-Cantorian elements. From this it follows that δ is Cantorian, from which follows $\mathrm{T}(A_\delta) = B_\delta$, which completes the proof.

Exactly the same argument used to get 1-Mahlos from inaccessibles can then be used to get 2-Mahlos from 1-Mahlos: extend the definition of G and use it to prove that each semi-natural set which has non-Cantorian elements and is closed except possibly at its upper limit contains 1-Mahlos, then argue just as above that there are 2-Mahlos. The same argument gets $(n+1)$-Mahlos from n-Mahlos for each concrete n. Mathematical induction on n will not suffice to show that there are n-Mahlos for each n (fortunately, since this is not a theorem!), because of the role of the unstratified condition "there are n-Mahlos in each semi-natural set which has non-Cantorian elements and is closed except possibly at its upper limit" in the step of the argument which proves the existence of $(n+1)$-Mahlos.

4.1 The Axiom of Large Ordinals; T-sequences

The models of *NFUA* constructed by Solovay on minimal consistency strength hypotheses also satisfy the following additional axiom:

Axiom of Large Ordinals: For each non-Cantorian ordinal α, there is a natural number n such that $\alpha > \mathrm{T}^n(\Omega)$.

Although this axiom strengthens *NFUB* (the theory introduced in the next section) enormously, it adds no strength at all to *NFUA* (natural models of *NFUA* constructed with minimal consistency strength hypotheses have the concrete ordinals $\mathrm{T}^n(\Omega)$ coinitial (downward cofinal) in the non-Cantorian ordinals; such constructions will be published by Solovay). Some work is required, though, to see that this axiom can even be expressed in the language of *NFU*! The difficulty is with the appearance of T with a variable exponent.

Note that the Axiom of Large Ordinals is another natural simplifying assumption: it asserts that the endomorphism of the ordinals has the simplest possible structure.

We exhibit the formal underpinnings which make it possible to express the axiom:

Definition: A *T-sequence* is a (set) sequence s of ordinals, of finite or infinite length, indexed by natural numbers, such that $s_{i+1} = \mathrm{T}(s_i)$ whenever $i+1 \in \mathrm{dom}(s)$.

Note that the definition of "T-sequence" is unstratified. It is very important that a T-sequence is a *set*; the sethood of a T-sequence makes it possible to carry out induction on otherwise unstratified conditions as far as the T-sequence goes! It is easy to prove in *NFU* that all T-sequences are either monotone increasing, monotone decreasing, or constant.

Definition: $\mathrm{T}^n(\alpha)$, for n a natural number and α an ordinal, is the unique ordinal which appears as the nth term s_n of all T-sequences s of sufficient length whose 0th term s_0 is α.

It is a theorem of *NFUB* (the theory introduced in the next section) that $\mathrm{T}^n(\alpha)$ is well-defined for each natural number n and ordinal α (this is an easy application of unrestricted mathematical induction (for all conditions, stratified or not), which holds in *NFUB*); it is a meta-theorem of weaker systems that it is well-defined for each concrete natural number n. It is, of course, harder to reason about this concept in the weaker systems.

A nice theorem which is only significant for our weakest system (without Counting) is that if $\mathrm{T}^n(\alpha)$ is defined, then n must be strongly Cantorian; this follows from the easy observation that a finite T-sequence is itself a strongly Cantorian set: its "image" under the T operation is the result of dropping one term at the beginning and adding one term at the end, so must be of the same length.

An open question remains about T-sequences in weak theories. A constant T-sequence can of course be infinite in length, and a strictly decreasing T-sequence must be finite since it is a strictly decreasing sequence of ordinals, but there remains the following

Question: Is there a model of *NFU* in which there is a strictly increasing T-sequence of infinite length?

Such a model must satisfy the Axiom of Counting; we conjecture that some stronger hypotheses than this may be needed.

The following theorem was surprising to us; we didn't originally think that *NFU* gave enough expressive power.

Theorem (*NFU*): The Axiom of Large Ordinals implies the Axiom of Cantorian Sets.

Proof: Suppose the Axiom of Cantorian Sets to be false. Then there is a Cantorian ordinal α which dominates a non-Cantorian ordinal β (because any Cantorian ordinal which dominates only Cantorian ordinals is strongly Cantorian). Let s be a T-sequence with $s(0) = \Omega$. It is easy to prove by mathematical induction that $s(n) > \alpha$ for every n, from which it follows that $\mathrm{T}^n(\Omega) > \alpha > \beta$ for all n, which contradicts the Axiom of Large Ordinals. The proof is complete.

It turns out that using the Axiom of Large Ordinals instead of the Axiom of Cantorian Sets in conjunction with the strong Axiom of Small Ordinals introduced in the next section causes a considerable boost in consistency strength.

5 The Axiom of Small Ordinals; *NFUB* and *NFUB−*; Interpreting *KM* with a Weakly Compact Cardinal

We proposed the following extension of *NFU* ourselves, which Solovay calls *NFUB*. *NFUB* consists of *NFUA* plus the axiom scheme

Axiom of Small Ordinals: For each formula ϕ (stratified or unstratified), there is a set A such that the elements of A which are strongly Cantorian ordinals are precisely the strongly Cantorian ordinals x such that ϕ.

