The equivalence of *NF*-style set theories with "tangled" type theories; the construction of ω-models of predicative *NF* (and more)

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2 Abstract

An ω-model (a model in which all natural numbers are standard) of the predicative fragment of Quine's set theory "New Foundations" (*NF*) is constructed.

Marcel Crabbé has shown that a theory *NFI* extending predicative *NF* is consistent, and the model constructed is actually a model of *NFI* as well. The construction follows the construction of ω-models of *NFU* (*NF* with ur-elements) by R. B. Jensen, and, like the construction of Jensen for *NFU*, it can be used to construct α-models for any ordinal α. The construction proceeds via a model of a type theory of a peculiar kind; we first discuss such "tangled type theories" in general, exhibiting a "tangled type theory" (and also an extension of Zermelo set theory with Δ_0 comprehension) which is equiconsistent with *NF* (for which the consistency problem seems no easier than the corresponding problem for *NF* (still open)), and pointing out that "tangled type theory with ur-elements" has a quite natural interpretation, which seems to provide an explanation for the more natural behaviour of *NFU* relative to the other set theories of this kind, and can be seen anachronistically as underlying Jensen's consistency proof for *NFU*.

3 Introduction

We will present a class of type theories equivalent in consistency strength to *NF* (Quine's set theory "New Foundations", introduced in [9]; [2] is an all-purpose reference for *NF*) and the fragments of *NF* which are known to be consistent. We will apply this by giving an extension of a fragment of Zermelo set theory which is precisely equiconsistent with *NF* and by presenting a construction of ω-models of predicative *NF* (actually, of the extension *NFI* of predicative *NF* shown to be consistent by Marcel Crabbé in [1]), the construction being analogous to Jensen's construction of ω-models of *NFU* (*NF* with urelements) in [8].

NF is the first-order theory with equality and membership whose non-logical axioms are extensionality (objects with the same elements are the same) and the scheme of stratified comprehension (a formula ϕ is said to be "stratified" if it is possible to assign a non-negative integer type to each variable used in ϕ in such a way that it becomes a well-formed formula of the theory of types; the scheme of stratified comprehension asserts that $\{x \mid \phi\}$ exists for each stratified formula ϕ). The stratified comprehension scheme is equivalent to a finite collection of its instances (see [5]).

NF and related set theories differ most obviously from the usual set theory (*ZFC* and extensions) in asserting the existence of big sets like the universe; "$x = x$" is a stratified formula, so its extension, the universe, exists. "$x \notin x$" is not a stratified formula, so stratified comprehension does not give us the paradox of Russell. We call the universe V; we define the cardinal number $|A|$ of a set A as the set of all sets equinumerous with A; this definition is stratified, so every set, including V, has a cardinal number. We avoid Cantor's paradox of the largest cardinal because the usual form of Cantor's theorem, $|A| < |P(A)|$, is unstratified; the stratified form, a theorem of *NF*, is $|P_1(A)| < |P(A)|$, where

$P_1(A)$ is the set of one-element subsets of A. Since $|P_1(V)| < |P(V)| = |V|$ by Cantor's theorem, the map which takes each x to $\{x\}$ cannot exist in *NF*, which should not be too surprising, since its definition is not stratified. Ordinal numbers are defined as equivalence classes of well-orderings under similarity; the Burali-Forti paradox is evaded due to the fact that the similarity between the elements of an ordinal α and the set seg(α) of ordinals less than α with the natural order is proven in naive set theory by an induction on an unstratified condition, which fails in *NF*; there is an order type Ω of the set of all ordinals with the natural order, but seg(Ω) has an order type below Ω, which permits Ω to be less than the largest ordinals (there is no largest ordinal, of course).

Although the question of the consistency of *NF* remains open, the scheme of stratified comprehension is known to be consistent. In [8], Jensen demonstrated that *NFU*, *NF* with extensionality weakened to allow urelements, is consistent and has models in which the Axiom of Choice holds and the ordinals below α are standard for any fixed ordinal α. The solutions to the paradoxes above may seem odd, but they can all be seen to succeed in well-understood models of *NFU*. The problems with *NF* do not arise from the presence of "big" sets.

The difficulties with *NF* were first made clear when Specker proved in [10] that the Axiom of Choice is false in *NF* (which has the corollary that *NF* proves the "Axiom" of Infinity). Neither of these results holds for *NFU*. *NF* is not known to be any stronger than the theory of types with the axiom of infinity, but no headway has been made on constructing a model of the theory. Specker's disproof of the Axiom of Choice for *NF* cannot be carried out in any fragment of *NF* which is known to be consistent (but see the next paragraph).

