Department of Philosophy

University of Toronto 215 Huron Street 9th Floor Toronto Ontario Canada M5S 1A1

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Dr. T.E. Foster
Department of Pure Mathematics and Mathematical Statistics
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge
ENGLAND
CB3 0WB

Dear Thomas:

Here's a copy of Firestone's thesis. The original is quite a remarkable document, a carbon copy with hand-written symbols. I'm surprised once more by how formal set theory was in those days. I think the free-wheeling style of the present day only came in later, in the 1960s. I was interested by Firestone's scepticism about Gödel's claims (p. 5).

I've included a copy of the signature sheet at the beginning of the thesis. The first two signatures are those of Steven Orey and George E. Collins – the latter became famous later for his work on quantifier elimination and cylindric algebraic decomposition.

With best regards,

Alasdair

Alasdair Urquhart
urquhart@cs.toronto.edu

SUFFICIENT CONDITIONS FOR THE MODELLING OF AXIOMATIC SET THEORY

A Thesis

Presented to the Faculty of the Graduate School of Cornell University for the degree of

DOCTOR OF PHILOSOPHY

By

Clifford Dixon Firestone

September, 1947

BIOGRAPHY

C. D. Firestone was born May 27, 1921, in El Paso, Texas. He received the degree of Bachelor of Science from the University of New Mexico in 1941. In 1941 he began graduate work in mathematics at Cornell University. Since 1942, except for an interruption from 1944 to 1946, he has been a graduate student and assistant in the Department of Mathematics at Cornell University.

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Introduction

We say that a logic (formal logic, formal language) L_1 <u>is consistent relative to</u> a logic L_2, provided that if a contradiction can be derived in L_1 then a contradiction can be derived in L_2. We say that a logic L_1 <u>is modelled in</u> a logic L_2 if there has been defined a one-to-one correspondence between the propositions of L_1 and a subclass of the propositions of L_2 such that:

1) The correspondence is effectively defined; i.e., given a proposition of L_1, explicit directions are available which enable one to construct, in a finite number of steps, the unique correspondent in L_2 of the given proposition.

2) The correspondence preserves demonstrability; i.e., given a demonstration, according to the rules of L_1, of a proposition p of L_1, explicit directions are available which enable one to construct, in a finite number of steps, a demonstration, according to the rules of L_2, of the correspondent in L_2 of p.

3) The correspondence preserves contradictions; i.e., if a proposition p of L_1 is a contradiction in L_1, then the correspondent in L_2 of p is a contradiction in L_2.

If a logic L_1 is modelled in a logic L_2, we shall call the class of correspondents in L_2 of the propositions of L_1 a <u>model</u> of L_1 in L_2. Obviously if there is a model of L_1 in L_2, then L_1 is consistent relative to L_2.

In [5],[1] Gödel took L_2 to be the system of axiomatic set theory (with an axiom of infinity but not an axiom of choice) and L_1 to be L_2 plus an axiom from which he derived the axiom of choice, the generalized continuum hypothesis, and other theorems. The proof that L_1 is consistent relative to L_2 was then effected by setting up a model of L_1 in L_2. In constructing the model of L_1, Gödel made extensive use of the resources of L_2, and in particular of theorems of L_2 which depend on the distinction between sets and classes.

We shall show that a model of axiomatic set theory can be set up in any logic which contains the lower functional calculus plus a certain minimum of the theory of classes and relations, and of transfinite ordinal theory. This implies that set theory is consistent relative to any system having these minimum resources (which will be specified in Part I, §2). Further, it will be seen that set theory itself has these minimum resources, so that while our model differs considerably from that of [5], it can nevertheless be set up within axiomatic set theory.

Part I of this paper will be devoted to specifying the necessary properties of L_2, and to the construction of the model.

In Part II we shall be concerned with the problem of

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1. Numbers in brackets refer to the bibliography.

determining whether the simple theory of types, or a suitable modification thereof, is adequate for the construction of the model of Part I. To this end a considerable development of certain parts of ordinal theory is given. In this development it is assumed that the reader is familiar with the basic properties of ordinals as defined in the theory of types. Furthermore, the ordinal theory developed is specifically directed toward the theorems which are needed in Part I, so that while it is of interest in itself, it does not constitute a systematic or complete development of ordinal theory.

We are unable to come to a definite conclusion in Part II as to whether the simple theory of types is adequate for the construction of the model of Part I. It appears, however, that in order to set up this model, one will have to add to the theory of types axioms from which two rather special results about ordinals can be derived. While one could add these special results themselves as axioms, they completely lack that character of "intuitive self-evidence" (whatever this means) which seems to be considered desirable in the axioms of systems of logic.

We proceed to show, therefore, that the two results needed can be derived from two well known propositions of classical ordinal theory, and we consider the addition to the theory of types of these propositions as axioms. Unfortunately, while one of these propositions is a weak form of the axiom of choice, the other contradicts strong forms of the axiom of choice. This is a convincing demonstration of the confusion which exists in classical ordinal theory, but it is hardly satisfying as far as the question of relative consistency of set theory with the theory of types is concerned.

Thus, we are unable to show that set theory is consistent relative to the simple theory of types. There remain then three possibilities:

1) A different method of modelling might require less ordinal theory, so that a model of set theory could be set up in the theory of types, augmented perhaps by one or more "acceptable" axioms.

2) It may be possible to prove the two results we need in the theory of types, using perhaps additional axioms which are in some sense "acceptable", and which do not contradict the axiom of choice. This seems rather unlikely.

3) We have perhaps done the best that can be done, and the special theorems which we need are actually necessary if set theory is to be modelled in the theory of types. There seems to be no reason to believe that this is true.

In any case, it should be pointed out that there is as much evidence that the axioms which we suggest adding to the theory of types are consistent with the other axioms of that system, as there is that the axiom of choice is consistent with the axioms of that system—namely, no evidence in either case. Gödel has stated[2] (but has not published a proof) that the axiom of choice is consistent with the ramified theory of types, but there has apparently been no corresponding investigation of the simple theory of types. In view of the results obtained in this paper, it is difficult to see how a proof can be carried out for either system along the lines used by Gödel in [5]. Some clarification of this matter by Gödel would be most welcome.

Part I

§1. The Logic L_1. (Axiomatic Set Theory.) In this section we state the primitive symbols, basic definitions, and axioms of the logic commonly known as axiomatic set theory. The system here defined is an attempt at a formalization of the system Σ described intuitively by Gödel in [5]. Since we are interested only in reproducing formally as close an approximation to the system Σ as possible, we neglect various opportunities for removal of redundancies which exist both in Σ and the system we define.

The primitive symbols of L_1 are the following constants: $|$, $($, $)$, ε, $\{$, $\}$.

Variables: x', x'', x''', ... (in alphabetical order).

Definition of noun and proposition.

2. See [3] and [4].

1) If x is a variable, then x is a noun.

2) If x and y are nouns, then ($x\varepsilon y$) is a proposition, and $\{xy\}$ is a noun.

3) If p and q are propositions and x is a variable, then $(p|q)$ and $((x)p)$ are propositions.

x, y, z, u, v, w are used as syntactical variables whose values are variables, and p and q as syntactical variables whose values are propositions.

We define "free occurrence of a variable", "bound occurrence of a variable", and "x bound part of a formula" as in [1]. The rules of inference of L_1 are the usual finitist rules (e.g., those of the system F^1 of [1]). The syntactical notation "$\vdash_1$" is defined as in [1]. The symbol "=df" between two formulas means that the formula on the left is an abbreviation of the formula on the right. The symbols $\sim, \vee, \supset, \equiv$ are defined in the usual manner. Parentheses will be omitted or replaced by dots as in Principia Mathematica.

Definitions of L_1.

DS1. $\langle xy\rangle$ = df $\{\{xx\}\{xy\}\}$

$\langle xyz\rangle$ = df$\langle x\langle yz\rangle\rangle$

etc.

DS2. $(Ex)p$ = df $\sim(x)\sim p$.

DS3. $M(x)$ = df$(Ey).x\varepsilon y$.

DS4. $(x)_s p$ = df$(x).M(x)\supset p$.

DS5. $(Ex)_s p = df \sim (x)_s \sim p.$

DS6. $x \subseteq y = df(z)_s . z\varepsilon x \supset z\varepsilon y.$

DS7. $x=y = df(z)_s . z\varepsilon x \equiv z\varepsilon y.$

DS8. $x \neq y = df \sim (x=y).$

DS9. $Un(x) = df(u,v,w)_s : \langle uv\rangle \varepsilon x . \langle wv\rangle \varepsilon x . \supset . u=w.$

Axioms of L_1.

$P_1, P_2, \cdots, P_n$. Axioms for the propositional calculus and the lower functional calculus (e.g., the axioms of the system F^1 of [1]).

A1. $(x,y,z) : x=y . \supset . x\varepsilon z \supset y\varepsilon z.$

A2. $(x,y,z)_s : z\varepsilon \{xy\} . \equiv . z=x \vee z=y.$

A3. $(x,y)_s (Ez)_s (u)_s : u\varepsilon z . \equiv . u=x \vee u=y.$

B1. $(Ez)(x,y)_s . \langle xy\rangle \varepsilon z \equiv x\varepsilon y.$

B2. $(x,y)(Ez)(u)_s : u\varepsilon z . \equiv . u\varepsilon x . u\varepsilon y.$

B3. $(x)(Ey)(u)_s . u\varepsilon y \equiv u \sim \varepsilon x.$

B4. $(x)(Ey(u)_s : u\varepsilon y . \equiv . (Ez)_s . \langle zu\rangle \varepsilon x.$

B5. $(x)(Ey)(u,v)_s . \langle vu\rangle \varepsilon y \equiv u\varepsilon x.$

B6. $(x)(Ey)(u,v)_s . \langle uv\rangle \varepsilon y \equiv \langle vu\rangle \varepsilon x.$

B7. $(x)(Ey)(u,v,w)_s . \langle uvw\rangle \varepsilon y \equiv \langle vwu\rangle \varepsilon x.$

B8. $(x)(Ey)(u,v,w)_s . \langle uvw\rangle \varepsilon y \equiv \langle uwv\rangle \varepsilon x.$

C1. $(Eu)_s :. (Ez)_s . z\varepsilon u . (x)_s : x\varepsilon u . \supset . (Ey)_s . y\varepsilon u . x \subseteq y . x \neq y.$

C2. $(x)_s (Ey)_s (u,v)_s : u\varepsilon v . v\varepsilon x . \supset . u\varepsilon y.$

C3. $(x)_s (z) :. Un(z) : \supset : (Ey)_s (u)_s : u\varepsilon y . \equiv . (Ev)_s . v\varepsilon x . \langle uv\rangle \varepsilon z.$

C4. $(x)_s (Ey)_s (u)_s . u \subseteq x \supset u\varepsilon y.$

D1. $(x):.(Ey)_s.y\varepsilon x:\supset:(Eu)_s:u\varepsilon x.\sim(Ez)_s.z\varepsilon u.z\varepsilon x.$

From the above axioms one can prove, by methods similar to those used in [5], the following theorem:

If

1) p is a proposition;

2) all the bound variables of p occur only in parts of the form $(z)_s q$;

3) all the free variables of p are contained in the set $y_1, y_2, \cdots, y_m, x_1, x_2, \cdots, x_n$;

4) u does not occur in this set: then

$$\vdash_1 (y_1, y_2, \cdots, y_m)(Eu)(x_1, x_2, \cdots, x_n)_s . \langle x_1 x_2 \cdots x_n \rangle \varepsilon u \equiv p.$$

Weak forms of the above theorem are stated in [5] on p. 8 and p. 14.

A1 does not appear in [5], but has been shown by A Robinsohn to be necessary.[3] A_2 is stated in [5] as a definition. (It is not clear, however, what is meant by the word "definition" in [5].) In [5], $(x)_s.Cls(x)$ (every set is a class) appears as an axiom, but is superfluous. DS3 appears in [5] as an axiom.

We note, for future reference, that L_1 has essentially two universes: first, the class of all <u>sets</u> (i.e., the class of all x such that M(x)); second, the "class" of all <u>classes</u>, whether sets or not. This latter "class" is not, strictly

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3. [9].

speaking, a class in L_1, since the definition of M(x) is such that any member of a class is automatically a set. Thus, while we may define a class for the universe of sets, the universe of classes can be defined only by means of a statement.