This axiom scheme is given in a weaker form than in [10] or [11]; the form given there includes Cantorian Sets as a consequence, and is equivalent to what is given here in the presence of the Axiom of Cantorian Sets.

We now demonstrate that the system *NFUB*− = *NFU* + Counting + Axiom of Small Ordinals (note the omission of the full Axiom of Cantorian Sets) interprets *ZFC*.

The natural way to interpret Zermelo-style set theory in *NFU*, to which we have already alluded above, is to consider the isomorphism classes of well-founded extensional relations "with top" (a notion which we will define shortly). The idea is to consider relations which can serve as "pictures" of the membership relation restricted to the transitive closure of a set in a Zermelo-style theory.

Definition: We define the *full domain* of a relation R as the union of the domain of R and the range of R.

Definition: A relation R is *well-founded* if for every subset S of the full domain of R there is an element s of S such that there is no $t \in S$ such that $t\,R\,s$; such an s is called an "R-minimal" element of S. A relation R is *extensional* iff distinct elements of the range of R have distinct preimages under R. A well-founded extensional relation R is said to have t as its "top" if for every x in the full domain of R there is a finite sequence s with its first element $s_0 = x$, $s_i\,R\,s_{i+1}$ for each appropriate index i, and its last element $s_n = t$. The empty relation has anything at all as its top; a nonempty well-founded extensional relation has a unique top by well-foundedness (if it had more than one top there would be a nontrivial cycle in the relation, whose domain would have no minimal element).

A well-founded extensional relation with top is suited to be a picture of the transitive closure of a set in a set theory with Extensionality and Foundation. The top stands in for the set actually being represented. With each element x of the full domain of a well-founded extensional relation with top is associated a maximal subrelation of which x is the top; this is called the *component* associated with x. (Notice that this works correctly for empty relations; every nonempty well-founded extensional relation with top has a unique element of its domain with empty preimage by well-foundedness, and the component associated with this unique element is the empty relation, of which the unique element (and everything else) is a top). The components associated with the preimages of the top of a well-founded extensional relation R are called *immediate* components and are pictures of the elements of the set pictured by the relation R.

In *NFU*, the isomorphism classes of well-founded extensional relations with top make up a set Z. There is a natural "membership" relation (also a set) on

Z, which we call E: for $x, y \in Z$, $x\,E\,y$ iff some (and thus any) element of y has an immediate component which belongs to x. Just as the natural order on the ordinals is a well-ordering, so the "membership" relation E turns out to be a well-founded extensional relation (it does not have a top). Each element x of Z has an associated component of E, a well-founded extensional relation with x as top. One might suppose that this relation would itself have isomorphism type x, but it actually has isomorphism type $\mathrm{T}^2(x)$: the T operation on elements of Z takes the isomorphism type of a relation R to the isomorphism type of the relation $\{(\{x\}, \{y\}) \mid x\,R\,y\}$, which is easily seen to be a well-founded extensional relation with top if R is, but which is not necessarily isomorphic to R. It is straightforward to show that the "membership" relation E commutes with this T operation.

The comprehension principle satisfied by the "membership" relation E is not restricted by stratification (E, after all, is a set, so "unstratified" formulas in E define subsets of Z as good as those defined by "stratified" formulas in E). The restriction, as in Zermelo-style set theory, is one of "limitation of size". The question for each set $S \subseteq Z$ is whether there is an element $s \in Z$ such that the preimage of s under E is precisely S. Clearly this cannot be true for Z itself (by well-foundedness of E). But for any set $S \subseteq Z$, there is an element s which has as its preimages under E exactly the $\mathrm{T}(s)$'s for $s \in S$. The reason for this is as follows: the obstruction to finding an element s for a set S which is "too large" is (roughly speaking) that one cannot form disjoint representatives of the isomorphism classes belonging to S if S is "too large". For any element R of a class $\mathrm{T}(s)$, one can choose an element R' of the class s (note that the type of R' is one less than the type of R, and the same as the types of the elements of the domain and range of R) and replace each element a of the domain and range of R with the pair (a, R'); this (well-typed) process, carried out on each $\mathrm{T}(s)$ with $s \in S$, gives disjoint representatives of each of the classes $\mathrm{T}(s)$; one can take their union, add a new element as top, and perform an extensional collapse to construct a uniquely determined element of Z with exactly the $\mathrm{T}(s)$'s for $s \in S$ as its "elements" in the sense of E.

The set Z with the "membership" relation E is a model of *ZFC* − Power Set + "there is a largest cardinal"; it can be thought of as a model of the sets hereditarily of the cardinality of the universe (of the model of *NFU*)

For a more detailed discussion of these considerations, see Roland Hinnion's Ph. D. thesis [7] (where these ideas were first treated in the context of *NF*), our recent book [11], or Solovay's preprint [19]. In our book [11], we concern ourselves primarily with a set Z_0 which may be characterized as the largest complete rank V_α coded in Z.