The fragments *NFP* (predicative *NF*) and *NFI* were defined and shown to be consistent by Marcel Crabbé in [1]. In *NFI*, stratified comprehension is restricted to those instances "$\{x \mid \phi\}$ exists" in which an assignment of types to the variables of ϕ can be made in such a way that no variable is assigned a type higher than the type which would be assigned to $\{x \mid \phi\}$ itself (one type higher than the type assigned to x). In *NFP*, the further restriction is imposed that variables of the same relative type as the set being defined must be parameters. These theories are very weak; Con(*NFI*) is a theorem of third-order arithmetic, while Con(*NFP*) is a theorem of Peano arithmetic! The arithmetic of *NFI* is at least second-order arithmetic, while the arithmetic of *NFP* is at least bounded arithmetic with exponentiation. The author showed in [6] how to construct a model of an extension of *NFI* in which it is possible to interpret the theory of types with infinity; this result will be improved below. These theories have the curious feature that they prove the "Axiom" of Infinity in a way related to Specker's proof of the negation of Choice; the point is that if the axiom of Union is adjoined to either theory, one gets *NF*; if Union holds, then Specker's proof goes through, but if Union does not hold, there must be an infinite set, because finite sets can be shown to have unions.

Another fragment of *NF* which is known to be consistent is NF_3, in which comprehension is restricted to those formulae which can be stratified using no more than three types. This was shown to be consistent by Grishin in [4], where he also showed that $NF_4 = NF$.

Specker showed in [11] that the problem of constructing a model of NF is equivalent to the problem of constructing a model of the theory of types with an external bijection sending each type onto the next higher type and respecting membership (a "type-raising isomorphism"), which is in turn equivalent to the consistency with the theory of types of an axiom scheme of "ambiguity". If ϕ is a formula of the language of the theory of types, let ϕ^+ be the formula obtained by raising each type index in ϕ by one. It is obviously true that ϕ^+ is a theorem of the theory of types if ϕ is a theorem; the axiom scheme of ambiguity makes the stronger assertion "$\phi \iff \phi^+$" for each formula ϕ. The theory of types with the axiom scheme of ambiguity is equivalent in consistency strength to NF; it remains entirely unclear how to construct a model of this theory! There are precisely analogous results for the fragments of NF which are known to be consistent: NFU, NFP and NFI correspond to type theories TTU, TTP, and TTI, respectively, and are each equivalent in consistency strength to the corresponding type theory with the scheme of ambiguity. TTU weakens extensionality to allow urelements in each type; TTP is simply predicative type theory, in which the definition of a set cannot involve any set of higher type, and can only involve sets of the same type as parameters; TTI is the "mildly impredicative" type theory in which the only restriction on impredicative comprehension is that the definition of a set cannot mention sets of higher type. Similarly, if we define TT_n as type theory with n types, NF_n is consistent exactly if TT_n is consistent with the ambiguity scheme (for all formula ϕ for which ϕ^+ makes sense).

4 "Tangled" Type Theories

We refer to type theory hereafter as TT. For the sake of completeness, we give a formal definition of the theory TT: it is the many-sorted first-order theory with sorts indexed by the non-negative integers and primitive relations of equality and membership, in which "$x = y$" is well-formed iff the type of x is the same as the type of y and "$x \in y$" is well-formed iff the type of y is the successor of the type of x. Its axioms are extensionality (sets of the same positive type are equal if they have the same elements) and comprehension ($\{x \mid \phi\}$ exists, one type higher than x, for any formula ϕ). The theories TTU, TTP, and TTI are defined similarly.

We now define the theory TTT (tangled type theory, or tangled TT). TTT has the same sorts and primitive relations as TT. A formula "$x = y$" is well-formed iff x and y have the same type, as in TT, but a formula "$x \in y$" is well-formed iff the type of x is less than the type of y, a more general condition than in TT. If ϕ is a formula of TT and s is a strictly increasing function from non-negative integers to non-negative integers, define ϕ^s as the well-formed formula of TTT which results if each type index τ in ϕ is replaced by $s(\tau)$. The axioms of TTT are exactly the formulas ϕ^s for formulas ϕ which are axioms of TT.

It is necessary to pause and think about what this means. An object of type τ is understood not only as a set of objects of type $\tau - 1$, but also as a set of objects of type σ for each $\sigma < \tau$. Axioms of extensionality of TTT assert that an object of type τ is uniquely determined by *each* of its extensions (one extension for each type below τ): the precise form of the axioms (one for each pair of types $\sigma < \tau$) is

$$(\forall A^\tau)(\forall B^\tau)((\forall x^\sigma)(x^\sigma \in A^\tau \leftrightarrow x^\sigma \in B^\tau) \rightarrow A^\tau = B^\tau).$$

If A^τ and B^τ differ, they have different members with respect to *each* type $\sigma < \tau$. A predicative formula (without parameters of the type of the set being defined) of the form ϕ^s determines an extension $\{x|\phi\}$ in *each* type higher than the type of x; impredicativity commits one to specific higher types; formulas not of the form ϕ^s cannot generally be expected to have extensions. Each ascending sequence of types (each is determined by a function s) is a model of TT, if the type $s(\tau)$ is used to interpret type τ of TT for each τ.