It may be of interest that Gödel's proof of the consistency of the axiom of choice with L_1 can be carried out if Cl is replaced by (Ex).M(x), but not if Cl is omitted without replacement by some axiom asserting the existence of at least one set. Also, Gödel's proof is easily carried out without use of Dl, and Dl can then be proved for the model defined in [5], thus giving a simple consistency proof for this axiom.

§2. <u>The Logic</u> L_2. We do not attempt to define L_2 with complete precision, since this would be possible only if we restricted L_2 to be a particular system of logic, and we wish our results to be valid at least for a modification of the simple theory of types and for L_1. We shall, instead, state for L_2 certain general requirements which are satisfied by most systems of logic, list expressions assumed to be definable in L_2, and list theorems assumed to be provable in L_2. It will follow from results obtained in [5] that L_2 can be taken to be L_1, and it will be shown in Part II that L_2 can be taken to be the simple theory of types plus appropriate additional axioms (p. 64).

We assume that among the symbols, primitive or defined, of L_2 are the constants (,), ε, | and an infinite set of variables, possibly arranged in types. We assume that any formula constructed from these symbols and variables which is a proposition of type theory is a proposition of L_2.

We assume that the expressions $(x)p$, $(Ex)p$, $(E_1x)p$, $\sim p$, $p \vee q$, $p \cdot q$, $p \supset q$, $p \equiv q$ have been defined in the usual manner, and that L_2 contains the propositional calculus and the lower functional calculus, together with the usual finitist rules of inference. We define the notation "$\vdash$" analogously to the notation "$\vdash_1$" of L_1.

We define free and bound variables as in L_1, and the notation $\{S\,x\,p\}\,(y)$ and the phrase "confusion of bound variables in $\{S\,x\,p\}\,(y)$" as in [10]. However, if $p(x)$ denotes a proposition, we shall write $p(y)$ for $\{S\,x\,p\}\,(y)$ when convenient. Syntactical variables are defined for L_2 as for L_1, but we also use $P, Q, R, \ldots, \alpha, \beta, \gamma, \ldots$ as syntactical variables whose values are variables.

We assume that a symbol ,=, has been defined in L_2 such that

T1. $\vdash (x,y) : x=y . \equiv . (z) . z \varepsilon x \equiv z \varepsilon y$.

T2. If there is no confusion of bound variables in $\{S x\ p(x)\}\,(y)$, then $\vdash (x,y) : p(x) . x=y . \supset . p(y)$.

We assume that L_2 contains <u>abstractions,</u> $\hat{x}p$, and <u>descriptions,</u> ιxp, such that whenever p is a proposition of the theory of types then T3, T4, and T5 are provable.

T3. If there is no confusion of bound variables in $\{S x p\}(y)$, then

$$\vdash (y):.(Ez).y \varepsilon z : \supset : y \varepsilon \hat{x}p . \equiv . \{S x p\}(y).$$

T3 is valid in both the theory of types and L_1. It will be noted that in every case (in Part I) where we assert $y \varepsilon \hat{x}p . \equiv . p(y)$, p is a proposition of type theory, and one can prove $(Ez).y \varepsilon z$ in L_1.

T4. If neither z nor v occurs free in p and there is no confusion of bound variables in $\{Syp\}(\iota xp)$ or $\{Syp\}(v)$, then $\vdash (Ez)(v) : v=z . \equiv . \{Syp\}(v) :. \supset :. \{Syp\}(\iota xp)$.

T5. If there is no confusion of bound variables in $\{Sxp\}(\hat{y}q)$, then

$$\vdash (x)p . \supset . \{Sxp\}(\hat{y}q).$$

$$\vdash (x)p . \supset . \{Sxp\}(\iota yq).$$

The second part of T5 is frequently stated as $\vdash (x)p . (E_1 y)q . \supset . \{Sxp\}(\iota yq)$. However, from this one can obtain the theorem in T5 by taking ιyq to be anything convenient (for example, 0) in those cases where $\sim (E_1 y)q$. See [8], §27.

We assume that the ordered pair of x and y, $\langle x,y \rangle$, has been defined in such a way that the type (if this is meaningful) of $\langle x,y \rangle$ is the same as the type of x and y,[4] and such that:

$$(x,y,u,v) : \langle x,y \rangle = \langle u,v \rangle . \equiv . x=u . y=v.$$

The comma in $\langle x,y \rangle$ will be omitted whenever this

4. See [7].

omission causes no ambiguity.

The following are definitions of L_2:

D1. $\langle xyz\rangle =_{df} \langle\langle xy\rangle z\rangle$.

D2. $rel =_{df} \hat{R}((x): x\varepsilon R . \supset . (Eu,v) . x=\langle uv\rangle)$.

D3. $Sv =_{df} \hat{R}((u,v,w): \langle uv\rangle\varepsilon R . \langle uw\rangle\varepsilon R . \supset . v=w)$.

D4. $x\cup y =_{df} \hat{z}(z\varepsilon x \vee z\varepsilon y)$.

D5. $x\cap y =_{df} \hat{z}(z\varepsilon x . z\varepsilon y)$.

D6. $\overline{x} =_{df} z(z \sim\varepsilon x)$.

$x-y =_{df} x\cap\overline{y}$.

D7. $Fnc =_{df} rel \cap Sv$.

D8. $Arg(R) =_{df} \hat{z}((Ey) . \langle zy\rangle\varepsilon R)$.

D9. $Val(R) =_{df} \hat{z}((Ey) . \langle yz\rangle\varepsilon R)$.

D10. $C(R) =_{df} Arg(R) \cup Val(R)$.

D11. $U(x) =_{df} \hat{z}(z=x)$.

D12. $\hat{x}\,\hat{y}\; p =_{df} \hat{u}((Ex,y) . u=\langle xy\rangle . p)$.

D13. $Can(x) =_{df} \hat{z}((Ey) . y\varepsilon x . z=U(y))$

$Can^2(x) =_{df} Can(Can(x))$.

D14. $r\,Can(R) =_{df} \hat{u}\,\hat{v}\,((Ez,w) . \langle zw\rangle\varepsilon R . u=U(z) . v=U(w))$

$r\,Can^2(R) =_{df} r\,Can(r\,Can(R))$.

D15. $R^{-1} =_{df} \hat{u}\,\hat{v}\,(\langle vu\rangle\varepsilon R)$

$\breve{R} =_{df} R^{-1}$.

D16. $\Lambda =_{df} \hat{x}(x\neq x)$.

D17. $V =_{df} \hat{x}(x=x)$.

D18. $R^{\text{‘}}x =_{df} \iota z((E_1 y) . \langle xy\rangle\varepsilon R . \langle xz\rangle\varepsilon R : \vee : z=V . \sim(E_1 y) . \langle xy\rangle\varepsilon R)$.

D19. $1\text{-}1 =_{df} \hat{R}(R\varepsilon Fnc . R^{-1}\varepsilon Fnc)$.

D20. $xRy =_{df} \langle xy\rangle\varepsilon R$.

D21. $x\ sm_R y =df\ R \epsilon 1\text{-}1 . Arg(R)=x . Val(R)=y.$

D22. $x\ sm\ y =df\ (ER) . x\,sm_R y.$

D23. $trans =df\ \hat{R}((x,y,z) : xRy . yRz . \supset . xRz).$

D24. $antisym =df\ \hat{R}((x,y) : xRy . yRx . \supset . x=y).$

D25. $connex =df\ \hat{R}((x,y) : x,y \epsilon C(R) . \supset . xRy \vee yRx).$

D26. $ref =df\ \hat{R}((x) . x \epsilon C(R) \supset xRx).$

D27. $ser =df\ trans \cap antisym \cap connex \cap ref \cap rel.$

D28. $min_R u =df\ \iota z((x) :. x \epsilon u \cap C(R) . (y) . y \epsilon u \cap C(R) \supset$
$xRy : \equiv : x=z :: \vee :: z=V . \sim (E_1 x) : x \epsilon u \cap C(R) . (y) . y \epsilon u \cap C(R) \supset xRy).$

D29. $bord =df\ \hat{R}((u) : u \neq \Lambda . u \subseteq C(R) . \supset . (Ey) . y \epsilon u . y = min_R u).$

D30. $\Omega =df\ bord \cap ser.$

D30.1. $P\,smor_R Q =df\ R \epsilon 1\text{-}1 . Arg(R)=C(P) . Val(R)=C(Q).$
$(x,y) . xPy \supset R\text{‘}xQR\text{‘}y . xQy \supset \breve{R}\text{‘}xP\breve{R}\text{‘}y.$

D31. $P\,smor\ Q =df\ (ER) . P\,smor_R Q.$

D32. $max_R u =df\ \iota z(z \epsilon u \cap C(R) . (x) . x \epsilon u \cap C(R) \supset xRz).$

D33. $x \upharpoonleft P =df\ \hat{z}((Eu,v) . z = \langle uv \rangle . u \epsilon x . z \epsilon P).$

Most of the above symbols are more or less standard notation in logic. For those that are not we give the following intuitive explanation.

rel is the class of relations, Sv the class of single valued classes, Fnc the class of functions (single valued relations). Arg(R) and Val(R) are the classes of arguments and values respectively of R. Note that the arguments of R occur as the first element of ordered pairs in R, and the values as the second element. (Just the reverse holds in L_1.)

Can(x) is the "Cantorian" of x, the class of unit classes of members of x. r Can(R) is the relational Cantorian of R, with an analogous interpretation.

R x, the value of R at x, is defined so as to be V if x is not an argument of R or R is not single valued at x. This becomes convenient following T37. $\min_R u$ is similarly defined for the same reason. We do not define $\max_R u$ in this way, as it turns out we are never interested in $\max_R u$ except when we have already proved that there is a last element in u according to R.

Most of the assumed theorems which follow are standard theorems of type theory, set theory, and other systems, and will not be proved in this paper. Those which are not so well known are marked with an asterisk (*) and will be proved in Part II. T27 depends on a weak form of the axiom of choice (Axiom A, p. 64), and T27 and all theorems which depend on T27 are marked with a dagger (†). T84 (p. 42) depends on an axiom (Axiom B, p. 64) which apparently can not be proved in the theory of types, and T84 and theorems which depend on it will be marked with two daggers (††).

We assume that a relation, $\leq$, has been defined in L_2 such that:

T6. $\vdash \leq \epsilon \Omega$.

T7. $\vdash V \sim \varepsilon C(\leq)$.

T7 could be replaced by $\vdash (Ex).x \sim \varepsilon C(\leq)$, but since in all known treatments of ordinal number theory one can prove $\vdash V \sim \varepsilon C(\leq)$, we assume this theorem for convenience.

D34. $NO =df\ C(\leq)$.

D35. $< =df\ \hat{\alpha}\hat{\beta}(\alpha \leq \beta . \alpha \neq \beta)$.

We assume that expressions 0, 1, 2, 3, 4, 5, 6, 7, 8 have been defined in L_2 such that:

T8. $\vdash 0, 1, \cdots, 8 \varepsilon NO$.

T9. $\vdash 0 = min_{\leq}\ NO$.

T10. $\vdash \sim (E\alpha).0 < \alpha < 1$

$\vdash \sim (E\alpha).1 < \alpha < 2$

etc.

T11. $\vdash 0 < 1 . 1 < 2 . 2 < 3 ... 7 < 8$.

We assume that an expression, ω_0, and an operation, $+$, have been defined in L_2 such that:

T12. $\vdash \omega_0 \varepsilon NO$.

T13. $\vdash 8 < \omega_0$.

T14. $\vdash (\alpha, \beta).\alpha, \beta \varepsilon NO \supset \alpha + \beta \varepsilon NO$.

T15. $\vdash (\alpha, \beta).\alpha, \beta < \omega_0 \supset \alpha + \beta < \omega_0$.

We assume that a relation, $\leq_t$, has been defined in L_2 such that:

*T16. $\vdash \leq_t \varepsilon \Omega$.