The comprehension principle for collections of elements $\mathrm{T}(s)$ of Z is sufficient to establish that every set of Cantorian elements of Z (fixed points of the T operation) is itself represented by an element of Z, and, further, that every collection of strongly Cantorian elements of Z (these are exactly the fixed points of T every iterated preimage under E of which is also fixed under T) is represented by an element of Z, which is itself easily shown to be strongly Cantorian itself (this last is not necessarily true for collections of Cantorian elements). To see this last point, note that there is a function f which is a set such that $f(\mathrm{T}(s)) = \{s\}$ for every $s \in Z$. Note that for any strongly Cantorian element x of Z, the isomorphism type of the component of E associated with x is x itself; the map f witnesses the strongly Cantorian character of the full domain of all such components uniformly. It is clear that if the component of E associated with x has all immediate components strongly Cantorian, f will witness the strongly Cantorian character of all these components at once, and of the component associated with x itself as well.

We now prove an important

Theorem: *NFUB*− proves that the (proper class) collection of the strongly Cantorian elements of Z with the membership relation E interprets *ZFC*; an obvious corollary is that the same is true for *NFUB* with the qualifier "strongly" omitted.

Proof: The axioms of Foundation and Extensionality obviously hold.

The interpretations of the axioms of Pairing and Union in the strongly Cantorian elements of Z involve easy "graph manipulations": for pairing, it suffices to make disjoint copies of two relations representing the two elements of Z to be "collected", add a new top, and apply an extensional collapse; the resulting relation is easily seen to be strongly Cantorian. For union, take a relation representing the element of Z of which a "union" is to be taken, delete the links between its top and the tops of its immediate components, and link the top with the tops of the immediate components of its immediate components. It is again easy to show that the resulting relation is strongly Cantorian, and its type will be the desired "union".

Showing that Choice holds, given Pairing, is strictly a technical exercise: one builds a relation coding a collection of pairs defined by reference to an actual well-ordering of the immediate components of a relation representing the element of Z which we are "well-ordering" in the interpreted set theory.

Infinity is provided because we have the Axiom of Counting (which gives us infinite strongly Cantorian sets).

Separation is a special case of Replacement, which we now consider.

First, we observe that the Axiom of Small Ordinals can be "transferred" from the strongly Cantorian ordinals to the strongly Cantorian elements of Z. To see this, observe that any map from an initial segment of the ordinals onto a non-Cantorian rank in Z which is increasing (not strictly, of course) with respect to rank will put the strongly Cantorian ordinals in one-to-one correspondence with the strongly Cantorian elements of Z. It is also important to note that any rank at which a new strongly Cantorian element appears has all elements strongly Cantorian. This bijection can be used to ensure that any definable (possibly proper) class of strongly Cantorian elements of Z is the intersection of the class of strongly Cantorian elements of Z with some set.

Now suppose that for each "element" a of a strongly Cantorian element A of Z there is a unique strongly Cantorian element b of Z such that ϕ, where ϕ is an arbitrary formula in the language of *NFU* (this will cover the cases of all formulas in the interpreted set theory); the Axiom of Replacement asserts that all such elements b make up a set. The class of codes in Z of all pairs (a, b) (using the Kuratowski pair as coded in Z) of an "element" a of A and the corresponding b such that ϕ is a class of Cantorian elements of Z. By the Axiom of Small Ordinals (transferred to Z), this class is the intersection of some set C with the class of strongly Cantorian elements of Z. First, let C' be the intersection of C with the collection of elements of Z which code pairs with first projection an element of A. Now let C'' be the function with domain A obtained by taking for each element a of A only those codes of pairs with first projection a for which the second projection is of minimal rank; this will pick out just the unique b such that ϕ for that choice of a, since all other second projections associated with a will be non-strongly-Cantorian, so of higher rank. The "range" (in the obvious sense) of C'' will be the desired set; any set of strongly Cantorian elements of Z is coded by a strongly Cantorian element of Z.

The proof is complete.

NFUB− is even stronger than this result indicates, as the next two theorems will show. Recall that Kelley-Morse set theory *KM* is the theory extending *ZFC* in which proper classes are admitted as first-class objects: instances of Replacement (and so of Separation) and of class comprehension are allowed to contain quantifiers over all classes.

Theorem: The strongly Cantorian elements of Z with "membership relation" E are the sets of an interpretation of Kelley-Morse set theory.

Proof: The subsets of Z in *NFU* may be used as (non-unique) codes for their intersections with the class of strongly Cantorian elements of Z, and are our candidates to code the classes of *KM*. Equality of classes will be interpreted by the relation holding between subsets of Z with the same strongly Cantorian elements. Every proper class of strongly Cantorian elements of Z is coded by some set, by the Axiom of Small Ordinals. Thus, we can code assertions about proper classes of the interpreted set theory as assertions about sets of *NFU*; these assertions can then be used themselves to define proper classes (by a direct use of the Axiom of Small Ordinals) or sets by Replacement (note that any formula of the language of *NFU* is allowed in our proof of Replacement).