Clearly, the model theory of TTT is unsatisfactory; it is not possible for all the "power sets" to be genuine! However, $TTTU$, the version of TTT in which extensionality is weakened to allow urelements in each type with respect to membership relative to each lower type, has a natural interpretation which we now present with an eye to motivation.

For each ordinal α, let $\mathrm{V}(\alpha)$ be stage α of the cumulative hierarchy in the usual set theory (*ZFC* or extensions). Our interpretation of $TTTU$ will have types indexed by general ordinals rather than by non-negative integers (note that any linearly ordered set could be used as the types of a version of tangled type theory, since a successor operation on the types is no longer needed. A compactness argument shows that a version of TTT with types indexed by a linearly ordered set is equiconsistent with TTT as long as the linearly ordered set is infinite.) Objects of type α will be the elements of $V(\alpha) \times \{\alpha\}$. Membership "$(x, \alpha) \in_{TTTU} (y, \beta)$" of an object of type α in an object of type β (for $\alpha < \beta$) is defined as "$x \in y$ and $y \in V(\alpha + 1)$". Whenever $\beta > \alpha + 1$, objects in $V(\beta) - V(\alpha + 1)$ are interpreted as urelements with respect to the membership relation of type α in type β. This interpretation of $TTTU$ is honest, in the sense that all the power sets involved are genuine; this is made possible because the power sets are padded with urelements. Below, we will give a more involved technique for modelling $TTTI$ (and so for modelling $TTTP$).

We will now prove

Theorem 1 *TTT is equiconsistent with NF.*

Proof: We will use Specker's result, described in the Introduction, that *NF* is equiconsistent with TT with the axiom scheme of ambiguity. The same argument can be adapted immediately to prove that $TTTU$ is equiconsistent with *NFU*, thus establishing the consistency of *NFU* relative to the usual set theory (actually, Con(NFU) is provable in arithmetic); the version of the argument for NFU is essentially Jensen's proof in [8].

A model of NF is easily converted to a model of TTT by letting the types of the model of TTT be disjoint copies of the model of NF labelled by non-negative integers and letting the membership relations between appropriate types be induced by the membership relation of NF in the obvious, trivial way. Thus, Con(NF) implies Con(TTT).

We show that Con(TTT) implies Con(TT + ambiguity), which implies Con(NF) by the results of Specker, completing our proof. Let Σ be a finite collection of formulae of TT using no type higher than $n-1$. With each collection A of n-element subsets of the set of non-negative integers, associate a strictly increasing map $s(A)$ which sends the numbers from 0 to $n-1$ to the elements of A. The collection Σ induces a partition of the n-element sets of non-negative integers into finitely many parts induced by the truth values of the formulae $\phi^{s(A)}$ for ϕ a formula in Σ and A an n-element set. By the Ramsey theorem, this partition has an infinite homogeneous set H; let h be the strictly increasing map which has the non-negative integers as its domain and H as its range. The model of TT obtained by interpreting type τ of TT as type $h(\tau)$ of our model of TTT satisfies ambiguity for formulae in Σ; compactness enables us to conclude that TT is consistent with the full ambiguity scheme. The proof of Theorem 1 is complete.

The problem of the consistency of NF thus reduces exactly to the problem of constructing models of TTT. However, models of TTT must be nonstandard, in the sense that the "power set" operations on types cannot be genuine. It is possible to define an alternative version of TTT, in which the types form a directed graph rather than a sequence, and in which "successor" types are unique and can be construed as genuine power sets in a version of Zermelo set theory. Ultimately, we show that NF is equiconsistent with this version of Zermelo set theory.

5 Untangling "Tangled" Type Theories: Type Theory on Graphs and MacLane Set Theory

Let G be a directed graph, with each element τ of the underlying set of G having a unique "parent" $p(\tau)$; elements of the underlying set of G will be called "nodes" of G, and nodes of G which have τ as their "parent" will be called "children" of τ; the arrows from children to parents are the edges of the graph. Such a graph is said to be "well-founded" if every set S of nodes of G has an element which has no children in S. It is said to have "no loops" (a weaker condition) if no node is an iterated image of itself under p.