*T17. $\vdash C(\leq_t) = \hat{z}((E\alpha, \beta, \gamma) : \alpha \leq 8 . \beta, \gamma \varepsilon NO . z = \langle \alpha\beta\gamma \rangle)$.

*T18. $\vdash (\alpha,\beta,\gamma,\mu,\nu,\xi) :: \alpha,\mu \leq 8 . \beta,\gamma,\nu,\xi \, \varepsilon \mathrm{NO} :.$
$\supset :. \langle \alpha\beta\gamma \rangle \leq_t \langle \mu\nu\xi \rangle : \equiv : \max_{\leq}(U(\beta) \cup U(\gamma)) <$
$\max_{\leq}(U(\nu) \cup U(\xi)) : \vee : \max_{\leq}(U(\beta) \cup U(\gamma)) =$
$\max_{\leq}(U(\nu) \cup U(\xi)) : \gamma < \xi . \vee . \gamma = \xi . \beta < \nu . \vee . \gamma = \xi . \beta = \nu . \alpha \leq \mu .$

*T19. $\vdash \leq_t \mathrm{smor} \leq .$

We assume that the following are provable in L_2:

T20. $\vdash (P,Q,R,S) : P,Q \varepsilon \Omega . P \, \mathrm{smor}_R Q . P \, \mathrm{smor}_S Q . \supset . R = S .$

T21. $\vdash (P) . P \varepsilon \Omega \supset r\mathrm{Can}(P) \varepsilon \Omega .$

T22. $\vdash (P) . C(r \, \mathrm{Can}(P)) = \mathrm{Can}(C(P))$

T23. $\vdash (x,y,z) : x \, \mathrm{sm} y . y \, \mathrm{sm} z . \supset . x \, \mathrm{sm} z .$

*+T27. $\vdash (x) : x \subseteq \mathrm{NO} . \sim (x \, \mathrm{sm} \, \mathrm{NO}) . \supset . (E\alpha) . \alpha \varepsilon \mathrm{NO} .$
$(\beta) . \beta \varepsilon x \supset \beta < \alpha .$

*T28. $\vdash (x) :. x \subseteq \mathrm{NO} . (E\alpha) . \alpha \varepsilon \mathrm{NO} . (\beta) . \beta \varepsilon x \supset \beta < \alpha :$
$\supset : \sim (x \, \mathrm{sm} \, \mathrm{NO}) .$

From T19 and T20 we obtain at once:

T29. $\vdash (E_1 P) . \leq_t \mathrm{smor}_P \leq .$

Hence we can define:

D34. $J =_{df} \iota P(\leq_t \mathrm{smor}_P \leq) .$

D35. $J_0 =_{df} \hat{x}\hat{y}((E\alpha,\beta) . \alpha,\beta \varepsilon \mathrm{NO} . x = \langle \alpha\beta \rangle . y = J ' \langle 0\alpha\beta \rangle)$

$J_1 =_{df} \hat{x}\hat{y}((E\alpha,\beta) . \alpha,\beta \varepsilon \mathrm{NO} . x = \langle \alpha\beta \rangle . y = J ' \langle 1\alpha\beta \rangle)$

. . . .

$J_8 =_{df} \hat{x}\hat{y}((E\alpha,\beta) . \alpha,\beta \varepsilon \mathrm{NO} . x = \langle \alpha\beta \rangle . y = J ' \langle 8\alpha\beta \rangle) .$

D35.1. $J_\mu =_{df} \iota z(\mu = 0 . z = J_0 . \vee . \mu = 1 . z = J_1 . \vee . \cdots . \vee .$
$\mu = 8 . z = J_8) .$

D36. $K_1 =_{df} \hat{\gamma}\hat{\alpha}(\gamma, \alpha \varepsilon NO . (E\mu, \beta) . \beta \varepsilon NO . \mu \leq 9 . \gamma = J ` \langle \mu \alpha\beta \rangle)$

$K_2 =_{df} \hat{\gamma}\hat{\beta}(\gamma, \beta \varepsilon NO . ((E\mu, \alpha) . \alpha \varepsilon NO . \mu \leq 9 . \gamma = J ` \langle \mu \alpha\beta \rangle)$.

We assume that the following theorems are provable in L_2:

T30. $\vdash Val(J_0) \cup Val(J_1) \cup \cdots \cup Val(J_8) = NO$.

T31. $\vdash (\mu, \nu) : \mu, \nu \leq 8 . \mu \neq \nu . \supset . Val(J_\mu) \cap Val(J_\nu) = \Lambda$.

*T32. $\vdash (\alpha, \beta, \mu) : \alpha, \beta \varepsilon NO . \mu \leq 8 . \supset . max_{\leq}(U(\alpha) \cup U(\beta)) \leq J ` \langle \mu \alpha\beta \rangle$.

*T33. $\vdash (\alpha, \beta, \mu) : \alpha, \beta \varepsilon NO . 0 < \mu \leq 8 . \supset . max_{\leq}(U(\alpha) \cup U(\beta)) < J ` \langle \mu \alpha\beta \rangle$.

*T34. $\vdash (\alpha) : \alpha \varepsilon NO . \supset . K_1 ` \alpha \leq \alpha . K_2 ` \alpha \leq \alpha$.

*T35. $\vdash (\alpha) : \alpha \varepsilon NO . \alpha \sim \varepsilon Val(J_0) . \supset . K_1 ` \alpha < \alpha . K_2 ` \alpha < \alpha$.

*T36. $\vdash \omega_o \varepsilon Val(J_0)$.

We assume further that T37 (p. 25) is provable in L_2. This amounts to assuming that a certain type of definition by induction, of which T37 is an instance, can be carried out in L_2. Theorem 7.5 of [5] is easily generalized to obtain a theorem from which T37 follows, and a theorem guaranteeing the possibility of such a definition by induction in the theory of types will be proved in Part II.

We assume finally that T84 (p. 42) is provable in L_2. T84 will be proved in Part II, from the axioms mentioned in the introduction.

§3. <u>Construction of the Model</u>. We motivate the method which will be used to model L_1 in L_2 by an intuitive discussion of the method used by Gödel to model a logic Δ (obtained by adding to L_1 an axiom from which the axiom of choice and the generalized continuum hypothesis are provable) in L_1.

Gödel defines $\leq_t, J, J_0, \cdots, J_8, K_1,$ and K_2 as in §2. It follows from these definitions that if $\alpha \varepsilon Val(J_i)$ then $\alpha+1 \varepsilon Val(J_{i+1})$ if $i+1<9$, and $\alpha+1 \varepsilon Val(J_0)$ if $i+1=9$. Thus the $Val(J_i)$ function, in a sense, as congruence classes mod 9; any sequence of ordinals, $\alpha, \alpha+1, \cdots, \alpha+8$ will be respectively in a cyclic permutation of $Val(J_0), \cdots, Val(J_8)$.

A function F is then defined by transfinite induction such that

$$\alpha \varepsilon Val(J_0) . \supset . F`\alpha = Val(\alpha \upharpoonleft F)$$

$$\alpha \varepsilon Val(J_1) . \supset . F`\alpha = U(F`K_1`\alpha) \cup U(F`K_2`\alpha)$$

$$\alpha \varepsilon Val(J_2) . \supset . F`\alpha = E \cap (F`K_1`\alpha)$$

$$\alpha \varepsilon Val(J_3) . \supset . F`\alpha = (F`K_1`\alpha) \cap (F`K_2`\alpha)$$

$$\alpha \varepsilon Val(J_4) . \supset . F`\alpha = (F`K_1`\alpha) \upharpoonleft (F`K_2`\alpha)$$

$$\alpha \varepsilon Val(J_5) . \supset . F`\alpha = (F`K_1`\alpha) \cap Val(F`K_2`\alpha)$$

$$\alpha \varepsilon Val(J_6) . \supset . F`\alpha = (F`K_1`\alpha) \cap (F`K_2`\alpha)^{-1}$$

$$\alpha \varepsilon Val(J_7) . \supset . F`\alpha = (F`K_1`\alpha) \cap Cnv_2(F`K_2`\alpha)$$

$$\alpha \varepsilon Val(J_8) . \supset . F`\alpha = (F`K_1`\alpha) \cap Cnv_3(F`K_2`\alpha)$$

where

$$Cnv_2(x) =df\ \hat{u}\ \hat{v}\ \hat{w}(\langle vwu \rangle \varepsilon x)$$

$$Cnv_3(x) =df\ \hat{u}\ \hat{v}\ \hat{w}(\langle uwv \rangle \varepsilon x)$$

$$E\ =df\ \hat{x}\hat{y}(x \varepsilon y).$$

(Note that E is not definable in a system based on the theory of types. However, in L_1 the existence of E follows from Axiom B1.)

The universe of sets is then defined to be L=Val(F). To define the universe of classes we consider the following:

1). The members of a class should be sets; hence, if x is a class we should have $x \subseteq L$.

2). One of the essential properties of sets in L_1 is that if y is a set and $x \subseteq y$, then x is a set (see 5.12 in [5]). Since a subclass of a set can be obtained by intersecting the set with an appropriate class, this can be stated as a property of classes: if x is a set and y is a class, then $x \cap y$ is a set.

The above considerations suggest that the universe of classes be defined by the statement $\mathcal{L}(x) =df\ x \subseteq L.$ $(z).z\varepsilon L \supset z \cap x\varepsilon L.$

If Δ is to be modelled in the universes defined above, then L_1 itself must be modelled in these universes; this will be accomplished if the axioms of L_1, when set and class quantification are restricted to the appropriate universes, remain provable in L_1; i.e., if in the axioms every expression of the form $(x)_s p$ is replaced by $(x).x\varepsilon L \supset p$, and every expression of the form $(x)p$ is replaced by $(x).\mathcal{L}(x) \supset p$, then the resulting propositions should remain provable in L_1. (We might also in this process of "relativization" use some other relation in place of ε, but F is defined in such a way that this is unnecessary.) To indicate how this happens, we consider two examples.

1) Consider Axiom A3. This says that for any sets x and y, there is a set z such that for every set u, $u\varepsilon z.\equiv.u=x \vee u=y$. We first note the fact, easily proved by induction and intuitively obvious from the definition of F, that $(x,z):x\varepsilon L.z\varepsilon x.\supset.z\varepsilon L$. From this it follows that the relation $=$ between sets is the same for the model as for L_1. Thus, what we need to show is that for any $x,y\varepsilon L$ there is a $z\varepsilon L$ such that for all u, $u\varepsilon z.\equiv.u=x \vee u=y$; i.e., $U(x)\cup U(y)\varepsilon L$.

So suppose $x,y\varepsilon L$. Then $x=F`\alpha, y=F`\beta$. Now consider the triple $\langle 1\alpha\beta\rangle$ and let $J`\langle 1\alpha\beta\rangle=\gamma$; i.e., $J_1`\langle\alpha\beta\rangle=\gamma$. Obviously $\gamma\varepsilon Val(J_1)$, and $K_1`\gamma=\alpha$, $K_2`\gamma=\beta$. So by the definition of F, $F`\gamma=U(F`\alpha)\cup U(F`\beta)$. Hence, $F`\gamma=U(x)\cup U(y)$. Thus, $U(x)\cup U(y)\varepsilon L$.

2) Consider Axiom B1. As in the first example, it is shown that any ordered pair, $\langle x,y\rangle$, of sets is the same in the model as in L_1. So to prove B1 for the model we must show that there is a class z such that $\mathcal{L}(z)$ and $(x,y):x,y\varepsilon L.\supset.\langle x,y\rangle\varepsilon z\equiv x\varepsilon y$.

It is first shown that $\mathcal{L}(E\cap L)$. Obviously $E\cap L\subseteq L$. Suppose $u\varepsilon L$. Then as pointed out in the first example, $(w).\ w\varepsilon u\supset w\varepsilon L$; i.e., $u\subseteq L$. Hence, $u\cap E\cap L=u\cap E=E\cap u$. Since $u\varepsilon L$, let $u=F`\alpha$, and let $J`\langle 2\alpha 0\rangle=\gamma$. So $\gamma=J_2`\langle\alpha 0\rangle$. Then $F`\gamma=E\cap(F`K_1`\gamma)=E\cap(F`\alpha)=E\cap u$. Thus $E\cap u\varepsilon L$. We have now shown that $\mathcal{L}(E\cap L)$.