The proof is complete.

Theorem: The proper class ordinal in the interpreted Kelley-Morse set theory is weakly compact.

Proof: The proper class ordinal κ in the interpreted Kelley-Morse set theory is obviously an inaccessible cardinal (using the definition of cardinal appropriate to *ZFC* and *KM*).

To show that it is weakly compact, it suffices to show that any binary tree of size κ has a branch of length κ. A binary tree of size κ is coded in *NFU* by a class of pairs of strongly Cantorian ordinals (as represented in Z). This class is the intersection of the class of strongly Cantorian elements of Z with some set T. There are arbitrarily high strongly Cantorian ranks in Z whose intersection with T is a binary tree on ordinals; thus, there must be a non-Cantorian rank in Z whose intersection T' with T is a binary tree on ordinals (otherwise the set of ordinals dominated by the index of some such rank would be the proper class of strongly Cantorian ordinals, which is absurd). Take any element of non-strongly-Cantorian rank in the tree T'; the branch through T' to that element codes a branch of length κ in the coded binary tree of size κ.

We proved this independently of Solovay and in a very similar way, but the details of this nice proof are due to Solovay in [19].

Solovay points out in [19] that one can simulate the construction of L in Kelley-Morse set theory sufficiently to obtain an interpretation of *ZFC* $-$ Power Set + $V = L$ + "there is a weakly compact cardinal" in *KM* + "the proper class ordinal is weakly compact".

Solovay has proved the following result:

Theorem (Solovay): The consistency strength of *NFUB* (and, by our results above, of *NFUB*$-$) is exactly that of the theory *ZFC* $-$ Power Set + "there is a weakly compact cardinal".

For the interpretation of *NFUB* in *ZFC* $-$ Power Set + "there is a weakly compact cardinal", see Solovay's electronically available preprint [19]. Note that our development above combined with Solovay's result show that *NFUB* and *NFUB*$-$ have precisely the same consistency strength.

6 Putting it all together: *NFUM*

The final system we consider in this paper is the extension *NFUM* (M for "measure", for reasons which will become evident) of *NFUB* obtained by adding the Axiom of Large Ordinals (this is equivalent to adding the Axiom of Large Ordinals to *NFUB*$-$, since the Axiom of Large Ordinals implies the Axiom of Cantorian Sets). The results about models of this system and its consistency strength are ours. *NFUM* is the full system of our book [11].

6.1 A model of *NFUM*

In this section, we construct a model of *NFUM*, known as the "BEST model" (both because it is the best model and because it was presented at a session of the Boise Extravaganza in Set Theory (BEST)), on the assumption that there is a measurable cardinal κ.

We work in *ZFC* with a measurable cardinal κ. We recall the construction of an elementary embedding from the universe V into an inner model M: using the measure on subsets of κ as an ultrafilter, build an ultrapower of V. Since the ultrafilter is countably complete, the ultrapower will be well-founded; we can thus carry out a transitive collapse of the ultrapower to an inner model M of *ZFC*. The usual elementary embedding of a set or class into its ultrapowers via equivalence classes of constant functions yields an elementary embedding j of the universe V into the inner model M. The measurable κ is sent by j to a larger ordinal $j(\kappa)$. See [12], after p. 305, for details.

We define a nonstandard model of the full theory of V_λ, where $\lambda = \lim j^i(\kappa)$. The elements are all those functions s with domain a tail $[n, \infty)$ of $\mathcal{N}$ (or all of $\mathcal{N}$) with the property that $s(n) \in j^n(V_\lambda)$ and $s(n+1) = j(s(n))$ whenever $s(n)$ is defined, and that if $j^{-1}(s(n))$ exists and $n > 0$, then $s(n-1)$ is defined and equal to $j^{-1}(s(n))$. The membership relation "$s \in t$" on this structure is defined by "for all sufficiently large n, $s(n) \in t(n)$"; all other predicates of the full language of V_λ can be defined analogously. It should be clear that the set of s for which $s(0)$ is defined is isomorphic to V_λ and an elementary substructure of the whole: the whole structure is a direct limit of the $j^i(V_\lambda)$'s using j as the embedding of each in the next.

The map J defined by $J(s)(n) = s(n+1)$ is an automorphism of this model which moves the element α with $\alpha(0) = \kappa$ (for example) upward. The model of *NFU* that we construct has as elements the "elements" in terms of this structure of the model element s with $s(0) = V_\kappa$ (i.e., V_α) and as membership relation "$x \in_{NFU} y$" the relation (in terms of this nonstandard structure) $x \in J(y) \wedge J(y) \in V_{\alpha+1}$; this is a case of the general model construction defined for *NFU* above.

It should be clear that this structure is a model of *NFU*; it remains to demonstrate that it is a model of *NFUM*.