The type theory on G, or TT_G, is the many sorted theory with equality and membership whose sorts are indexed by the nodes of G. "$x = y$" is a well-formed formula if the types of x and y are the same; "$x \in y$" is a well-formed formula if the type of y is the parent of the type of x. Thus, each type now has a unique "power set" type, its parent as a node. If ϕ is a formula of TT and τ is a node of G, let ϕ^τ be the formula of TT_G which results if each type index n in ϕ is replaced with $p^n(\tau)$, the n-th iterated parent of τ (with the 0th iterated parent being τ itself). The axioms of TT_G are exactly the formulae ϕ^τ for ϕ an axiom of TT.

TT_G has a natural interpretation in the usual set theory (*ZF* – remarks here apply to situations inconsistent with choice) whenever it is possible to find for each node τ in G a cardinal $\kappa(\tau)$ in such a way that $2^{\kappa(\tau)} = \kappa(p(\tau))$ for each τ. One can then interpret each type τ using a set of cardinality $\kappa(\tau)$ in a natural way; note that since type $p(\tau)$ has the cardinality of the power set of type τ in this interpretation, the "power set types" in this interpretation contain (representations of) all subsets of the types of which they are "power sets"; the difference from the usual situation is that several types may share the same "power set type". TT_G is consistent if it is possible to make an assignment of cardinals to each finite subgraph of G in this way, by a compactness argument (no proof mentions more than finitely many types = nodes of G), and it follows that any TT_G with no loops in G is consistent exactly if TT itself is consistent, while all TT_G's are consistent if *NF* is consistent. By a theorem of Forster (see [3] or [2], pp. 47-8), a TT_G with a natural interpretation must have G well-founded.

We now specify the particular kind of graph G we wish to use as our system of types. We define an "n-tree" as a graph G whose nodes are labelled with non-negative integers, with exactly one node being labelled with $n-1$, no node labelled with any higher integer, and a node labelled with i having exactly i children, one labelled with each integer less than i. Note that an n-tree is easily obtained from the $\in$-diagram of the von Neumann natural number n. We define an "$< \omega$-tree" as a countable graph with nodes labelled with non-negative integers, some unbounded ascending path in the graph, and each node labelled with i having exactly i children, one labelled with each integer less than i. Let G be an $< \omega$-tree; an example of an $< \omega$-tree (following a suggestion of Solovay) is the tree whose nodes are the ascending sequences of non-negative integers with cofinite range, with the label of a sequence s being $s(0)$ and the parent of a sequence s being the sequence $p(s)$ such that $p(s)(n) = s(n+1)$.

The theory we are interested in is TT_G for this G with an additional axiom scheme asserting that the truth value of each concrete formula of TT_G depends only on the non-negative integer labels of the types occuring in the formula. The proof of equiconsistency of this theory with *NF* is essentially the same as the proof of equiconsistency for TTT with *NF*. A model of *NF* clearly gives us a model of this theory (make one copy of the model of *NF* per node of G), and a set of sentences of TT induces a partition of the n-element sets of *labels* in this theory in the same way it induced a partition of the n-element sets of *types* in TTT. We refer to this theory hereafter as $TTTG$ (the graph version of tangled type theory).

We define MacLane set theory, *Mac* for short, as Zermelo set theory with the comprehension scheme restricted to those formulae in which every quantifier is restricted to a set (Δ_0 comprehension). Saunders MacLane described this theory in [7]. We are further interested in MacLane set theory with the foundation axiom weakened to allow objects which are their own sole elements, which we will call (Quine) atoms.

We call an n-tree an "n-tree of cardinals" if its nodes τ are labelled with cardinals $\kappa(\tau)$ in such a way that $\kappa(p(\tau)) = 2^{\kappa(\tau)}$ and with the property that the truth value of each concrete sentence of the language of TT in the natural model of an initial segment of the theory of types in which type i is represented by a set of cardinality $\kappa(p^i(\tau))$ for τ a node in the n-tree and $0 < i < j$ for some $j < n$ is determined solely by the identity of the sentence and the non-negative integer labels attached to the nodes $p^i(\tau)$ involved.

The assertion that a certain n-tree is an n-tree of cardinals is not expressible by a single formula in the language of *Mac*, but by an infinite collection of formulas, one for each concrete sentence of the language of TT. A stronger assertion can be expressed by quantifying over all Gödel numbers of formulas of TT as described internally to *Mac*; this is a stronger assertion because some of these "Gödel numbers" may be nonstandard. We refer to this latter assertion as asserting the existence of an n-tree of cardinals "internally" to *Mac*.

Definition: We define the theory *MacQ* as *Mac* (Zermelo set theory with comprehension restricted to Δ_0 formulas) with foundation weakened to allow Quine atoms and an axiom scheme asserting the existence of an n-tree of cardinals for each concrete natural number n (recall that this already requires an infinite collection of axioms for each individual n).

