The usual model construction for *NFU* preserves information

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1 Introduction

This paper was originally motivated by a question of Thomas Forster about the urelements in a model of Jensen's modification *NFU* of Quine's New Foundations, though that question seems to be far from the main point of this paper, which is a strengthening of the sense in which (a version of) *NFU* and (a version of) the usual set theory are mutually interpretable.

Forster asked whether there are models of *NFU* in which the urelements are homogeneous (in which for any formula $\phi(a_1, \ldots, a_n)$, in which the a_i's are the only parameters, $(\forall a_1 \ldots a_n.(\forall b_1 \ldots b_n.U(a_1) \wedge \ldots U(a_n) \wedge U(b_1) \wedge \ldots U(b_n) \rightarrow \phi(a_1, \ldots, a_n) \leftrightarrow \phi(b_1, \ldots, b_n)))$ holds, where $U(x)$ means "x is an urelement") (it is known that this is true if ϕ is stratified). Of course a model of *NF* would be such a model of *NFU*.

The author's (incorrect!) response was that of course there are such models, because all the urelements in the "usual" models of *NFU* (described in detail below) are indistinguishable in the suitable sense, because all information about the extensions of the urelements of the model is discarded in the construction.

But it turns out that this is not the case, and for a reason which perhaps has more interest than the answer to the original question (which remains unanswered, and may have some of the flavor of the unsolved problem of the consistency of *NF*). The original models of *NFU* are in fact quite rigid, because (surprise!) the information about extensions of objects which become urelements is *not* discarded in the usual construction of a model of *NFU*. The usual construction of a model of *NFU* starts with a model of (some fragment of) ordinary set theory with an automorphism which moves a rank of the cumulative hierarchy. The domain of the model of *NFU* is a rank in this model moved by the automorphism. Most elements of this rank are treated as urelements in the model of *NFU* and it *appears* that information is being discarded in this process. But it is not: the membership relation of the original model of set theory with automorphism turns out to be first-order definable in the model of *NFU*, which quite incidentally gives a negative answer to the question as to whether the urelements in the model of *NFU* are homogeneous.

2 *NFU*

We briefly describe the theory *NFU*, along with the related theories *TST* (simple type theory) and *NF* (New Foundations).

Simple type theory *TST* is a strongly streamlined version of the type theory of Russell and Whitehead ([20]). One follows Ramsey in eliminating the orders in [20] ([14]). One follows Wiener ([18]) in noting that since there is a definable ordered pair in typed set theory, one does not need relation types. Theories of this kind were apparently first proposed about 1930 (see [19] for historical remarks).

TST is a first-order theory with sorts (called types) indexed by the natural numbers. The typing conditions for atomic formulas are briefly given by the templates $x^n = y^n$, $x^n \in y^{n+1}$. The axioms of *TST* are

Extensionality:

$$(\forall A^{n+1} B^{n+1}.A^{n+1} = B^{n+1} \leftrightarrow (\forall x^n.x^n \in A^{n+1} \leftrightarrow x^n \in B^{n+1}))$$

is an axiom for each n.

Comprehension: For each formula ϕ in which A^{n+1} is not free, and each variable x^n,

$$(\exists A^{n+1}.(\forall x^n.x^n \in A^{n+1} \leftrightarrow \phi))$$

is an axiom.

Axioms of Infinity and Choice are usually adjoined, whose exact form need not be investigated at this point.

In [13], W. v. O. Quine proposed that *TST* could be collapsed to an unsorted first-order theory with equality and membership whose axioms are exactly the axioms of *TST* with distinctions of type between variables ignored (in a way which does not introduce any identification between variables of different types). This theory is called *NF* (New Foundations, after the name of the paper [13]). It is traditional to present it in a way which does not depend on the definition of another theory:

Extensionality:

$$(\forall AB.A = B \leftrightarrow (\forall x.x \in A \leftrightarrow x \in B))$$

is an axiom for each n.

Definition: Let ϕ be a formula in the language of first-order logic with equality and membership. We say that a function σ from variables to natural numbers is a *stratification* of ϕ iff for each atomic subformula '$x = y$' of ϕ we have $\sigma(\text{'}x\text{'}) = \sigma(\text{'}y\text{'})$ and for each atomic subformula '$x \in y$' we have $\sigma(\text{'}x\text{'}) + 1 = \sigma(\text{'}y\text{'})$. We say that ϕ is *stratified* iff there is a stratification of ϕ.

Stratified Comprehension: For each stratified formula ϕ in which A is not free, and each variable x,

$$(\exists A.(\forall x.x \in A \leftrightarrow \phi))$$

is an axiom.