This model has two kinds of elements: elements s with $s(0)$ defined, which will then have $s(0) \in V_\kappa$ and so have $s(0) = j(s(0))$; these elements make up a structure isomorphic to the standard V_κ. These elements are fixed points of J and dominate only fixed points of J (and so are Cantorian and indeed strongly Cantorian in the model of *NFU*). The other elements are elements s with $\text{dom}(s) = [n, \infty)$ for some $n > 0$; call the number n $\text{char}(s)$.

We verify that the Axiom of Large Ordinals holds in the model. The ordinals of the model will be in one-to-one correspondence with (though they will not be the same objects as) the ordinals of the underlying structure which are in the model. For simplicity's sake, we deal with the latter. Also for simplicity, we refer to the ordinal represented by the model element s with $s(1) = \kappa$ as Ω, though it does not correspond to the true Ω of the model; we will explain below why this is harmless. There are, of course, ordinals of the first kind, with $s(0) = \alpha < \kappa$. These are fixed by J and are the strongly Cantorian ordinals of the model. The other ordinals s of the model have $\text{char}(s) > 0$ and $s(\text{char}(s)) \geq \kappa$, so $s(\text{char}(s) + 1) \geq j(\kappa)$. Such an element will be greater than $J^{-\texttt{char}(s)-1}(\Omega)$, which has value κ at $\text{char}(s) + 1$ and is undefined below that point. Since $J^n(\Omega)$ interprets $\mathrm{T}^n(\Omega)$ for each n, this shows that the ordinals

$\mathrm{T}^i(\Omega)$ are coinitial in the non-Cantorian ordinals of the model, so in fact the ordinals $\mathrm{T}^i(\alpha)$ are coinitial in the non-Cantorian ordinals of the model for any non-Cantorian ordinal α of the model, and so the same holds true for the true Ω (the order type of the ordinals), whether that is to be identified with our Ω of convenience or not (in fact, it is not). This establishes the claim of the paragraph.

We verify that the Axiom of Small Ordinals holds in the model. It is sufficient to show that every subclass of the elements of the first kind (those with $s(0)$ defined) is the intersection of the class of elements of the first kind with some set. Let C be any subset of the true V_κ; there is an element of the underlying structure with $c(0) = C$. c is in the model only if $C \in V_\kappa$ itself, but the object $J^{-1}(c)$ with $c(1) = C$ is always a model element ($C \in j(V_\kappa)$ holds for any $C \in V_{\kappa+1}$) and has exactly the correct elements of the first kind.

This model is the source for most of our "intuition" about *NFUM*, though it turns out that the assumption of a true measurable cardinal is more than is needed, as we will see below. An interesting aside about this model: it models something very much like the "theory of concepts" which Wang sketches as the possible realization of a project of Gödel in [22], p. 310, though there seem to be infelicities in Wang's formulation. Wang's idea is to implement "concepts" as elements of a model of stratified comprehension without extensionality in which there is a "downward closed" extensional well-founded part of the membership relation whose domain is a model of *ZFC* (he calls this the domain of "sets" as opposed to concepts in general); the "best" model fulfils this description (the analogue in the model of the standard V_κ is the maximal downward closed extensional well-founded part of the membership relation excluding urelements, and it is of course a model of *ZFC*). This result can probably be achieved in an extension of *NFU* with lower consistency strength, but Wang makes the additional stipulation that the replacement scheme for the "sets" include instances for all formulas in the full language of the system of concepts (not just the restricted language of the domain of sets), which appears to require considerable strength, though maybe not quite that of the Axiom of Small Ordinals. Any model of *NFUM* can be converted to a model with these characteristics by a suitable application of permutation techniques standard in the area of *NF*-like theories (see [3] for a treatment) to convert the E relation on the Cantorian part of Z to a "downward closed" part of the true membership relation.

6.2 Interpreting *KMU* in *NFUM*

The results of this section come from an attempt to "reverse engineer" the construction of the previous section and recover the measurable cardinal κ of the construction of the "best" model starting with an arbitrary model of *NFUM*.

Of course, since *NFUM* is an extension of *NFUB*−, it interprets *KM* in the same way that *NFUB*− does. It turns out that we can do rather more in *NFUM*, because the Axiom of Large Ordinals enables us to refine our coding of classes of Cantorian objects (since the Axiom of Cantorian Sets holds, "Cantorian" and "strongly Cantorian" are equivalent notions, and we will use the shorter term).

Definition: On any set which supports a T operation (ordinals, cardinals, elements of Z) we define a *natural set* as a subset A of the T-carrying set with the property that for each x in the T-carrying set, $x \in A$ iff $T(x) \in A$. (Compare with the definition of "semi-natural set" above).

Theorem (*NFUM*): Each class of Cantorian ordinals is coded by a unique natural set. (The same result applies to cardinals or elements of Z).

Proof: Certainly any natural set codes some uniquely determined class.