Theorem 2: *MacQ* is equiconsistent with *NF*.

Proof: The proof is analogous to the natural proof that TT itself is equiconsistent with *Mac* with Quine atoms (for which references seem to be hard to come by; Jensen refers to it in [8] as part of the folklore of the subject). Clearly, *MacQ* provides a model of the finite collection of types mentioned in any proof in $TTTG$, and so its consistency implies the consistency of $TTTG$. The idea of the proof that the consistency of $TTTG$ implies the consistency of *MacQ* is to choose the ascending path of types in G starting at a base type τ; identify each object of type τ with its singleton in type $p(\tau)$; once objects of type $p^i(\tau)$ have been identified with elements of type $p^{i+1}(\tau)$, identify objects of type $p^{i+1}(\tau)$ with their element-wise images under this identification in type $p^{i+2}(\tau)$. The direct limit formed by these identifications, with the membership relation determined by the membership relations among types in the sequence, is easily seen to be an interpretation of *Mac* with Quine atoms (one gets Δ_0 comprehension because each instance of comprehension can be interpreted as having all quantifiers restricted to the set associated with the highest type mentioned in its definition). Types not on the ascending path in G (i.e., not of the form $p^i(\tau)$) are realized via their sets of iterated singletons in types on the ascending path (we can assume w. l. o. g. that there were no nodes in G without iterated parent nodes on the ascending path; the part of a more general graph which was thus connected to the ascending path would be an $< \omega$-tree in its own right). Each type σ in G can be associated with a cardinal number, the cardinal $\kappa(\sigma)$ of the set of j-fold iterated singletons of objects of that type in some type $p^i(\tau) = p^j(\sigma)$ in the ascending sequence (all such cardinals are identified in the interpretation); it is straightforward to show that $\kappa(p(\sigma)) = 2^{\kappa(\sigma)}$. To get an n-tree of cardinals, take a type labelled with an integer $m \geq n$ on the ascending path in G; label an m-tree (which includes an n-tree) with the cardinals of the sets of iterated singletons in this type of all the types below this type, on or off the ascending sequence, in the obvious way, to obtain an m-tree of cardinals. The proof of Theorem 2 is complete.

6 Consequences for the Study of *NF*

The truly interesting thing about this result is that the issue of "big" sets like the universe has been completely eliminated; *MacQ* is a set theory of the familiar type (with a very strange combinatorial axiom scheme added). It should already have been apparent that the issue of "big" sets ceased to be the problem when *NFU* was shown to be consistent. Another feature of this approach to *NF* is that it gives us natural extensions and weakenings of (a theory equivalent to) *NF* to consider. There is an obvious hierarchy of possibly weaker theories TTT_n (TTT with n types) to consider; any theory of TTT is actually a theorem of one of these systems. We can consider axioms which assert the existence of n-trees of cardinals in a sense internal to the theory, and, in this context, we can generalize the concept of an n-tree to trees labelled with ordinals: an α-tree is a tree with nodes labelled with ordinals less than or equal to α, a node labelled with α, and in which a node labelled with β has exactly one child labelled with each ordinal below β and no other children. We can then consider MacLane set theory with Quine atoms and axioms asserting the existence of α-trees of cardinals for various ordinals α as inducing strengthenings of *NF* (this is interesting only if the α-trees are detectable inside the theory). Obviously, it is interesting to consider the effects of strengthening the MacLane set theory to full Zermelo set theory, *ZFC* or extensions thereof.

The combinatorial issues seen in *MacQ* are already well-known to those who work in *NF*. They are related to the properties of "Specker trees" (the construction of these is found in [2], p. 29, and results about it appear for example on pp. 47-8). The Specker tree of a cardinal κ (in any set theory) is the set of all iterated inverse images of κ under the map exp which sends each cardinal λ to 2^λ; the children of a cardinal which is a node of the tree are its inverse images under exp. In *NF*, exp sends the cardinality of $P_1(A)$ rather than the cardinality of A to the cardinality of $P(A)$, to preserve stratification. Specker's disproof of the Axiom of Choice and proof of the Axiom of Infinity, as well as other peculiar combinatorial results in *NF*, can be derived from the study of the Specker trees of "big" cardinals like that of the universe; most other work in *NF* can be duplicated in the unproblematic theory *NFU* (with suitable axioms of infinity). Observe that an n-tree of cardinals is a fragment of a Specker tree. It might appear that an n-tree of cardinals is easily constructed in *NF* for each concrete n: attach the cardinal $|P_1^{n-i}(V)|$ (the cardinality of the set of $(n-i)$-fold singletons) to each node labelled with i in the n-tree; in fact, an additional assumption such as Rosser's Axiom of Counting is needed for this to work (see [2], pp. 29-31 and later points, for a discussion of this axiom; thanks to Marcel Crabbé for correcting an error I made on this point). A theorem of Forster (see [3] or [2], pp. 47-8) implies, in *NF* and in the usual set theory (*ZF*) alike, that Specker trees must be well-founded, and so have an ordinal rank. The Axiom of Choice implies that the rank of a Specker tree must be finite (this easy result is proved in [3]); in the natural extension of *NF* afforded by Rosser's Axiom of Counting (which must hold if all natural numbers are standard), the rank of the Specker tree of $|V|$ is infinite. It is not known whether the existence of cardinals with Specker trees of infinite rank is consistent with *ZF* (obviously without Choice).