Since it is obvious that $(x,y):x,y\varepsilon L.\supset.\langle xy\rangle\varepsilon E\ L\equiv x\varepsilon y$, we can take the desired z to be E L, and we have proved B1 for the model.

It is not too surprising that with quantification restricted to the appropriate universes, Axioms B1-B8 remain provable. It is clear that F is defined in such a way that starting with Λ a sequence of sets is produced such that for any <u>set in the sequence</u> there appears later in the sequence each of the classes whose existence is asserted by B1-B8. It is perhaps less obvious that this holds also for <u>classes</u> of sets in the sequence, which satisfy the condition $\mathcal{L}$; however, comparison of the definitions of F and $\mathcal{L}$ indicates that both are designed for this purpose (among others), and one can at least hope that everything will work out all right (as it does for L_1).

It is certainly not obvious that Axioms C1-C4 will remain provable for the model; in fact, as will be seen when we model L_1 in L_2, the proofs of C3 and C4 for the model depend essentially on certain highly non-trivial properties of ordinals, properties that unfortunately seem not to be provable in the theory of types. In L_1, however, such disturbing difficulties do not arise, and all the axioms of L_1, plus the additional axiom of Δ, are provable for the model.

Now let us consider how we might modify the procedure described above so that we can model L_1 in L_2. We do not wish to define the specific function F in L_2, since the class E occurs in the definition of F, and this class is not definable in the theory of types. Thus we wish to define a function, G, by means available in the theory of types, which will have approximately the same properties as F. To see how to do this, we reason heuristically as follows.

For every ordinal, α, there is a correspondent in L, namely F'α. However, different ordinals may have the same correspondent in L; e.g., F'0=F'2=Λ. Thus the correspondence between the ordinals and L is not one-to-one. However, we can define the <u>index of u</u>, uεL, as the least ordinal, α, such that F'α=u. This establishes a one-to-one correspondence between L and a subclass of the ordinals. Hence, for any statement about members of L we should be able to define a corresponding statement about the indices of these members, which is, in a sense, a translation of the given statement. This suggests the possibility of constructing a model of L_1 in the ordinals and classes of ordinals, by means of a function, G, obtained from F by using as values of G the classes of ordinals defined by the translations of the statements which defined the various values of F. More precisely, it is suggested that we define a function, G, such that for any x, if xεL (and hence, as pointed out previously, x$\subseteq$L) there will be a class of ordinals, A, AεVal(G), such that the members of A are the indices of the members of x. This will, of course, have to be done without explicit reference to F, since we are not assuming that F is definable in L_2.

Assuming that we can define such a G, we can, just as with F, define the index of x, for $x\varepsilon Val(G)$, as the least ordinal, α, such that $G`\alpha=x$.

Now if we expect to model L_1 by using Val(G) as the universe of sets, we must define a relation, ε_g, to function as the membership relation for the model. Further, if we are to prove Axiom B1 for the model, ε_g must be defined in such a way that if $x\varepsilon_g y$ then x and y are of the same type. Also, if $x\varepsilon_g y$, then x should be a set; i.e., $x\varepsilon Val(G)$. It should now be clear how this can be accomplished. We shall take $x\varepsilon_g y$ to mean that the index of x is a member of y. Then $x\varepsilon_g y$ will be meaningful when x and y are of the same type, and will in addition be defined by a statement which contains parts of the form $u\varepsilon w$ only when u is one type lower than w.

If G is to be defined so as to have the properties just described, the values of G will have to be classes of ordinals. Since in the theory of types the arguments and values of a function must be of the same type, this means that G will have to be defined over a well-ordered class whose elements are classes of ordinals, rather than ordinals. Hence we shall define G over Can(NO) rather than NO, and in the preceding discussion we should have written $G`U(\alpha)$ rather than $G`\alpha$.

A function having the above properties will be defined in T37.[5] Before defining this function G, we first introduce some definitions which will shorten the formulas occurring in the definition of G, and will perhaps also make clearer the way in which G is obtained from F.

D37. $\mathrm{Ind}_w(x) =_{df} \min_{\leq} \hat{\gamma}(\gamma \varepsilon NO . w\text{‘}U(\gamma) = x)$.

D38. $\{\gamma\delta\}(w) =_{df} \min_{\leq}\beta(\beta \varepsilon NO . w\text{‘}U(\beta) =$

$U(\mathrm{Ind}_w(w\text{‘}U(\gamma))) \cup U(\mathrm{Ind}_w(w\text{‘}U(\delta))))$.

D39. $\{\gamma\}(w) =_{df} \{\gamma\gamma\}(w)$.

D40. $\langle\gamma, \delta\rangle(w) =_{df} \{\{\gamma\}(w), \{\gamma\delta\}(w)\}(w)$.

D41. $\langle\alpha\beta\gamma\rangle(w) =_{df} \langle\alpha, \langle\beta\gamma\rangle(w)\rangle(w)$.

D42. $\hat{\alpha}_w(p) =_{df} \hat{\alpha}(\alpha \varepsilon NO . \alpha = \mathrm{Ind}_w(w\text{‘}U(\alpha)) . p)$.

D43. $(\alpha)_w p =_{df} (\alpha)(\alpha \varepsilon NO . \alpha = \mathrm{Ind}_w(w\text{‘}U(\alpha)) . \supset . p)$.

We make use of the following temporary abbreviations, for T37 only:

$w =_{df} (\mathrm{Can}(\hat{\mu}(\mu < \beta))) \upharpoonleft G$

$Z(\langle\gamma\delta\rangle(w)) =_{df} \gamma, \delta, \{\gamma\}(w), \{\gamma\delta\}(w), \langle\gamma\delta\rangle(w) < \beta$

$Z(\langle\gamma\delta\eta\rangle(w)) =_{df} \gamma, \{\gamma\}(w), \{\gamma, \langle\delta\eta\rangle(w)\}(w), \langle\gamma\delta\eta\rangle(w) < \beta .$

$Z(\langle\delta\eta\rangle(w))$.

5. Actually T37 is not a definition but a theorem which asserts the existence of a function having certain recursive properties. However, it is customary to call such theorems "definitions by induction".

T37. $\vdash$ (EG):.GεFnc.Arg(G)=Can(NO).G$^{\text{‘}}$U(0)=Λ.

(β):βεNO.β≠0.⊃.G$^{\text{‘}}$U(β)=

ι z[βεVal(J_0).z=$\hat{\alpha}_w$(α<β).∨.

βεVal(J_1).z=U(Ind_w(w$^{\text{‘}}$U($K_1^{\text{‘}}$β)))∪ U(Ind_w(w$^{\text{‘}}$U($K_2^{\text{‘}}$β))).∨.

βεVal(J_2).z=w$^{\text{‘}}$U($K_1^{\text{‘}}$β)∩$\hat{\alpha}_w$((E γ,δ)$_w$.Z(<γδ>(w)).

α=<γδ>(w).γεw$^{\text{‘}}$U(δ)).∨.

βεVal(J_3).z=w$^{\text{‘}}$U($K_1^{\text{‘}}$β)∩ $\overline{\text{w}^{\text{‘}}\text{U}(K_2^{\text{‘}}\beta)}$.∨.

βεVal(J_4).z=w$^{\text{‘}}$U($K_1^{\text{‘}}$β)∩$\hat{\alpha}_w$((E γ,δ)$_w$.Z(<γδ>(w)).

α=<γδ>(w).δεw$^{\text{‘}}$U($K_2^{\text{‘}}$β)).∨.

βεVal(J_5).z=w$^{\text{‘}}$U($K_1^{\text{‘}}$β)∩$\hat{\alpha}_w$((Eδ)$_w$.Z(<δα>(w)).

<δα>(w)εw$^{\text{‘}}$U($K_2^{\text{‘}}$β)).∨.

βεVal(J_6).z=w$^{\text{‘}}$U($K_1^{\text{‘}}$β)∩$\hat{\alpha}_w$((E γ,δ)$_w$.Z(<γδ>(w)).

α=<γ δ>(w).<δ γ>(w)εw$^{\text{‘}}$U($K_2^{\text{‘}}$β)).∨.

βεVal(J_7).z=w$^{\text{‘}}$U($K_1^{\text{‘}}$β)∩$\hat{\alpha}_w$((E γ,δ,η)$_w$.Z(<γδη>(w)).

Z(<δηγ>(w)).α=<γδη>(w).<δηγ>(w)εw$^{\text{‘}}$U($K_2^{\text{‘}}$β).∨.

βεVal(J_8).z=w$^{\text{‘}}$U($K_1^{\text{‘}}$β)∩$\hat{\alpha}_w$((E γ,δ,η)$_w$.Z(<γδη>(w).

Z(<γη δ>(w)).α=<γδη>(w).<γη δ>(w)εw$^{\text{‘}}$U($K_2^{\text{‘}}$β))].

Throughout the rest of this paper we let G be a function satisfying the proposition in T37.

D44. Ind(x)=df Ind_G(x).

T38. (α,β):βεNO.αεG$^{\text{‘}}$U(β).⊃.α<β.α=Ind(G$^{\text{‘}}$U(α)).

Proof by induction on β. If β=0, then G$^{\text{‘}}$U(β)= Λ and the theorem is vacuously true. Assume the theorem for all ordinals less than β.

Case 1. βεVal(J_0). Then G$^{\text{‘}}$U(β)=$\hat{\alpha}_w$(α<β), where w=(Can($\hat{\mu}$(μ<β)))⥿G. Hence, G$^{\text{‘}}$U(β)=$\hat{\alpha}_G$(α<β).

But $(\alpha):\alpha\varepsilon\hat{\alpha}_G(\alpha<\beta).\supset.\alpha<\beta.\alpha=\mathrm{Ind}(G`U(\alpha))$.

Case 2. $\beta\varepsilon\mathrm{Val}(J_1)$. Then $\beta=J_1`<\gamma\delta>$. Hence $G`U(\beta)=U(\mathrm{Ind}_w(w`U(\gamma)))\cup U(\mathrm{Ind}_w(w`U(\delta)))$, where $w=(\mathrm{Can}(\hat{\mu}(\mu<\beta)))\upharpoonright G$. But by T35, $\gamma<\beta$, $\delta<\beta$, so $G`U(\beta)=U(\mathrm{Ind}(G`U(\gamma)))\cup U(\mathrm{Ind}(G`U(\delta)))$. Hence, $\alpha\varepsilon G`U(\beta).\supset.\alpha=\mathrm{Ind}(G`U(\gamma))\vee\alpha=\mathrm{Ind}(G`U(\delta))$. Obviously, $\mathrm{Ind}(G`U(\gamma))\leq\gamma$, $\mathrm{Ind}(G\ U(\delta))\leq\delta$, so $\alpha\varepsilon G`U(\beta).\supset.\alpha<\beta$. Now suppose $\alpha=\mathrm{Ind}(G`U(\gamma))=\min_{\leq}\hat{\nu}(G`U(\nu)=G`U(\gamma))$, and that $(E\eta).\eta<\alpha.G`U(\eta)=G`U(\alpha)$. Then $G`U(\eta)=G`U(\mathrm{Ind}(G`U(\gamma)))=G`U(\gamma)$, contradicting $\alpha=\mathrm{Ind}(G`U(\gamma))$. Thus, $\alpha=\mathrm{Ind}(G`U(\gamma)).\supset.\alpha=\mathrm{Ind}(G`U(\alpha))$. Similarly, $\alpha=\mathrm{Ind}(G`U(\delta)).\supset.\alpha=\mathrm{Ind}(G`U(\alpha))$. So $\alpha\varepsilon G`U(\beta).\supset.\alpha=\mathrm{Ind}(G`U(\alpha))$.

Case 3. $\beta\varepsilon\mathrm{Val}(J_\mu)$, $\mu=2,3,\cdots,8$. Then $G`U(\beta)=w`U(K_1`\beta)\cap\hat{\alpha}_w(\ldots)$, where $w=(\mathrm{Can}(\hat{\mu}(\mu<\beta)))\upharpoonright G$. By T35, $K_1`\beta<\beta$, so $G`U(\beta)=G`U(K_1`\beta)\cap\hat{\alpha}_w(\ldots)$. Hence, $\alpha\varepsilon G`U(\beta).\supset.\alpha\varepsilon G`U(K_1`\beta)$. So by hypothesis of induction, $\alpha\varepsilon G`U(\beta).\supset.\alpha<\beta.\alpha=\mathrm{Ind}(G`U(\alpha))$.