We need to show that two distinct natural sets A and B cannot code the same class of Cantorian ordinals. To see this, consider the smallest element x of the symmetric difference of A and B. $\mathrm{T}(x)$ will also belong to the symmetric difference of A and B, and so must either be greater than x or equal to x by minimality. If $\mathrm{T}(x) = x$, then x is Cantorian and the two natural sets do not code the same class. If $\mathrm{T}(x) > x$, then $\mathrm{T}^{-1}(x)$ exists, is less than x, and also belongs to the symmetric difference of A and B (by naturality of these sets), contradicting minimality of x.

We need to show that every class is coded by at least one natural set (and so, by the previous paragraph, by exactly one natural set). Choose a class C. By the Axiom of Small Ordinals, the class C is coded by some set A. If A is natural, we are done. If A is not natural, then consider the minimal element x of the symmetric difference of A and $\mathrm{T}[A]$ (the elementwise image of A under the T operation). By the Axiom of Large Ordinals, there is a natural number n such that $\mathrm{T}^n(\Omega) < x$. The set $\mathrm{T}^{-(n+1)}[A]$ is the desired natural set (this last assertion relies on the effective reasoning about T-sequences made possible by the Axiom of Small Ordinals).

The proof is complete.

This result enables us to define a "measure" on classes of Cantorian ordinals, which translates to a "measure" on the proper class ordinal in the interpreted *KM* (it is not hard to show that there is a precise correspondence between the ordinals in a model of *NFUM* and the codes for von Neumann ordinals in Z, which preserves the property of being strongly Cantorian; the set of codes of von Neumann ordinals in Z is the natural set which interprets the proper class ordinal of *KM*).

Definition: We say that a class C of Cantorian ordinals is "of measure 1" if the natural set of ordinals coding C contains Ω (the order type of the ordinals). If a class is not of measure 1, we say that it is "of measure 0".

The choice of Ω is natural but immaterial; any non-Cantorian ordinal would serve as well.

We claim that this measure on the proper class ordinal κ is a κ-complete nonprincipal ultrafilter; it requires a little work to show that we can even *say* this, since a measure in the usual sense would belong to $V_{\kappa+2}$, and we only have access to $V_{\kappa+1}$ (the sets of the interpreted *KM* make up V_κ, and the proper classes fill out $V_{\kappa+1}$).

The trick is to observe that we can, in KM, code sequences of length κ of elements of $V_{\kappa+1}$: we code a sequence $\{A_\alpha\}_{\alpha<\kappa}$ as the class $\{\langle\alpha, x\rangle \mid x \in A_\alpha\}$, where $\langle\alpha, \beta\rangle$ represents a coding of pairs of Cantorian ordinals as Cantorian ordinals (such a pair is easy to define; there is a description of one such in [11]).

Given this machinery, we can assert (as a definition) that a superclass (predicate of classes) U is a κ-complete nonprincipal ultrafilter on κ iff the following conditions hold:

1. Each A such that $U(A)$ is an infinite class of ordinals.
2. If $A \subseteq B$ and $U(A)$, then $U(B)$.
3. If each element of a sequence $\{A_\alpha\}_{\alpha<\gamma}$, where $\gamma < \kappa$, satisfies $U(A_\alpha)$, then $U(\bigcap A_\alpha)$. (It is easy to see that $\bigcap A_\alpha$ is definable in KM and must be a class).
4. For every class A, either $U(A)$ or $U(A^c)$.

We define KMU as the theory KM + "there is a κ-complete nonprincipal ultrafilter on the proper class ordinal κ"; KMU has an additional primitive predicate U of classes (the ultrafilter) which can be used freely in definitions of classes and sets. We hope that the use of the suffix "U" for "ultrafilter" here will not conflict in anyone's mind with its use for "urelement" in the name "*NFU*".

We now complete the demonstration that *NFUM* interprets KMU, with "is of measure 1" as the predicate of classes coding the ultrafilter. We use the same headings as above.

1. Each A such that $U(A)$ is an infinite class of ordinals: clearly each natural set containing Ω codes an infinite class of Cantorian ordinals (the requirement that every A such that $U(A)$ be infinite rules out principal ultrafilters).
2. If $A \subseteq B$ and $U(A)$, then $U(B)$: the natural set coding a superclass of a given class will be a superset of the natural set coding the given class: a superset of a set containing Ω also contains Ω.
3. If each element of a sequence $\{A_\alpha\}_{\alpha<\gamma}$, with $\gamma < \kappa$, satisfies $U(A_\alpha)$, then $U(\bigcap A_\alpha)$: the sequence of classes, because it is not of full length κ, is coded by a *set* of natural sets containing Ω, whose intersection will be a natural set containing Ω coding the intersection of the sequence of classes: the natural set coding the subclass of $\gamma \times \kappa$ (as coded into the ordinals) which codes the sequence will have γ natural sets as "rows" (this would not work if the sequence went all the way out to κ, because the natural set coding the sequence would have additional "rows" with non-Cantorian indices which would not necessarily be natural).
4. For every class A, either $U(A)$ or $U(A^c)$: the natural set coding the complement of a class is the complement of the natural set coding the given class; exactly one of these natural sets will contain Ω.

This establishes the claim of this section: *NFUM* interprets *KMU*.