The effect of the equiconsistency result is to show that the Specker trees of the "big" cardinals (the cardinality of the universe of *NF* is interpreted using cardinals in n-trees) encode *all* of the peculiarities of *NF*. It also shows that *NF* is a very strange theory; the author has abandoned his earlier certainty that *NF* is consistent. The equiconsistency result should make it possible to investigate the consistency of *NF* in the more familiar environment of a set theory of the usual type without choice, which may help to solve this problem at last.

We present a proof that the Axiom of Choice fails in TTT, as our final contribution to the discussion of *NF* proper here. This proof is modelled on Specker's proof in *NF* proper. Suppose that the Axiom of Choice holds. Let n be the index of a sufficiently high type in TTT. Let μ be the smallest cardinal (in the same type $n+2$ as the cardinal of type n) which has only finitely many iterated images under exp (because the iterated images eventually become too large; in the MacLane set theory interpretation, this is the smallest cardinal which has an iterated image under exp which is greater than or equal to the cardinality of type n). Now consider what happens to cardinals when one shifts type in TT or TTT with the Axiom of Choice: each cardinal κ found in type n, for instance, corresponds to a cardinal which we will call $T\kappa$ in type $n+1$, namely, the cardinality of sets $P_1(A)$ for A with cardinality κ. A "new" cardinal in type $n+1$ must be strictly greater than all the cardinals $T\kappa$. Now observe that the smallest cardinal in type $n+3$ which has finitely many iterated images under exp must be $T\mu$; $T\mu$ has finitely many iterated images under exp which are less than or equal to the cardinality of type n, whose image under T in type $n+3$ is a preimage under exp of the cardinality of type $n+1$; it is easy to see that $T\mu$ has either one or two more iterated images under exp in type $n+3$ than μ does in type $n+2$, depending on whether the image under exp of the last iterated image of μ "projects" to the cardinality of type $n+1$ or to a cardinal between the cardinality of type n and the cardinality of type $n+1$ whose image under exp is the cardinality of type $n+1$. Thus, the number of iterated images under exp of the smallest cardinal with finitely many iterated images under exp increases by either 1 or 2 when one goes up one type. Now consider the situation in TTT when some types are skipped; we compare the sequence of types 0-12 with the sequence of types 0-3 followed by 9-12. exp is first definable as a map from type 3 to type 3 (it sends $|P_1(A)|$ to $|P(A)|$; one needs a type of elements of A, a type of A, a type of $P(A)$, and a type for $|P(A)|$; thus, the number we are interested in (the number of iterated images of μ under exp) is first defined for μ in type 3. The value of this number is not affected by what type is taken to be next above 3 (whatever type it is, it will see the same exp map in type 3), so it is the same in both type sequences. In the type sequence 0,1,2,3,4,5,6,7,8,9,10,11,12 the number for μ in type 12 exceeds the number for μ in type 3 by between 9 and 18; in the sequence 0,1,2,3,9,10,11,12 the number for type 12 exceeds the number for type 3 by between 4 and 8. But the numbers for type 3 and type 12 must be the same relative to both sequences; the number of interest for type n in a type sequence is completely determined by the three predecessors of n in the sequence. To avoid any problems which might arise from the possibility that the numbers might be nonstandard, consider the residues of the numbers mod 19, which are standard objects; the same problem is seen. The failure of choice occurs in TTT for essentially the same reasons it occurs in *NF*. In *MacQ*, the argument would involve cardinals in a 12-tree, and the sequences above would be sequences of labels on types rather than sequences of types themselves; the reference to residues mod 19 would be essential, since the numbers of interest for the two sequences would have the same elementary properties, but would not need to be equal.

Marcel Crabbé informs me that the number of interest can go up by 2 instead of 1 *only once*, and so the number of types in the above argument can be reduced somewhat.