D45. $E_g=\mathrm{df}\ \hat{\alpha}_G((E\gamma,\delta)_G.\alpha=<\gamma\delta>(G).\gamma\varepsilon G`U(\delta))$.

D46. $\{x,y\}_g=\mathrm{df}\ G`U(\{\mathrm{Ind}(x),\mathrm{Ind}(y)\}(G))$.

D47. $<xy>_g=\mathrm{df}\ \{\{x,x\}_g,\{x,y\}_g\}_g$.

D48. $<xyz>_g=\mathrm{df}\ <x,<yz>_g>_g$.

D49. $(V\times y)_g=\mathrm{df}\ \hat{\alpha}_G((E\gamma,\delta)_G.\alpha=<\gamma\delta>(G).\delta\varepsilon y)$.

D50. $D_g(y)=\mathrm{df}\ \hat{\alpha}_G((E\delta)_G.<\delta\alpha>(G)\varepsilon y)$.

D51. $\mathrm{Cnv}_{1g}(y)=\mathrm{df}\ \hat{\alpha}_G((E\gamma,\delta)_G.\alpha=<\gamma\delta>(G).<\delta\gamma>(G)\varepsilon y)$.

D52. $\mathrm{Cnv}_{2g}(y)=\mathrm{df}\ \hat{\alpha}_G((E\gamma,\delta,\nu)_G.\alpha=<\gamma\delta\nu>(G).<\delta\nu\gamma>(G)\varepsilon y)$.

D53. $Cnv_{3g}(y) =_{df} \hat{\alpha}_G((E\gamma,\delta,\nu)_G . \alpha = \langle\gamma\delta\nu\rangle(G) . \langle\gamma\nu\delta\rangle(G) \varepsilon y)$.

T39. $\vdash (\alpha,\beta)_G (E\gamma)_G . G`U(\gamma) = U(\alpha) \cup U(\beta)$.

Proof. By T37, $\gamma = Ind(G`U(J_1\langle\alpha\beta\rangle))$ is such an ordinal.

From T39 and D46 we get

T40.1. $\vdash (\alpha,\beta)_G . \{G`U(\alpha), G`U(\beta)\}_g = G`U(\{\alpha,\beta\}(G)) = U(\alpha) \cup U(\beta$

By similar methods we have

T40.2. $\vdash (\alpha,\beta)_G . \langle G`U(\alpha), G`U(\beta)\rangle_g = G`U(\langle\alpha\beta\rangle(G))$ $= U(\{\alpha\}(G)) \cup U(\{\alpha,\beta\}(G))$.

The above theorems indicate how $\{x,y\}_g$ and $\langle xy\rangle_g$ will function as the unordered pair and ordered pair respectively in the model.

T40.3. $\vdash (\alpha,\beta)_G . \alpha < \{\alpha\beta\}(G) . \beta < \{\alpha\beta\}(G)$.

Proof. By T40.1, $G`U(\{\alpha\beta\}(G)) = U(\alpha) \cup U(\beta)$, so $\alpha,\beta \varepsilon G`U(\{\alpha\beta\}(G))$. Hence by T38, $\alpha,\beta < \{\alpha\beta\}(G)$.

T40.4. $\vdash (\alpha,\beta)_G . \alpha < \langle\alpha\beta\rangle(G) . \beta < \langle\alpha\beta\rangle(G)$.

Proof. Use T40.3.

The following table of certain selected values of G shows the manner in which G produces the indices of the sets of L_1.

Ordinals (α)	Corresponding triples by $\bar{J}$	Members of $G`U(\alpha)$	Index of $G`U(\alpha)$	Remarks
0	<000>	none	0	
1	<100>	0	1	$G`U(1)=U(0)=\{G`U(0)\}_g$;
2	<200>	none	0	i.e., $1=\{0\}$ (G)
3	<300>			
4	<400>			
5	<500>			
6	<600>			
7	<700>			
8	<800>	none	0	
9	<010>	0,1	9	$G`U(9)=U(0)\cup U(1)$
10	<110>	0,1	9	$=\{G`U(0),G`U(1)\}_g$;
11	<210>	none	0	i.e., $9=\{0,1\}$ (G)
12	<310>	0	1	
13	<410>	none	0	
14	<510>			
15	<610>			
16	<710>			
17	<810>	none	0	
18	<001>	0,1,9	18	
19	<101>	0,1	9	
20	<201>	none	0	

21	<301>			
22	<401>			
23	<501>			
24	<601>			
25	<701>			
26	<801>	none	0	
27	<011>	0, 1, 9, 18	27	
28	<111>	1	28	$G`U(28)=U(1)=$
29	<211>	none	0	$\{G\ U(1)\}_g = \{\{G\ U(0)\}_g\}_g$
30	<311>			$=\langle G`U(0), G`U(0)\rangle_g$;
31	<411>			i.e., $28=\langle 0,0\rangle(G)$
32	<511>			
33	<611>			
34	<711>			
35	<811>	none	0	
36	<020>	0,1,9,18,27,28	36	
37	<120>	0	1	
38	<220>	none	0	
39	<320>			
40	<420>			
41	<520>			
42	<620>			
43	<720>			
44	<820>	none		

45	<021>	0,1,...,28,36	45
46	<121>	0,1	9
47	<221>	none	0
48	<321>		
49	<421>		
50	<521>		
51	<621>		
52	<721>		
53	<821>	none	0
54	<002>	0,1,...,36,45	54
55	<102>	0	1
56	<202>	none	0
57	<302>		
58	<402>		
59	<502>		
60	<602>		
61	<702>		
62	<802>	none	0
63	<012>	0,1,...,45,54	63
64	<112>	0,1	9
65	<212>	none	0
66	<312>	0	1
67	<412>	none	0
68	<512>		
69	<612>		

70	<712>			
71	<812>	none	0	
72	<022>	0,1,···,54,63	72	
73	<122>	0	1	
74	<222>	none	0	
75	<322>			
76	<422>			
77	<522>			
78	<622>			
79	<722>			
80	<822>	none	0	
738	<091>	0,···,729,730	738	
739	<191>	1,9	739	$G`U(739)=U(1)\cup U(9)$ $=\{G`U(1),G`U(9)\}_g$ $=\{\{G`U(0)\}_g,\{G`U(0),G`U(1)\}_g\}_g$ $=\langle G`U(0),G`U(1)\rangle_g$; i.e., $739=\langle 0,1\rangle(G)$
7056	<0,28,0>	0,1,···	7056	
7057	<1,28,0>	28,0	7057	$G`U(7057)=U(28)\cup U(0)$ $=\{G`U(28),G`U(0)\}_g$; i.e., $7057=\{28,0\}(G)$

7058	$\langle 2,28,0\rangle$	1,739	7058	$G`U(7058) =$ $U(1)\cup U(739) =$ $\{G`U(1),G`U(739)\}_g$; i.e., $7058= \{1,739\}\ (G)$
7309	$\langle 1,28,28\rangle$	28	7308	$G`U(7309)=U(28)$ $= \{G`U(28)\}_g$ $= \{\{G`U(1)\}_g\}_g$ $=\langle G`U(1),G`U(1)\rangle_g$; i.e., $7309=\langle 1,1\rangle(G)$

T41. I. $\vdash (\mu,\eta):\mu,\eta\ \varepsilon \mathrm{NO}.\supset.G`U(J_0`\langle\mu\eta\rangle)$
$=\hat{\alpha}_G(\alpha\langle J_0`\langle\mu\eta\rangle)$.

II. $\vdash (\mu,\eta):\mu,\eta\ \varepsilon \mathrm{NO}.\supset.G`U(J_1`\langle\mu\eta\rangle)$
$=\{G`U(\mu),G`U(\eta)\}_g$.

III. $\vdash (\mu,\eta):\mu,\eta\ \varepsilon \mathrm{NO}.\supset.G`U(J_2`\langle\mu\eta\rangle)$
$=G`U(\mu)\cap E_g$.

IV. $\vdash (\mu,\eta):\mu,\eta\ \varepsilon \mathrm{NO}.\supset.G`U(J_3`\langle\mu\eta\rangle)$
$=G`U(\mu)\cap\overline{G`U(\eta)}$.

V. $\vdash (\mu,\eta):\mu,\eta\ \varepsilon \mathrm{NO}.\supset.G`U(J_4`\langle\mu\eta\rangle)$
$=G`U(\mu)\cap(V\times G`U(\eta))_g$.

VI. $\vdash (\mu,\eta):\mu,\eta\ \varepsilon \mathrm{NO}.\supset.G`U(J_5`\langle\mu\eta\rangle)$
$=G`U(\mu)\cap D_g(G`U(\eta))$.

VII. $\vdash (\mu,\eta):\mu,\eta\ \varepsilon \mathrm{NO}.\supset.G`U(J_6`\langle\mu\eta\rangle)$
$=G`U(\mu)\cap \mathrm{Cnv}_{19}(G`U(\eta))$.

VIII. $\vdash(\mu,\eta):\mu,\eta\,\varepsilon NO.\supset.G\text{‘}U(J^{\text{‘}}_{7}\langle\mu\eta\rangle)$ $=G\text{‘}U(\mu)\cap Cnv_{2g}(G\text{‘}U(\eta)).$

IX. $\vdash(\mu,\eta):\mu,\eta\,\varepsilon NO.\supset.G\text{‘}U(J^{\text{‘}}_{8}\langle\mu\eta\rangle)$ $=G\text{‘}U(\mu)\cap Cnv_{3g}(G\text{‘}U(\eta)).$

Proof. Use T37 and T40.1-T40.4.

D54. $L{=}df\ Val(G).$

D55. $NO_G{=}df\ \hat{\alpha}_G(\alpha\varepsilon NO).$

D56. $\mathcal{L}(x){=}df\ x\subseteq NO_G.(z).z\varepsilon L\supset x\cap z\varepsilon L.$

D57. $\mathcal{L}(x,y){=}df\ \mathcal{L}(x).\mathcal{L}(y).$

T42. $\vdash(x,y):x,y\varepsilon L.\supset.\{x,y\}_g\varepsilon L.x\cap E_g\varepsilon L.$ $x\cap\overline{y}\varepsilon L.x\cap(V\times y)_g\varepsilon L.x\cap D_g(y)\varepsilon L.$ $x\cap Cnv_{1g}(y)\varepsilon L.x\cap Cnv_{2g}(y)\varepsilon L.x\cap Cnv_{3g}(y)\varepsilon L.$

Proof. Suppose $x,y\varepsilon L$. Let $Ind(x)=\alpha, Ind(y)=\beta$. The theorem follows from T41.

T43. $\vdash(x).x\varepsilon L\supset x\subseteq NO_G.$

Use T38.

T44. $\vdash V\sim\varepsilon L.$

Proof. Use T38.

T45. $\vdash(x,y).x,y\varepsilon L\supset x\cap y\varepsilon L.$

Proof. Suppose $x,y\varepsilon L$. Then by T42, $x\cap\overline{y}\varepsilon L$. So by T42 again, $x\cap\overline{x\cap\overline{y}}\varepsilon L$. But $x\cap y=x\cap\overline{x\cap\overline{y}}$. Hence $x\cap y\varepsilon L$.

T46.1. $\vdash(x,y).x,y\varepsilon L\equiv\{x,y\}_g\varepsilon L.$

Proof. By T42, $x,y\varepsilon L\supset\{x,y\}_g\varepsilon L$. Suppose $\{x,y\}_g\varepsilon L$; i.e., $G\text{‘}U(\{Ind(x),Ind(y)\}(G))\varepsilon L$. Suppose $x\sim\varepsilon L$. Then by T44, $Ind(x)=V$, so $G\text{‘}U(Ind(x))=V$. Hence

$\{ Ind(x), Ind(y) \} (G) = \min_{\leq} \hat{\alpha}(G{}^{\text{c}}U(\alpha) = U(V) \cup U(G{}^{\text{c}}U(Ind(y)))) = V$
by T38 and T7. So $G{}^{\text{c}}U(\{ Ind(x), Ind(y) \} (G)) = V = \{ x,y \}_g$, contradicting T44 and the assumption that $\{ xy \}_g \varepsilon L$.