It should be clear that this is an equivalent description of the theory, as the stratified formulas are exactly those which can be obtained from well-formed formulas of *TST* by dropping type distinctions without introducing identifications between variables.

In fact, the notion of stratification can be eliminated from the definition of the theory and with it the last reference to even relative notions of type: the axiom scheme of stratified comprehension is equivalent to the conjunction of finitely many of its instances. The usually referenced though far from the nicest presentation of this is in [5].

The consistency of *NF* remains an open question. In 1969 ([9]) R. B. Jensen proved the consistency of *NFU* (New Foundations with urelements), which differs from *NF* only in its formulation of Extensionality:

Weak Extensionality:

$$(\forall AB.(\exists y.y \in A) \rightarrow A = B \leftrightarrow (\forall x.x \in A \leftrightarrow x \in B))$$

is an axiom for each n.

Notice that *NFU* has the same comprehension scheme as *NF*. The idea is that *NFU* allows the existence of many urelements without elements in addition to the empty set. Strictly speaking, the formulation above does not allow one to specify an empty set. An inessential and very convenient modification of the theory is to add the empty set as a primitive (which allows one further to define a sethood predicate). We present a full axiomatization of *NFU* with empty set as a primitive notion:

Empty Set: $(\forall x.x \notin \emptyset)$

Definition:

$$\texttt{set}(x) \equiv_{\texttt{def}} x = \emptyset \vee (\exists y.y \in x)$$

Weak Extensionality:

$$(\forall AB.\texttt{set}(A) \wedge \texttt{set}(B) \rightarrow A = B \leftrightarrow (\forall x.x \in A \leftrightarrow x \in B))$$

is an axiom for each n.

Stratified Comprehension: For each stratified formula ϕ in which A is not free, and each variable x,

$$(\exists A.\texttt{set}(A) \wedge (\forall x.x \in A \leftrightarrow \phi))$$

is an axiom.

Specker showed in [17], 1953 that *NF* disproves Choice (and so proves Infinity). Jensen showed that *NFU* is consistent with Infinity and Choice, and further that *NFU* is consistent with the negation of Infinity.

3 The Boffa model construction

We do not describe the original consistency proof of *NFU* due to Jensen. Instead, we give a related model construction due to Maurice Boffa (in [1]). This is appropriate as the aim of this paper is to prove a result about models of the sort described by Boffa (the class of models we consider is actually slightly less general than the class of models discussed by Boffa).

Let M be a model of Mac Lane set theory (Zermelo set theory with separation restricted to bounded formulas; so-called because it was suggested as a foundation for mathematics by Saunders Mac Lane in [10]) with the additional axiom that every set belongs to a rank V_α (this axiom provides no essential additional strength: see [11]). We can further suppose, using the Ehrenfeucht-Mostowski theorem of model theory ([3]), that M has an external automorphism j which moves a "rank" V_α (and of course its "ordinal index" α, which is not a standard ordinal). We can further suppose without loss of generality that $M \models \alpha > j(\alpha)$ (as we could replace j with j^{-1} to make this true in the worst case). We stipulate that the structure we are talking about is of the form $\langle M, \in_M, j, \alpha \rangle$, so that we can talk about the automorphism. $\langle M, \in_M, j, \alpha \rangle$ is a model of Mac Lane set theory + "every set belongs to a rank" with the following modifications: j and α are added to the language, along with axioms asserting that α is an infinite ordinal, V_α exists, and $j(\alpha) < \alpha$, and with Separation restricted to formulas in which j does not appear (which can be extended to formulas in which j appears in parameters).

We construct a Boffa model $\langle B, \in_B, j \rangle$ of *NFU* (with an additional function symbol j which cannot appear in instances of stratified comprehension, except in parameters; we include it merely so that j makes sense in B-formulas) as follows. $B = \{x \mid M \models x \in V_\alpha\}$: B is the collection of M-elements of V_α. $x \in_B y$ is defined as $j(x) \in_M y \wedge y \in_M V_{j(\alpha)+1}$. Of course $B \models x = j(y)$ iff $M \models x = j(y) \wedge y \in V_\alpha$. Notice that if $M \models u \in V_\alpha - V_{\alpha+1}$ then $B \models (\forall x . x \notin u)$ (the empty set of M is also elementless in the model of *NFU*, and naturally taken to be the empty set of B). The M-elements of $V_\alpha - V_{\alpha+1}$ are the urelements of the model B. It should be evident that weak extensionality holds in B. The proof that Stratified Comprehension holds in B is found in [4]). In general, the translation of a formula in the language of B into terms of M introduces occurrences of j, which cannot occur in instances of Separation (except in parameters); the strategy of the proof is to show that the problematic occurrences of j in the translation of a stratified formula can be eliminated.