Solovay has convinced us that enough of the construction of $L[U]$ can be carried out in *KMU* to obtain an interpretation of *ZFC* $-$ Power Set + "the largest cardinal has a class measure U" + "$V = L[U]$" (with the proviso that the class measure may be alluded to in instances of Separation and Replacement). Details of this are not given here.

6.3 Interpreting *NFUM* in *KMU*

Our approach to interpreting *NFUM* in *KMU* will be to reproduce the construction of the "best" model of *NFUM* with the more restricted set-theoretic resources available in *KMU*.

The superclass (!) structure which we will use to interpret *NFUM* will be a direct limit of ultrapowers. Of course, since superclasses are not really objects in *KMU*, this construction needs to be carried out with extreme care.

On any superclass A with an associated notion of equivalence on its elements, we construct an ultrapower as follows: the elements of the ultrapower will be elements of "A^κ" (the superclass of "sequences of length κ" of elements of A; we have seen above how to code sequences of classes of length κ as single classes); two elements A and B of the ultrapower will be considered equivalent iff the class of all indices α such that A_α is equivalent to B_α belongs to U. For technical reasons, we eliminate all κ-sequences with all terms eventually the empty set, except for the empty set itself (this ensures that the iterated ultrapowers U_i below will be disjoint (except for the shared empty set), and it does no damage to the ultrapower, since the omitted objects all represent the empty set anyway).

All structure on the superclass A is inherited by the ultrapower on A in the usual way; every predicate and relation of the full language on A which respects the associated equivalence relation has a natural analogue on the ultrapower, and the "map" sending elements of A to their constant κ-sequences is an elementary embedding of A (with its full language mod the equivalence relation) into the ultrapower (with the same language).

We let $U_0 = V_\kappa$ (with equality as the associated notion of equivalence), while $U_{n+1} =$ the ultrapower on U_n for each $n \in \mathcal{N}$ (with its natural equivalence relation defined above). Each U_i except the first is a superclass, but all elements of any U_i's are classes. The direct limit U_∞ of the U_i's can be considered as the union of the U_i's; it inherits the equivalence relation of each U_i and also stipulates that each element of a U_i is equivalent to the associated constant sequence in U_{i+1}.

All structure on V_κ describable in its full language is inherited by each U_i and by U_∞ for the usual model-theoretic reasons. We need to confirm that membership in the "superclass" U_∞ and equivalence in U_∞ are definable by formulas in the language of *KMU*, since superclasses are not really objects in *KMU*.

It is clear that for any class A and ordinal subscript α, A_α is definable. Let s be a finite sequence of ordinals. We define A_s as A if s is the empty sequence; otherwise we define s^- as the sequence obtained by dropping the last term s_n

of s and define A_s as $(A_{s^-})_{s_n}$. We can then express $A \in U_n$ as "either A is empty or every A_s with the length of s equal to n is in V_κ, while every A_s with shorter s is either empty or a κ-sequence with κ nonempty terms." It is clear then that membership in U_∞ is definable in KM, since membership in the U_n's is uniformly definable.

Defining the equivalence relation on U_n uniformly involves a trick. If A and B are both elements of the same U_n, we proceed as follows: let K be the class of all finite sequences of ordinals s of length n such that $A_s = B_s$. Now define a class S of pairs with first projection a natural number and second projection a finite sequence of ordinals as follows: $(0, x) \in S$ iff $x \in K$; we define x conc y, where x is a finite sequence and y is an ordinal, as the result of extending the sequence x by adding y at the end, and provide that for each n, $(n+1, x) \in S$ iff $\{y \mid (n, x \text{ conc } y) \in S\} \in U$. This kind of inductive definition of a class succeeds in KMU (because it is allowed in KM with any predicate of classes in place of U). It is straightforward to verify that A is equivalent to B in the sense of U_n iff the pair $(n, ())$ of n and the sequence $()$ of length 0 belongs to the class S, which is uniformly definable in terms of A and B, so the equivalence on U_n is expressible in terms of KMU.

Two elements A and B of U_m and U_n, with $m < n$, are equivalent in the sense of U_∞ iff the $(n - m)$-fold iterated constant sequence of A in U_n is equivalent to B in the sense of U_n; it is clear from the discussion above that this can be expressed in the language of KMU.

It follows that the superclass U_∞ and its associated notion of equivalence are represented by predicates of the language of KMU. It should be clear that all predicates of the full language of V_κ are represented by predicates in U_∞ which respect the notion of equivalence, and that U_∞ mod its equivalence relation is an elementary superstructure of V_κ.