Hence $x \varepsilon L$, and by a similar proof, $y \varepsilon L$.

T46.2. $\vdash (x,y).x,y \varepsilon L \equiv \langle xy \rangle_g \varepsilon L$.

Proof. Use T46.1.

T47. $\vdash (x,y).\langle xy \rangle_g \varepsilon L \equiv \langle yx \rangle_g \varepsilon L$.

Proof. Use T46.2.

T48. $\vdash (\alpha,\beta,\gamma,\delta):.\alpha,\beta,\gamma,\delta \varepsilon NO_G : \supset : \langle \alpha\beta \rangle(G) = \langle \gamma\delta \rangle(G). \equiv . \alpha = \gamma . \beta = \delta$.

Proof. Suppose $\langle \alpha\beta \rangle(G) = \langle \gamma\delta \rangle(G)$. Then $G{}^{\text{c}}U(\langle \alpha\beta \rangle(G)) = G{}^{\text{c}}U(\langle \gamma\delta \rangle(G))$. Hence,

(1) $U(\{ \alpha \}(G)) \cup U(\{ \alpha\beta \}(G)) = U(\{ \gamma \}(G)) \cup U(\{ \gamma\delta \}(G))$.

Case 1. $\alpha = \beta$. Then $U(\{ \alpha \}(G)) = U(\{ \gamma \}(G)) \cup U(\{ \gamma\delta \}(G))$, so $\{ \alpha \}(G) = \{ \gamma\delta \}(G)$. Hence, $U(\alpha) = U(\gamma) \;\; U(\delta)$, so $\alpha = \gamma = \delta = \beta$.

Case 2. $\alpha \neq \beta$.

Subcase 1. $\gamma = \delta$. Then by Case 1, $\alpha = \beta$, a contradiction.

Subcase 2. $\gamma \neq \delta$. If $\{ \gamma\delta \}(G) \varepsilon U(\{ \alpha \}(G))$, then $\gamma = \delta$, a contradiction. So by (1), $\{ \gamma\delta \}(G) \varepsilon U(\{ \alpha\beta \}(G))$. Hence,

(2) $U(\gamma) \cup U(\delta) = U(\alpha) \cup U(\beta)$.

If $\{ \gamma \}(G) \varepsilon U(\{ \alpha\beta \}(G))$, then $\alpha = \beta$, a contradiction. So by (1), $\{ \gamma \}(G) \varepsilon \{ \alpha \}(G)$. Hence,

(3) $\alpha = \gamma$.

Then by (2), (3), and the hypothesis $\alpha \neq \beta$,

(4) $\beta = \delta.$

T49. $\vdash (x):.x \subseteq NO_G.(E\alpha).\alpha \varepsilon NO.(\beta).\beta \varepsilon x \supset \beta<\alpha: \supset :(Ey).y \varepsilon L.x \subseteq y.$

Proof. Assume $x \subseteq NO_G.\alpha \varepsilon NO.(\beta).\beta \varepsilon x \supset \beta<\alpha.$ Let $\gamma \varepsilon Val(J_G).\alpha<\gamma.$ Then by T41, $x \subseteq G`U(\gamma)$. So let $y=G`U(\gamma)$.

T50. $\vdash (x):\mathcal{L}(x).(E\alpha).\alpha \varepsilon NO.(\beta).\beta \varepsilon x \supset \beta<\alpha: \supset :x \varepsilon L.$

Proof. By T49, $(Ey).y \varepsilon L.x \subseteq y.$ Then since $\mathcal{L}(x)$, $x \cap y \varepsilon L.$ But $x \cap y=x.$ So $x \varepsilon L.$

T51. $\vdash \bigwedge \varepsilon L.$

T52. $\vdash \mathcal{L}(NO_G).$

Proof. Obviously, $NO_G \subseteq NO_G.$ Suppose $x \varepsilon L.$ Then by T43, $x \subseteq NO_G.$ So $NO_G \cap x=x.$ Hence, $NO_G \cap x \varepsilon L.$

T53. $\vdash \mathcal{L}(E_g).$

Proof. By D45, $E_g \subseteq NO_G.$ Suppose $z \varepsilon L.$ Then $z \cap E_g \varepsilon L$ by T41.

T54. $\vdash (x,y).\mathcal{L}(x,y) \supset \mathcal{L}(x \cap \overline{y}).$

Proof. Suppose $\mathcal{L}(x,y).$ $x \cap \overline{y} \subseteq x$, so since $\mathcal{L}(x)$, $x \cap \overline{y} \subseteq NO_G.$ Suppose $z \varepsilon L.$ Then since $\mathcal{L}(x,y)$, $z \cap x \varepsilon L$ and $z \cap y \varepsilon L.$ Hence, by T42, $(z \cap x) \cap \overline{(z \cap y)} \varepsilon L.$ But $(z \cap x) \cap \overline{(z \cap y)} = z \cap (x \cap \overline{y}).$ So $z \cap (x \cap \overline{y}) \varepsilon L.$

T55. $\vdash (x,y).\mathcal{L}(x,y) \supset \mathcal{L}(x \cap y).$

Proof. $x \cap y = x \cap \overline{x \cap \overline{y}}.$ Use T54.

T56. $\vdash (y).\mathcal{L}(y) \supset \mathcal{L}((V \times y)_g).$

Proof. By D49, $(V \times y)_g \subseteq NO_G.$ Suppose $z \varepsilon L.$ We wish to show that $z \cap (V \times y)_g \varepsilon L.$

Let $u=\hat{\nu}_G((E\eta,\mu)_G.\eta\varepsilon z.\eta=\langle\mu\nu\rangle(G).\nu\varepsilon y)$, so that $u\subseteq y$.

Let $\gamma\varepsilon NO_G.z=G`U(\gamma)$. Then by T38, $(\beta).\beta\varepsilon z\supset\beta<\gamma$. Suppose $\nu\varepsilon u$. Then $\eta\varepsilon z$. $\eta=\langle\mu\nu\rangle(G)$. By T40.4, $\nu<\eta$, so $\nu<\gamma$. Hence $(E\gamma).\gamma\varepsilon NO.(\beta).\beta\varepsilon u\supset\beta<\gamma$. So by T49, $(Ew).w\varepsilon L.u\subseteq w$. $\mathcal{L}(y)$, so by D56, $w\cap y\varepsilon L$. Obviously $u\subseteq w\cap y\subseteq y$, so $(Ew_o).w_o\varepsilon L.u\subseteq w_o\subseteq y$.

Since $w_o\subseteq y$, $z\cap(V\times w_o)_g\subseteq z\cap(V\times y)_g$. Now suppose $\alpha\varepsilon z\cap(V\times y)_g$. Then $\alpha\varepsilon z.(E\gamma,\delta)_G.\alpha=\langle\gamma\delta\rangle(G).\delta\varepsilon y$. Hence, $\alpha\varepsilon z.(E\gamma,\delta)_G:\alpha=\langle\gamma\delta\rangle(G).(E\eta,\mu).\eta\varepsilon z.\eta=\langle\mu\delta\rangle(G).\delta\varepsilon y$. That is, $\alpha\varepsilon z\cap(V\times u)_g$. But $u\subseteq w_o$, so $\alpha\varepsilon z\cap(V\times w_o)_g$. Hence, $z\cap(V\times y)_G\subseteq z\cap(V\times w_o)_g$.

Thus $z\cap(V\times y)_g=z\cap(V\times w_o)_g$. By T42, $z\cap(V\times w_o)_g\varepsilon L$. Hence $z\cap(V\times y)_g\varepsilon L$.

† T57. $\vdash(y).\mathcal{L}(y)\supset\mathcal{L}(D_g(y))$.

Proof. Assume $z\varepsilon L$, and let $u=\hat{\nu}_G((E\gamma,\delta).\gamma\varepsilon z.\delta=\min_{\leq}\hat{\delta}_G(\langle\delta\gamma\rangle(G)\varepsilon y).\nu=\langle\delta\gamma\rangle(G))$. Then $(Eu_o).u_o\subseteq z.u\ sm\ u_o$.

Let $\gamma\varepsilon NO_G.z=G`U(\gamma)$. Then by T38, $(\beta).\beta\varepsilon z\supset\beta<\gamma$, so by T28, $\sim(z\ sm\ NO)$. But $z\subseteq NO$, hence $\sim(u_o\ sm\ NO)$. So since $u\ sm\ u_o$, $\sim(u\ sm\ NO)$. Then by T27, $(E\gamma).\gamma\varepsilon NO.(\beta).\beta\varepsilon u\supset\beta<\gamma$, so by T49, $(Ew_o).w_o\varepsilon L.u\subseteq w_o$. From $\mathcal{L}(y)$ we have $w_o\cap y\varepsilon L$, and obviously $u\subseteq w_o\cap y\subseteq y$. Thus $(Ew).w\varepsilon L.u\subseteq w\subseteq y$.

Since $w\subseteq y$, $z\cap D_g(w)\subseteq z\cap D_g(y)$. Suppose $\alpha\varepsilon z\cap D_g(y)$. Then $\alpha\varepsilon z.(E\delta)_G.\langle\delta\alpha\rangle(G)\varepsilon y$. Hence, $\alpha\varepsilon z.(E\delta)_G.\delta=\min_{\leq}\hat{\delta}_G(\langle\delta\alpha\rangle(G)\varepsilon y)$, so $(E\delta,\gamma)_G.\gamma\varepsilon z.\delta=\min_{\leq}\hat{\delta}_G(\langle\delta\gamma\rangle(G)\varepsilon y).\langle\delta\alpha\rangle(G)=\langle\delta\gamma\rangle(G)$. So $(E\mu)_G(E\delta,\gamma)_G.\gamma\varepsilon z.\delta=\min_{\leq}\hat{\delta}_G(\langle\delta\gamma\rangle(G)\varepsilon y).\langle\mu\alpha\rangle(G)=\langle\delta\gamma\rangle(G)$. That is, $\alpha\varepsilon z\cap D_g(u)$. So $\alpha\varepsilon D_g(w)$. Hence $z\cap D_g(y)\subseteq z\cap D_g(w)$.

Thus $z\cap D_g(y)=z\cap D_g(w)$. But $z\cap D_g(w)\varepsilon L$. So $z\cap D_g(y)\varepsilon L$.

T58. $\vdash (y).\mathcal{L}(y)\supset\mathcal{L}(Cnv_{1g}(y))$.

Proof. Let $u=\hat{\nu}_G(\nu\varepsilon y.(E\gamma,\delta)_G.\nu=\langle\gamma\delta\rangle(G).\langle\delta\gamma\rangle(G)\varepsilon z)$. Then proceed as in T56.

T59. $\vdash (y).\mathcal{L}(y)\supset\mathcal{L}(Cnv_{2g}(y))$.

Proof. Similar to that of T58.

T60. $\vdash (y).\mathcal{L}(y)\supset\mathcal{L}(Cnv_{3g}(y))$.

Proof similar to that of T58.

T61. $\vdash (x,y):\mathcal{L}(x,y).\supset.\hat{\alpha}_G((E\gamma)_G.\gamma\varepsilon x.\langle\alpha\gamma\rangle(G)\varepsilon y)=D_g(Cnv_{1g}(y\cap(V\times x)_g))$.

Proof. Use D49, D50, D51, and T48.

T62. $\vdash (x,y).\mathcal{L}(x,y)\supset\mathcal{L}(\hat{\alpha}_G((E\gamma)_G.\gamma\varepsilon x.\langle\alpha\gamma\rangle(G)\varepsilon y)$.

Proof. Use T56, T55, T58, T57, and T61.

T63. $\vdash (\alpha).\alpha\varepsilon Val(J_0)\supset\alpha\varepsilon NO_G$.