It is straightforward to establish that B will satisfy Infinity and that B will satisfy Choice iff M does. If we modified the construction by providing that $M \models$ "α is a finite ordinal", the model B would satisfy *NFU* with the negation of the Axiom of Infinity.

4 Recovering information from a Boffa model

One might be able at this point to get a general idea of why the author made the mistake of intuition described in the Introduction. It appears that all information about the M-extensions of the M-elements of the object which M calls $V_\alpha - V_{j(\alpha)+1}$ is discarded in the definition of $\in_B$. But it turns out that all this information is still accessible: the restriction of $\in_M$ to the domain of the Boffa model B is first-order definable in the language of *NFU*. In this section, we describe how to do this.

If $M \models A \in V_{\alpha+1}$, then for each y, $M \models x \in A$ iff $M \models j(x) \in j(A)$ iff $B \models x \in j(A)$: informally, we say that the M-set A is implemented by the B-set $j(A)$. An unordered pair $\{a, b\}$ is implemented by $\{j(a), j(b)\}$ (the second object being named here in the language of M; in the language appropriate to B, it would be called $\{a, b\}$: the extremely precise statement of this is that for any u, a, b, $B \models u = \{a, b\}$ iff $M \models u = \{j(a), j(b)\}$). The ordered pair $\langle a, b\rangle$ in the sense of B is then $\langle j^2(a), j^2(b)\rangle$ in the sense of M. Finally, if f is a function in M, we see that $j^3(f)$ is the function with the same extension in B: $M \models y = f(x)$ iff $M \models \langle x, y\rangle \in f$ iff $M \models \langle j^2(x), j^2(y)\rangle \in j^2(f)$ iff $B \models \langle x, y\rangle \in j^3(f)$ (clearly $M \models \langle j^2(x), j^2(y)\rangle \in V_{j(\alpha)+1}$). So we see that every M-function from V_α to V_α is implemented in B as a function from the universe to the universe (note that $j(V_\alpha)$ is the universal set in the sense of B: $M \models x \in V_\alpha$ implies $M \models j(x) \in j(V_\alpha)$ which in turn implies $B \models x \in j(V_\alpha)$. We denote the universal set by V and the collection of all sets by $\mathcal{P}(V)$ in B-formulas: the same objects are $j(V_\alpha)$ and $j^2(V_{\alpha+1})$ in M.

We introduce a nonstandard piece of notation common in *NF* studies (ultimately derived from [20]), because we will shortly use it. $\iota(x)$ is a notation for $\{x\}$, and so ι"A is notation for the collection of one-element subsets of A (the elementwise image of A under the singleton operation).

We introduce a specific function found in M, namely the function $\mathbf{S} = j^3(\{\langle\{x\}, x\rangle \mid x \in j(V_\alpha)\})$. $M \models$ "the domain of $j^{-3}(\mathbf{S})$ is ι"$j(V_\alpha)$ and its range is $j(V_\alpha)$", so of course $M \models j^{-3}(\mathbf{S}) : \iota\text{“}j(V_\alpha) \rightarrow j(V_{\alpha+1})$. An

M-element of $\iota``j(V_\alpha)$ is of the form $\{j(x)\}$, which $B$ sees as $\{x\}$, whereas $M$-elements of $j(V_{\alpha+1})$ are exactly the sets in the sense of $B$: $B \models \mathbf{S} : \iota``V \to \mathcal{P}(V)$, i.e., B says that $\mathbf{S}$ is a function from the set of all singletons into the set of all sets. Further, $M \models (\forall x \in V_\alpha . j^{-3}(\mathbf{S})(\{j(x)\}) = j(x))$: thus $B \models (\forall x . \mathbf{S}(\{x\}) = j(x))$. $\mathbf{S}$, which is a set map, codes the external automorphism j in a certain sense (j is not a set function in either M or B).

The function $\mathbf{S}$ has a further property of considerable interest in B. Suppose $M \models A \in j(V_{\alpha+1})$ (equivalently, $B \models$ "A is a set"). $B \models x \in A$ iff $M \models j(x) \in A$ iff $M \models j^2(x) \in j(A)$ iff $B \models j(x) \in j(A)$ (certainly $M \models j(A) \in V_{j(\alpha+1)}$) iff $B \models \mathbf{S}(\{x\}) \in \mathbf{S}(\{A\})$. Further, if $B \models y \in j(A)$, then $M \models j(y) \in j(A)$, so $M \models y \in A$, so $M \models y \in V_{j(\alpha)}$, so $M \models y = j(x)$ for some x such that $M \models x \in V_\alpha$, so $B \models y = \mathbf{S}(\{x\})$ for some such x. Thus $B \models$ "for any set A, $\mathbf{S}(\{A\}) = \{\mathbf{S}(\{x\}) \mid x \in A\}$".