We claim further that there is an endomorphism J on U_∞ which moves a nonstandard ordinal of U_∞ downward. The endomorphism is induced by the map J which sends all elements of $V_\kappa = U_0$ to their constant sequences in U_1 and sends each element $\{A_\alpha\}_{\alpha<\kappa}$ of U_n $(n > 0))$ to the element $\{J(A_\alpha)\}_{\alpha<\kappa}$ of U_{n+1}. The ordinal δ which is moved is the element of U_1 determined by the "diagonal sequence" $A_\alpha = \alpha$; it is moved downward to the element of U_2 determined by the sequence whose αth member is the constant sequence of α; it is straightforward to verify that this is a smaller ordinal: each of its terms is a "standard" ordinal in U_1, and so is less than the nonstandard ordinal in U_1 coded by the diagonal sequence. It should be obvious that J is an endomorphism. $J(A)$ can be more economically described: for any finite sequence s of ordinals, define s_- as the sequence obtained by dropping the first element: $J(A)$ is then characterized by $J(A)_s = A_{s_-}$; from this we can see that the superclass map J can be described in the language of KMU.

We include some informal discussion of what the "nonstandard model" U_∞ is like. U_1 is a nonstandard model of V_κ in which all new elements are of rank above all the standard ranks (U_1 is an end extension of V_κ). An example of a new ordinal added in U_1 is the ordinal δ determined by the "diagonal sequence" $A_\alpha = \alpha$. U_2 in its turn is a nonstandard model of U_1 (an elementary extension of U_1 for the properties of U_1 expressed in the full language of V_κ in its capacity as being itself an elementary superstructure of V_κ); the ordinal $J(\delta)$ such that $J(\delta)_\alpha$ is the constant sequence of α is (from the standpoint of U_1) a "new" nonstandard ordinal appearing above all the standard ordinals and below all the nonstandard ordinals in U_1. The same phenomenon occurs at each step of the construction: the ordinal $J^n(\delta) \in U_{n+1}$ appears above all the standard ordinals $< \kappa$ and below all nonstandard elements of U_n.

As we observed earlier in the paper, it takes only a slight modification of the model construction for *NFU* to use an endomorphism downward in place of an automorphism upward: if V_α is the domain of the model (with $J(\alpha) < \alpha$) define $x \in_{NFU} y$ as $J(x) \in y \wedge y \in V_{J(\alpha)+1}$. We use the ordinal δ defined above in the role of α in this definition.

The arguments for the Axiom of Large Ordinals and the Axiom of Small Ordinals which follow are essentially the same as the arguments that these axioms hold in our "best" model described above: U_∞ is essentially the same structure as the "best" model, constructed without the advantage of being able to carry out transitive collapses of ultrapowers onto inner models.

We claim that the Axiom of Large Ordinals holds in this model. An ordinal in U_0 is fixed by J and strongly Cantorian in the model (Cantorian because fixed and strongly Cantorian because everything below is fixed.) If δ is the ordinal in U_1 determined by the diagonal sequence, any ordinal of the model in $U_n - U_{n-1}$ (i.e., new in U_n) will dominate the ordinal $J^n(\delta)$ in U_{n+1} (which from the standpoint of U_n is "just larger" than the "standard" ordinals in U_0); this establishes the claim of this paragraph, since it shows that the iterated images under T of δ (or of any non-Cantorian ordinal) will be coinitial in the non-Cantorian ordinals.

We claim that the Axiom of Small Ordinals holds in this model. The strongly Cantorian ordinals of the model are exactly those in V_κ. Observe that any class of strongly Cantorian ordinals defined in terms of the model U_∞, its equivalence relation and the automorphism on it will be describable in the language of *KMU* and so will be a class of ordinals in *KMU*. It is sufficient to show that there is a set in our model with its strongly Cantorian members exactly those in an arbitrarily chosen class C of ordinals of *KMU*. The class C will not be in our model unless it is actually an element of V_κ itself. But the element of U_1 defined by $C_\alpha = C \cap \alpha$ will belong to the model and have precisely the Cantorian elements (i.e., elements in U_0) that C has, completing the proof of the claim.

6.4 Concluding Remarks about *NFUM*

We summarize the work of this section in a

Theorem: The consistency strength of *NFUM* is exactly the same as the consistency strength of *KMU*; each of these theories interprets the other.

Some work remains to be done. A pithy statement of the precise consistency strength of *NFUM* analogous to Solovay's theorems would be nice. It is quite high, but below *ZFC* + "there is a measurable cardinal": an argument from the construction of $L[U]$ along class well-orderings in *KM* appears to give the result that *NFUM* has the same strength as "*ZFC* − Power Set + there is a class measure U on the largest cardinal" with the proviso that U can be mentioned in instances of Separation and Replacement; this is analogous to Solovay's formulation for *NFUB*, but does not seem quite as satisfactory, since this theory has the same odd feature as *KMU* that the "measure" needs to be introduced as a primitive notion of the theory: the existence of such a "measure" is not directly expressible in the language of *ZFC*!

NFUM seems to be almost as strong a "natural" extension of *NFU* along these lines as one can come up with; the one further idea we can see is to exploit the unique coding of classes of ordinals of the interpreted *KM* as natural sets and try to formulate axioms about what collections of classes of ordinals should be coded by sets of *NFU*.

We present the whole paper as further evidence for our belief, which also motivated [11], that *NFU* (with extensions) is a nice supplement to the set theorist's usual view of the mathematical world via *ZFC* (with extensions).

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