Proof. The theorem is obvious when $\alpha=0$. Assume the theorem for all $\beta\lessgtr\alpha$, and suppose $\alpha\varepsilon Val(J_0).\alpha\sim\varepsilon NO_G$. Then $G`U(\alpha)=G`U(\gamma)$, $\gamma<\alpha$. So by hypothesis, $\gamma\varepsilon NO_G$. Then by T41, $\gamma\varepsilon G`U(\alpha)$. Hence $\gamma\varepsilon G`U(\gamma)$, contradicting T38.

T64. $\vdash (\alpha,\beta):\alpha\varepsilon NO_G.\beta\varepsilon Val(J_0).\alpha<\beta.\supset.G`U(\alpha)\subseteq G`U(\beta).G`U(\alpha)\neq G`U(\beta)$.

Proof. By T38 and T41, $G`U(\alpha)\subseteq G`U(\beta)$. By T41, $\alpha\varepsilon G`U(\beta)$, so by T38, $G`U(\alpha)\neq G`U(\beta)$.

T65. $\vdash (\alpha):\alpha<\omega_o.\supset.(E\beta).\beta\varepsilon Val(J_0).\alpha<\beta<\omega_o.$

Proof. Use T18, T15.

We are now in a position to construct a model of L_1 in L_2. We have already defined $\{x,y\}_g$ (D46). We now define:

D58. $x\varepsilon_g y$=df $x\varepsilon L.Ind(x)\varepsilon y.$

D59. $(x)_g p$=df $(x).\mathcal{L}(x)\supset p.$

D46, D58, and D59, respectively, furnish us with correspondents for the expressions $\{x,y\}$, $x\varepsilon y$, and $(x)p$ of L_1. We now define by induction a unique correspondent in L_2 for each noun and proposition of L_1.

Definition of L_2-correspondents of nouns and propositions of L_1.

1) If x is the n^{th} variable of L_1, the L_2-correspondent of x, x_g, is the n^{th} variable [of type 6] of L_2.

2) If x and y are nouns of L_1, then the L_2-correspondent of $(x\varepsilon y)$ is $(x_g\varepsilon_g y_g)$, and the L_2-correspondent of $\{x,y\}$ is $\{x_g,y_g\}_g$.

3) If p and q are propositions of L_1, p_g and q_g are the L_2-correspondents of p and q respectively, and x is a variable of L_1, then the L_2-correspondent of $(p|q)$ is $(p_g|q_g)$, and the L_2-correspondent of $((x)p)$ is $((x_g)_g p_g)$.

The bracketed expression in 1) above is relevant only if L_2 is a theory of types, and should otherwise be omitted. The restriction to type 6 is not essential, and any higher type would do as well (see p. 46).

If p is a proposition of L_1, p_g is the L_2-correspondent of p, and $x_1, x_2, \cdots, x_n$ are the L_2-correspondents of the free variables of p, then we call $\mathcal{L}(x_1, x_2, \cdots, x_n) \supset p_g$ the L_2-<u>image</u> of p. If a proposition of L_1 contains no free variables, then the L_2-correspondent and the L_2-image of the proposition are the same. We shall show that the L_2-images of the propositions of L_1 constitute a model of L_1 in L_2.

D60. $(Ex)_g p =df \sim (x)_g \sim p.$

D61. $M_g(x) =df (Ey)_g . x \varepsilon_g y.$

D62. $(x)_\sigma\ p =df (x) . M_g(x) \supset p.$

D63. $(Ex)_\sigma . p =df \sim (x)_\sigma \sim p.$

D64. $x \subseteq_g y =df (z)_\sigma . z \varepsilon_g x \supset z \varepsilon_g y.$

D65. $x =_g y =df (z)_\sigma . z \varepsilon_g x \equiv z \varepsilon_g y.$

D66. $x \neq_g y =df \sim (x =_g y).$

D67. $Un_g(x) =df (u,v,w)_\sigma : \langle uv \rangle_g \varepsilon_g x . \langle wv \rangle_g \varepsilon_g x . \supset . u =_g w.$

Note that (assuming the variables involved to be the appropriate variables of type 6, if L_2 is the theory of types), the definiens of D46, D60-D67 are the L_2-correspondents of the definiens of DS1-DS9. This is important in that in constructing the L_2-correspondent of a proposition of L_1 which contains non-primitive expressions (e.g., $\langle xy \rangle$, $p \equiv q$, etc.), we want actually to obtain the L_2-correspondent of that formula as expressed in primitive notation.

Since, except from the standpoint of having the L_2-correspondents of propositions of L_1 be unique, it is irrelevant which variable [of type 6] corresponds to a variable of L_1, we shall denote the L_2-correspondent of a variable x simply by x, it being assumed that the appropriate variable has been chosen.

The following two theorems follow immediately from T43.

T67. $\vdash (x,y):\mathcal{L}(x,y).\supset.x=y\equiv x=_g y.$

T68. $\vdash (x,y):\mathcal{L}(x,y).\supset.x\subseteq y\equiv x\subseteq_g y.$

T69. $\vdash (x).M_g(x)\equiv x\varepsilon L.$

Proof. Use T52.

We now prove the L_2-images of the axioms of groups A, B, C, and D of L_1.

T70. (L_2-image of A1). $\vdash (x,y,z)_g:x=_g y.\supset.x\varepsilon_g z\supset y\varepsilon_g z.$

Proof. Use T67.

T71. (L_2-image of A2). $\vdash (x,y,u)_\sigma:u\varepsilon_g \{x,y\}_g.\equiv.u=_g x\vee u=_g y.$

Proof. Start with $x=G`U(\alpha),y=G`U(\beta),u=G`U(\gamma),$ $\alpha,\beta,\gamma\varepsilon NO_G$. Then $u\varepsilon_g \{x,y\}_g.\equiv.\gamma\varepsilon U(\alpha)\cup U(\beta).\equiv.\gamma=\alpha\vee\gamma=\beta.$ $\equiv.u=x\vee u=y.\equiv.u=_g x\vee u=_g y.$

T72. (L_2-image of A3). $\vdash (x,y)_\sigma (Ez)_\sigma (u)_\sigma :u\varepsilon z.\equiv.u=_g x\vee u=_g y.$

Proof. Take z to be $\{x,y\}_g$ and use T71 and T42.

T73. (L_2-image of B1). $\vdash (Ez)_g(x,y)_\sigma .\langle xy\rangle_g\varepsilon_g z\equiv x\varepsilon_g y.$

Proof. Take z to be E_g. Then $\mathcal{L}(z)$ by T53, and $\langle xy\rangle_g\varepsilon_g z\equiv x\varepsilon_g y$ by D45, T40.4.

T74. (L_2-image of B2). $\vdash (x,y)_g(Ez)_g(u)_\sigma :u\varepsilon_g z.\equiv.u\varepsilon_g x.u\varepsilon_g y.$

Proof. Take z to be $x \cap y$, and use T55.

T75. (L_2-image of B3). $\vdash (x)_g (Ey)_g (u)_\sigma . u\varepsilon_g y \equiv u \sim \varepsilon_g x.$

Proof. Take y to be $NO_G \cap \overline{x}$, and use T52 and T54.

T76. (L_2-image of B4). $\vdash (x)_g (Ey)_g (u)_\sigma : u\varepsilon_g y. \equiv . (Ez)_\sigma . \langle zu \rangle_g \varepsilon_g x.$

Proof. Take y to be $D_g(x)$, and use T57.

T77. (L_2-image of B5). $\vdash (x)_g (Ey)_g (u,v)_\sigma . \langle vu \rangle_g \varepsilon_g y \equiv u\varepsilon_g x.$

Proof. Take y to be $(V \times x)_g$, and use T56.

T78. (L_2-image of B6). $\vdash (x)_g (Ey)_g (u,v)_\sigma . \langle uv \rangle_g \varepsilon_g y \equiv \langle vu \rangle_g \varepsilon_g x.$

Proof. Take y to be $Cnv_{1g}(x)$, and use T58.

T79. (L_2-image of B7). $\vdash (x)_g (Ey)_g (u,v,w)_\sigma . \langle uvw \rangle_g \varepsilon_g y \equiv \langle vwu \rangle_g \varepsilon_g x.$

Proof. Take y to be $Cnv_{2g}(x)$, and use T59.

T80. (L_2-image of B8). $\vdash (x)_g (Ey)_g (u,v,w)_\sigma . \langle uvw \rangle_g \varepsilon_g y \equiv \langle uwv \rangle_g \varepsilon_g x.$

Proof. Take y to be $Cnv_{3g}(x)$, and use T60.

T81. (L_2-image of C1). $\vdash (Eu)_\sigma :. (Ez)_\sigma . z\varepsilon_g u . (x)_\sigma : x\varepsilon_g u . \supset . (Ey)_\sigma . y\varepsilon_g u . x \subseteq_g y . x \neq_g y.$

Proof. Take u to be $G{}^{\backprime}U(\omega_0)$. Then by T36 and T41, $u = \hat{\alpha}_G(\alpha < \omega_0)$. By T51, $G{}^{\backprime}U(0)\varepsilon_g u$. Now suppose $x = G{}^{\backprime}U(\alpha), \alpha\varepsilon NO_G$, and $\alpha\varepsilon u$. Then $\alpha < \omega_0$. Using T65, let $\beta = \min_{\leq_G} \hat{\beta}(\beta\varepsilon Val(J_0) . \alpha < \beta < \omega_0)$. By T41, $G{}^{\backprime}U(\beta)\varepsilon_g u$, and by T64, $G{}^{\backprime}U(\alpha) \subseteq G{}^{\backprime}U(\beta) . G{}^{\backprime}U(\alpha) \neq G{}^{\backprime}U(\beta)$. Hence by T63, $(Ey)_\sigma . y\varepsilon_g u . x \subseteq_g y . x \neq_g y.$

T82. (L_2-image of C2). $\vdash (x)_\sigma (Ey)_\sigma (u,v)_\sigma :$ $u \varepsilon_g v . v \varepsilon_g x . \supset . u \varepsilon_g y$.

Proof. Suppose $x = G`U(\alpha), \alpha \varepsilon NO_G$. Let $\delta = \min_{\leq} \hat{\delta}(\delta \varepsilon Val(J_0) . \alpha < \delta)$, and take y to be $G`U(\delta)$. Now suppose $u = G`U(\mu), \mu \varepsilon NO_G$, $v = G`U(\nu), \nu \varepsilon NO_G$, and $\mu \varepsilon G`U(\nu) . \nu \varepsilon G`U(\alpha)$. Then by T38, $\mu < \alpha$, so by T41, $\mu \varepsilon y$. That is, $u \varepsilon_g y$.

†T83. (L_2-image of C3). $\vdash (x)_\sigma (z)_g :. Un_g(z) :$ $\supset : (Ey)_\sigma (u)_\sigma : u\varepsilon_g y . \equiv . (Ev)_\sigma . v \varepsilon_g x . \langle uv\rangle_g \varepsilon_g z$.

Proof. Suppose $x = G`U(\gamma), \gamma\varepsilon NO_G, \mathcal{L}(z)$, and $Un_g(z)$. Then $(\mu,\nu,\xi)_G : \langle\mu\nu\rangle(G)\varepsilon z . \langle\xi\nu\rangle(G)\varepsilon z . \supset . \mu = \xi$, by D67. Now let $y = \hat{\alpha}_G((E\delta)_G . \delta\varepsilon x . \langle\alpha\delta\rangle(G)\varepsilon z)$. Then $(Ew) . y \text{ sm } w . w \subseteq x$. By T38 and T28, $\sim(x \text{ sm } NO)$. Hence $\sim(y \text{ sm } NO)$. So by T62 and T50, $y\varepsilon L$. But $(\alpha)_G : \alpha\varepsilon y .$ $\equiv . (E\delta)_G . \delta\varepsilon x . \langle\alpha\delta\rangle(G)\varepsilon z$. Hence, $(u)_\sigma : u\varepsilon_g y . \equiv . (Ev)_\sigma . v\varepsilon_g x . \langle uv\rangle_g \varepsilon_g z$.

In order to prove the L_2-image of C4 we must now make the additional assumption mentioned previously. We assume then that the following theorem is provable in L_2:

††T84. $\vdash (\gamma) :. \gamma\varepsilon NO : \supset : (E\beta) : \beta\varepsilon NO . (\delta) . \delta\varepsilon\hat{\alpha}_G(G`U(\alpha)$ $\subseteq G`U(\gamma)) \supset \delta < \beta$.