We summarize this statement in the format of an axiom to adjoin to *NFU* (as we did in [7] where we first introduced these ideas with a rather different application in view).

Axiom of Endomorphism (*NFU*): There is an injective function $\mathbf{S} : \iota``V \to \mathcal{P}(V)$ such that for any set A, $\mathbf{S}(\{A\}) = \{\mathbf{S}(\{x\}) \mid x \in A\}$.

We now observe that in the Boffa model B it is possible to define the restriction of the membership relation of M to V_α in terms of the membership relation of B and the function $\mathbf{S}$: $M \models x \in y$ iff $M \models j(x) \in j(y)$ iff $B \models x \in j(y)$ iff $B \models x \in \mathbf{S}(\{y\})$.

Definition (*NFU*+ Endomorphism): $x \, E \, y$ is defined as $x \in \mathbf{S}(\{y\})$. Notice that as the relative types of x and y are the same in the definition of $x \, E \, y$, E is a set relation: $\{\langle x, y\rangle \mid x \, E \, y\}$ exists by stratified comprehension.

$B \models x \, E \, y$ iff $M \models x \in y$.

The final claim which establishes our main result is that if there is a function $\mathbf{S}$ which witnesses the truth of the Axiom of Endomorphism in a Boffa model, then there is exactly one such function. We can then define $x \in^* y$ as "there is a function $\mathbf{S}$ which witnesses the truth of the Axiom of Endomorphism, and $x \in \mathbf{S}(\{y\})$", a statement which can be expanded out into a sentence in the first-order language of *NFU*. Considerations shown above already show that (subject to our final claim) the relation $\in^*$ will coincide with $\in_M$ restricted to the domain of B.

Because the domain of B is a rank of the cumulative hierarchy in M, there is a notion of ordinal rank of elements of the domain (an element x has rank β iff β is the least ordinal such that $x \subseteq V_\beta$). It is the case that the ordinal rank of a set A is the successor of the supremum of the set of ranks of elements of A. The notion of "rank of the cumulative hierarchy" is definable in Mac Lane or Zermelo set theory (the definitions in [12] p. ??? will work), and the notion of ordinal can be defined in Mac Lane set theory with the Axiom of Rank using the trick introduced by Dana Scott in [15]: the order type of a well-orde ring W is the set of all well-orderings isomorphic to W and belonging to the smallest rank in the inclusion order which contains any well-ordering isomorphic to W. The same trick can be used to represent other sorts of isomorphism class as sets (a common application of Scott's trick is to define cardinals in ZF, where the usual von Neumann definition of cardinal number does not work). In Mac Lane or Zermelo set theory without rank there is no reasonable global way to represent cardinal or ordinal number: an extensive development of mathematics in a theory which is essentially Zermelo set theory with the Axiom of Rank, making extensive use of Scott's trick, is found in [12]. The Axiom of Rank does not essentially strengthen Zermelo or Mac Lane set theory; this can be seen in [11].

Suppose that B satisfies the assertion there is another map $\mathbf{S}^*$ such that $\mathbf{S}^* : \iota``V \to \mathcal{P}(V)$ and for any set A, $\mathbf{S}^*(\{A\}) = \{\mathbf{S}^*(\{x\}) \mid x \in A\}$. Define $x \, E^* \, y$ as $x \in \mathbf{S}^*(\{y\})$.

Suppose $\mathbf{S} \neq \mathbf{S}^*$. Then there is a minimal ordinal β such that there is an element x of V_α of ordinal rank β such that $\mathbf{S}(\{x\}) \neq \mathbf{S}^*(\{x\})$.

Now $B \models \mathbf{S}(\{\{x\}\}) = \{\mathbf{S}(\{x\})\} \neq \{\mathbf{S}^*(\{x\})\} = \mathbf{S}^*(\{\{x\}\})$, so if x is a counterexample, so is the object that the model B calls $\{x\}$ – which is the object that M calls $\{j(x)\}$, which has ordinal rank $j(\beta)+1$. It follows that $\beta \leq j(\beta)+1$. In fact, since $\beta = j(\beta)+1$ is impossible (consider the parity of the finite part of the ordinal β), it follows that $\beta < j(\beta)+1$, so $\beta \leq j(\beta)$.