7 The Construction of ω–models

We show that the theory *NFI* introduced by Marcel Crabbé in [1] has models in which all natural numbers are standard, using methods analogous to those used by Jensen in constructing ω–models of *NFU* in [8]. *NFI* is stronger than predicative *NF* (the theory *NFP* also introduced by Crabbé in [1]), thus the title. Just as was the case with Jensen's results of [8], we will be able to state our result in a more general form:

Theorem 3: For any ordinal α, there is a model of *NFI* which contains a well-ordering of order type α.

The proof of this theorem occupies the rest of the section and the paper.

NFI is the first-order theory with membership and equality whose axioms are extensionality and the instances of comprehension "$\{x \mid \phi\}$ exists" in which the formula ϕ is stratified and an assignment of types can be made to the variables in ϕ in such a way that no variable is assigned a type higher than the intended type of $\{x \mid \phi\}$ (one higher than the type assigned to x). The theory *NFP* (predicative *NF*) has the further restriction that variables assigned the exact type of $\{x \mid \phi\}$ cannot be bound in ϕ. *NFI* augmented with the axiom of union is exactly *NF* (for the easy proof, see [1] – the basic idea is that the set of n-fold singletons of objects x such that ϕ is predicatively defined for large enough n, and n applications of set union to this set give $\{x \mid \phi\}$). *NFI* interprets second-order arithmetic; the only hard part of showing this is proving that *NFI* proves Infinity; this is done by observing that Infinity is clearly true if Union is false (unions of finite sets can be shown to exist by induction) while if Union is true one can use the proof of Infinity in *NF* as a corollary of the result of Specker (in [10]) that the axiom of choice is false in *NF*. The axiom of choice is consistent with *NFI*, but clearly implies the existence of sets without unions.

In [1], Crabbe showed the consistency of *NFI* using techniques which could be emulated in third-order arithmetic; *NFI* is very weak, and Crabbe's methods did not show how to model strong extensions of *NFI*. The author showed in [6] how to model an extension of *NFI* as strong as the Theory of Types with Infinity. The theory of ω-models of *NFI* is even stronger (we will not try to determine exactly how strong; we suspect that *NFI* + "all natural numbers are standard" has the same strength as *NFU* + "all natural numbers are standard").

We begin the construction. Choose an ordinal α. Let κ be a strong limit cardinal of cofinality greater than $2^{|\alpha|}$ (where $|\alpha|$ is the cardinality of α).

TTI is the subtheory of the Theory of Types in which instances of comprehension "$\{x \mid \phi\}$ exists" are restricted to those in which no type higher than that of $\{x \mid \phi\}$ is mentioned. We define a further extension of *TTI*: let β be an ordinal; "tangled *TTI* of rank β" is the many sorted theory with sorts indexed by the ordinals $\gamma < \beta$ with equality in each type and membership relations $\in_{\gamma,\delta}$ for each pair of ordinals $\gamma < \delta < \beta$, "$x \in_{\gamma,\delta}$ y" being a well-formed formula precisely if x is of type γ and y is of type δ. The axioms of "tangled *TTI*" are simply the axioms of *TTI* for each ascending sequence of types.

We proceed to build a model of tangled *TTI* of rank κ. Type β will be the set $\kappa \times \{\beta\}$, for each $\beta < \kappa$; that is, the objects of type β will be pairs (γ, β) for $\gamma < \kappa$. This suffices to define type 0. When all types with index less than $\beta > 0$ have been constructed, we indicate how to construct type β. We assume that an ordered pair is defined in each type below β, that relations $\in_{\delta,\gamma}$ have been defined for $\delta < \gamma < \beta$, and that relations $R_{\alpha,\delta}$ realizing the natural order on the first coordinate of the elements (γ, δ) of type δ with $\gamma < \alpha$ have been provided for in each type ϵ whenever $\delta < \epsilon < \beta$. We *temporarily* interpret type β as the power set of κ, defining $(\gamma, \delta) \in_{\delta,\beta} A$ as $\gamma \in A$ for each $\gamma < \kappa$, $\delta < \beta$. Note that the temporary "type β" is too large. The pair in type β is interpreted using an arbitrarily chosen bijection between the power set of κ and the Cartesian product of the power set with itself. The relations $R_{\alpha,\delta}$ of type β for each $\delta < \beta$ are clearly provided for. We then augment the language of tangled TTI + pairs in each type + $R_{\alpha,\delta}$'s in each type + a constant for each object of type below β with Skolem functions: for each formula ϕ in variables y, $x_1, \ldots, x_n$, we provide a function symbol $S_{\phi,y}$ (and associate with it an actual function on the model) such that ϕ holds with $y = S_{\phi,y}(x_1, \ldots, x_n)$ exactly if "for some y, ϕ" holds (with the given values of the parameters). Iterate this process until every existential fact expressible in the language we are using, including facts expressed using Skolem functions, is witnessed by a term; the set of all objects of type β represented by terms in this language is a set of cardinality κ (the size of the language) and can be used as type β instead of the full power set of κ without changing any facts expressible in our language; a bijection between this set and the set $\kappa \times \{\beta\}$ is used to induce the final interpretation of the latter as type β. It is easy to see that the types up to type β satisfy the axioms of TTI in which no type higher than β appears, using the inductive hypothesis that this already held for each ordinal less than β.

We now indicate how to use each formula ϕ of non-tangled TTI with integer types, pairs in each type, constants $\beta < \alpha$ and relations R_α in each type to define a set of finite subsets of κ (in its role as the set of type indices): let $[m, n]$ be the interval of types used in ϕ; each set of $n - m + 1$ ordinals will belong to the set associated with ϕ if the formula is true in our model of tangled TTI when the ordinals are used in ascending order to replace the type indices in ϕ (with appropriate substitutions of membership relations). We augment the language of non-tangled TTI with constants $\beta < \alpha$ (intended to represent the objects (β, γ) in each type γ) and the relation R_α in each type, representing the relation $R_{\alpha,\delta}$, where δ is the next lower type in the sequence of types being used. In addition, we augment the language with Skolem functions, so that every existential fact will actually be witnessed by a term (Skolem functions are not functions in an "internal" sense; they do not augment the possibilities for defining sets). We supply these Skolem functions with an interpretation by taking our model of tangled TTI with the language of tangled TTI + pairs + $R_{\alpha,\delta}$'s + the constants (β, γ) for each $\beta < \alpha$ in each type γ, and augmenting this language with Skolem function symbols, associating each symbol with an actual function on the model. We will then be able to interpret each Skolem function symbol in the extended language for non-tangled TTI with integer types when we are given the sequence of ordinal type indices to be used to interpret the integer type indices in the formulae involved.

We will now construct a set A of sentences of the language of non-tangled TTI with integer types (we admit negative integers as types, an idea originally found in [12]; a compactness argument shows that this does not strengthen the theory), as extended in the previous paragraph, which is consistent, complete, and "typically ambiguous", meaning that any sentence in A is unaffected in truth-value if all type indices in the sentence are raised or lowered by a constant amount. Observe that the cardinality of the extended language for non-tangled TTI is $|\alpha|$, the cardinality of α. Consider the set of all sentences of the language of non-tangled TTI which involve n types. We can partition $n[\kappa]$, the set of all subsets of κ of cardinality n, into at most $2^{|\alpha|}$ subsets by considering the truth values of each of the $|\alpha|$ sentences with n types when the element of $n[\kappa]$ is used to interpret the n types in each sentence. By an extension of the Erdös-Rado partition theorem (see [8]; he uses the same version, and provides a proof), we can find a homogenous set H for this partition of cardinality λ for each $\lambda < \kappa$ (recall that κ is a strong limit cardinal greater than $|\alpha|$). Each such homogenous set determines a complete, consistent, typically ambiguous theory for the sentences with n types, namely, the set of sentences of TTI true for each sequence of n types taken from H. By considering the fact that there are no more than $2^{|\alpha|}$ such theories, and choosing a theory realized in a homogeneous set of cardinality λ for each $\lambda < \kappa$, we see that some such theory must be realizable in homogenous sets of each cardinality less than κ (recall that $\mathrm{cf}(\kappa) > 2^{|\alpha|}$). Call a typical such theory A_n. We show how to extend A_n to a theory A_{n+1} with the same properties relative to the larger index. If we choose $\lambda < \kappa$ large enough, a homogeneous set H of cardinality λ which realizes the theory A_n will itself have a homogeneous set H_2 with respect to the partition determined by the sentences of TTI with $n+1$ types. We can choose λ large enough so that the cardinality of H_2 can in turn be as close to κ as desired, and there must be some theory A_{n+1} extending A_n which is in turn realized in homogeneous sets of arbitrarily large cardinality below κ. A nested sequence of such theories A_n has as its union the desired theory A.

The theory A determines its own term model of TTI, using its Skolem functions. This model of TTI is not merely ambiguous, but has a type-shifting automorphism (each term has an analogous term produced by raising or lowering types uniformly, of which analogous sentences are true; the use of negative integers as types makes the map one-to-one); identification of the analogous objects in each type produces a model of *NFI*, as discussed by Specker in [4] (for *NF* rather than *NFI*). It should be clear that an object is in the domain of R_α (in each type, before the collapse to *NFI*) exactly if it is one of the objects represented by ordinals below α, since this is true in the underlying model of tangled TTI and the extended language for non-tangled TTI is strong enough to capture the fact, so the model of *NFI* will contain a well-ordering of type α, as originally required. Thus, we have proved Theorem 3 above.

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