Since $\beta \leq j(\beta)$ and $\beta < \alpha$, it is clear that $\beta \leq j(\beta) < j(\alpha)$, from which it follows that a counterexample must be a subset of $V_{j(\alpha)}$ so an element of $V_{j(\alpha)+1}$, so a B-set.

If x is a counterexample of minimal ordinal rank β, and $B \models y \in x$, then $M \models j(y) \in x$, so the ordinal rank of $j(y)$ is less than the ordinal rank of x: if the ordinal rank of y is γ, we have $j(\gamma) < \beta$, so we have $\gamma < j^{-1}(\beta) \leq \beta$. It follows that for any y such that $B \models y \in x$, we have $B \models \mathbf{S}(\{y\}) = \mathbf{S}^*(\{y\})$.

Now since x is a B-set, we have $B \models \mathbf{S}(\{x\}) = \{\mathbf{S}(\{y\} \mid y \in x\} = \{\mathbf{S}^*(\{y\} \mid y \in x\} = \mathbf{S}^*(\{x\})$, which contradicts our initial assumptions about

x. We conclude that there can be only one function witnessing the Axiom of Endmorphism in a Boffa model, from which it follows that the membership relation of the original model is definable in the Boffa model, which is what we set out to prove.

5 Further Remarks and Conclusions

It is worth noting that, while the notion of ordinal rank used in this argument is an M-notion, it is definable in B, and in fact we can formulate an assumption about B which is equivalent to the assertion that the universe of B is a rank of the cumulative hierarchy according to M. In any well-founded relation W, we can define in *NFU* as in ordinary set theory a notion of ordinal rank of elements of the domain of W. Further, we can say that an ordinal rank β in a relation W is *complete* if every subset of the collection of ordinals of rank $\leq \beta$ is the W-preimage of some element of the domain of W of rank $\leq \beta + 1$. Because of the way B is constructed, an assertion satisfied by B is "E is a well-founded relation and any ordinal rank in E is either complete or the entire domain of E". *NFU* with the Axiom of Endomorphism and this additional assertion proves that there is at most one function witnessing the truth of the Axiom of Endomorphism. It is much easier to formulate the argument as we have given it in terms of M's notion of rank, as we avoid technicalities about the way ordinals are defined in *NFU* and peculiar properties of the ordinals of *NFU*.

It is further worth remarking that, while it was already well-known that *NFU* + Infinity has precisely the consistency strength of Mac Lane set theory, the argument of this paper shows that there is a slight extension of *NFU* (*NFU* + Endomorphism + "E is a well-founded relation and any ordinal rank in E is either complete or the entire domain of E") which has a rather more intimate relationship of mutual interpretatibility with Mac Lane set theory (without Infinity) enhanced with an automorphism of the universe which moves a rank: we have seen above that we can derive a model of the extension of *NFU* from the extension of Mac Lane which defines all the concepts of Mac Lane set theory as restricted to its domain. It can further be noted that from the model of *NFU* we can recover an entire model of Mac Lane, not just a rank: the elements of the interpreted Mac Lane are pairs $\langle x, n\rangle$ where x is a set and n is a natural number, with the intention that $\langle x, n\rangle$ code $j^{-n}(x)$; we do not give the (easy) full details of this development here. The model of Mac Lane that is recovered is in effect truncated at the supremum of the ranks indexed by $j^{-n}(\alpha)$'s; this is of course not an ordinal of M, and it is important here that we are working in Mac Lane rather than Zermelo set theory, as unbounded quantifiers would be a problem in the latter context. There is another way to interpret Mac Lane set theory with an automorphism in *NFU* (or more accurately the theory of a rank in a model of Mac Lane moved downward by an automorphism): the objects of this interpretation are isomorphism classes of well-founded extensional relations with a top element. The methods used are derived from [6], and full details can be found in [8].

It is quite striking that in the usual models of *NFU*, the apparently featureless urelements are thus seen to be far from featureless: this seems far from obvious (it is relatively easy to show that the urelements of any model of *NFU are* homogeneous with respect to stratified formulas, though we do not give details here: one uses the permutation techniques adapted to *NF* by Dana Scott in [16] and discussed in the context of *NFU* in [2]). A predicate of an urelement u which distinguishes urelements from one another is easily described: $\emptyset\, E\, u$ is a very simple example (its full form in the language of *NFU* would be the expansion of "there is a function $\mathbf{S}$ witnessing the truth of the Axiom of Endomorphism such that $\emptyset \in \mathbf{S}(\{u\})$"). But this is not the main point of this result: the main point is that one has stronger interpretability of the ambient Mac Lane set theory in which a model of *NFU* is constructed by the Boffa procedure than one would expect.

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