Intuitively this says that the class of indices of the subclasses of a class in L is bounded. The theorem is easily proved in L_1 as follows (numerical references are to theorems in [5]):

It is easily shown (in L_1) that

$\hat{\alpha}_G(G`U(\alpha) \subseteq G`U(\gamma))$ sm $\hat{x}(x \varepsilon L . x \subseteq G`U(\gamma))$. Also, $\hat{x}(x \varepsilon L . x \subseteq G`U(\gamma)) \subseteq \hat{x}(x \subseteq G`U(\gamma))$. $G`U(\gamma)$ is a set, so by 5.121, $\hat{x}(x \subseteq G`U(\gamma))$ is a set. Hence by 5.12 and 5.1, $\hat{\alpha}_G(G`U(\alpha) \subseteq G`U(\gamma))$ is a set. The theorem follows by 7.451.

††T85. (L_2-image of C4.) $\vdash (x)_\sigma (Ey)_\sigma (u)_\sigma .$ $u \subseteq_g x \supset u \varepsilon_g y.$

Proof. Suppose $\gamma \varepsilon NO$, $x = G`U(\gamma)$. Using T38 and T84, let $\beta = min_{\leq} \hat{\beta}(\beta \varepsilon Val(J_0) . (\delta) . \delta \varepsilon \hat{\alpha}_G(G`U(\alpha) \subseteq G`U(\gamma)) \supset \delta < \beta)$. Take y to be $G`U(\beta)$, so that $y \varepsilon L$. Now suppose $u = G`U(\mu)$, $\mu \varepsilon NO_G$ and $u \subseteq_g x$. By T38, $G`U(\mu) \subseteq G`U(\gamma)$. Hence $\mu < \beta$, so $\mu \varepsilon G`U(\beta)$. That is, $u \varepsilon_g y$.

T86. (L_2-image of D1). $\vdash (x)_g : . (Ey)_\sigma . y \varepsilon_g x :$ $\supset : (Eu)_\sigma : u \varepsilon_g x . \sim (Ez)_\sigma . z \varepsilon_g u . z \varepsilon_g x.$

Proof. Let $x = G`U(\alpha)$, $\alpha \varepsilon NO_G$, and suppose $(E\beta) . \beta \varepsilon G`U(\alpha)$. Let $\gamma = min_{\leq} \hat{\delta}(\delta \varepsilon G`U(\alpha))$.

Now suppose $\varphi \varepsilon G`U(\gamma) . \varphi \varepsilon G`U(\alpha)$. Then by T38, $\varphi \varepsilon G`U(\alpha) . \varphi < \gamma$, contradicting $\gamma = min_{\leq} \hat{\delta}(\delta \varepsilon G`U(\alpha))$.

Thus we can take $G`U(\gamma)$ to be the required u.

There remain now only the axioms $P_1, P_2, \ldots, P_n$. Since the L_2-correspondent of a proposition of L_1 is a proposition of L_2, the L_2-correspondents of the axioms of the propositional calculus for L_1 are obviously provable in L_2.

Consider the following typical axioms for the lower functional calculus for L_1:

P_1. $(x).p\supset q:\supset:p\supset(x)q$, where x is a variable which does not occur free in p.

P_2. $(x)p(x).\supset.p(y)$, where x and y are variables, and x does not occur free in a y-bound part of p.

In L_2 the following are easily proved:

T87. $\vdash (x)_g.p\supset q:\supset:p\supset(x)_g q$, where ... .

T88. $\vdash (x)_g p(x).\supset.\mathcal{L}(y)\supset p(y)$, where ... .

That T88 is not the L_2-correspondent of P_2 is unimportant, since it is in any case adequate for the proof of the following theorem:

T89. If p is any proposition of L_1, $x_1,\cdots,x_h$ are the L_2-correspondents of the free variables of p, p_g is the L_2-correspondent of p, and $\vdash_1 p$, then $\vdash\mathcal{L}(x_1,\cdots,x_h)\supset p_g$.

Proof. Let $q_1,q_2,\cdots,q_n$, where q_n is p, be a demonstration in L_1 of p, and let $y_1,y_2,\cdots,y_s$ be the L_2-correspondents of all the free variables occurring in $q_1,q_2,\cdots,q_n$. We first prove by induction on i that $\vdash \mathcal{L}(y_1,y_2,\cdots,y_s)\supset(q_i)_g$ for $1\leq i\leq n$, where $(q_i)_g$ is the L_2-correspondent of q_i.

Suppose i=1. Then q_i is an axiom of L_1. We have two cases.

Case 1. q_i is not P_2. Then we have already shown that $\vdash(q_i)_g$. Hence, $\vdash \mathcal{L}(y_1,y_2,\cdots,y_s)\supset(q_i)_g$.

Case 2. q_i is P_2, i.e., q_i is $(x)r(x)\supset r(y_j)$. By T88, $\vdash (x)_g(r(x))_g.\supset.\mathcal{L}(y_j)\supset(r(y_j))_g$. Hence, $\vdash \mathcal{L}(y_1,y_2,\cdots,y_s)\supset(q_i)_g$.

Now suppose $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) \supset (q_i)_g$ for $1 \leq i < t$.

Case 1. q_t is an axiom of L_1. Then as above, $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) \supset (q_t)_g$.

Case 2. There are j and k, $1 \leq j < t$, $1 \leq k < t$, such that q_k is $q_j \supset q_t$. Then by hypothesis, $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) . \supset . (q_j)_g \supset (q_t)_g$ and $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) \supset (q_j)_g$. Hence, $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) \supset (q_t)_g$.

Case 3. There is a j, $1 \leq j < t$, such that q_t is $(z)q_j$. By hypothesis, $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) \supset (q_j)_g$. Hence, $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) . \supset . \mathcal{L}(z) \supset (q_j)_g$. If z is not one of $y_1, y_2, \cdots, y_s$ we obtain at once $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) . \supset . (z)_g (q_j)_g$. If z is y_k, we have $\vdash \mathcal{L}(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_s) . \supset . \mathcal{L}(z) \supset (q_j)_g$, so $\vdash \mathcal{L}(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_s) \supset (z)_g (q_j)_g$. Hence, $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) \supset (z)_g (q_j)_g$.

This completes the induction. Thus $\vdash \mathcal{L}(y_1, y_2, \cdots, y_s) \supset p_g$. Hence, $\vdash \mathcal{L}(y_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_k}) . \supset . \mathcal{L}(x_1, x_2, \cdots, x_h) \supset p_g$ where $y_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_k}$ are the variables of the set $y_1, y_2, \cdots, y_s$ which do not occur free in p. Then $\vdash (Ey_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_k}) \mathcal{L}(y_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_k}) . \supset . \mathcal{L}(x_1, \cdots, x_h) \supset p_g$. But $\vdash (Ey_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_k}) \mathcal{L}(y_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_k})$. Hence $\vdash \mathcal{L}(x_1, \cdots, x_h) \supset p_g$.

It follows from T89 and the definition of L_2-images that the L_2-images of the propositions of L_1 form a model of L_1 in L_2. So if L_1 is inconsistent, then L_2 is inconsistent; i.e., if L_2 is consistent, then L_1 is consistent.

Part II

§1. The Use of the Theory of Types for L_2. In this section we show that a logic obtained by adding two axioms (Axioms A and B, p. 64) to the simple theory of types is adequate for L_2; i.e., that in such a logic T1 to T88 are provable.

As an example of the simple theory of types we can take the system P of [6], supplemented by ι-axioms from which T3, T4, and T5 are provable. (That such supplementation does not introduce a contradiction may be shown by methods similar to those used in [10] and [8].)

For convenience we assume that non-negative integers have been defined of type 1 and type 3, and that the axiom of infinity is provable for individuals of type 1 (and hence of each type). Using Quine's device,[6] the ordered pair <xy> can then be defined so as to have the same type as x and y if x and y are of type 3. Relations can then be defined so that they are of type 4, and ordinals can be defined so that they are of type 5. Then since there are integers of type 3, ordered pairs of ordinals can be defined so as to be of the same type as the ordinals.

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6. [7].

Classes of ordinals and relations between ordinals or pairs of ordinals will be of type 6.

We shall denote the class of all ordinals by NO and the ordering relation for ordinals by $\leq$, and it is assumed that all occurrences of NO and $\leq$ are of type 6. Unless an explicit statement concerning types is made, it is assumed that all variables and constants other than NO and $\leq$ are of appropriate types to make the formulas in which they occur meaningful. It is assumed that ιxp is of the same type as x, and that the x appearing explicitly in ιxp is of the same type as the free occurrences of x in p (of which there will always be at least one). Similarly, $\hat{x}p$ is one type higher than x, and the x appearing explicitly in $\hat{x}p$ is of the same type as the free occurrences of x in p (of which there will always be at least one).

We make use of the definitions of §2 plus the following:

D101. $seg_x R = df\ \hat{y}\hat{z}(yRz.zRx.z \neq x).$

D102. $LE = df\ \hat{R}\hat{S}((x){:}x \epsilon C(R). \supset .(Ey).y \epsilon C(S).$
$(seg_x R)\, smor (seg_y S)).$

D103. $Nr(P)$ for $\hat{R}(P\ smor\ R).$

D104. $NO = df\ \hat{\alpha}((EP).P \epsilon \Omega . \alpha = Nr(P)).$

D105. $\leq = df\ \hat{\alpha}\hat{\beta}((EP,Q).P,Q \epsilon \Omega . \alpha = Nr(P). \beta = Nr(Q).P(LE)Q).$

D106. $Prod(x) = df\ \hat{z}((y).y \epsilon x \supset z \epsilon y).$

D107. $Sum(x) = df\ \hat{z}((Ey).y \epsilon x.z \epsilon y).$

D108. $R``x = df\ Val(x \upharpoonleft R).$

D109. $u \times w =_{df} \hat{z}((Ex,y).x \varepsilon u.y \varepsilon w.z = \langle xy \rangle)$.

D110. $Cn(\alpha) =_{df} \hat{x}((ER).R \varepsilon \Omega.\alpha = Nr(R).x \text{ sm } C(R))$.

D111. $Nc(x) =_{df} \hat{y}(y \text{ sm } x)$.

D112. $\leq_c =_{df} \hat{u}\hat{w}((x,y):x \varepsilon u.y \varepsilon w.\supset.(Ez).z \subseteq y.x \text{ sm } z)$.

D113. $<_c =_{df} \hat{u}\hat{w}(u \leq_c w.u \neq w)$.

D114. $SC(x) =_{df} \hat{z}(z \subseteq x)$.

D115. $Nc(u) \times_c Nc(w) =_{df} Nc(u \times w)$.

D116. $y \text{ Closed}(R)_2 =_{df} R``y \subseteq y$.

D117. $y \text{ Closed}(S)_3 =_{df} R``(y \times y) \subseteq y$.

D118. $\text{Closure}(y,(R)_2,(S)_3) =_{df} \text{Prod}(\hat{z}(y \subseteq z.$
$R``z \subseteq z.R``(z \times z) \subseteq z))$.

Closure $(y,(R)_2,(S)_3)$ is the smallest class containing y which is closed with respect to the dyadic relation R and the triadic relation S.

We assume without proof all theorems of §2 which are not marked with asterisks, and also †801- †835 of [11] (with obvious modifications so that they will be applicable to the theory of types). Of †801- †835, the following are used frequently:

†812. $\vdash (R,S,x):R \varepsilon \Omega.S \subseteq seg_x R.x \varepsilon C(R).\supset.\sim(S \text{ smor } R)$.

†818. $\vdash (R,S):R,S \varepsilon \Omega.(R)LE(S).(S)LE(R).\supset.R \text{ smor } S$.

†821. $\vdash (R,S):R,S \varepsilon \Omega.\supset.(R)LE(S) \vee (S)LE(R)$.

†822. $\vdash (R,S):.R,S \varepsilon \Omega:\supset:\sim(S)LE(R).\equiv.(Ey).y \varepsilon C(S).$
$R \text{ smor } (seg_y S)$.

†835. $\vdash \leq \varepsilon \Omega$.

Well known theorems concerning classes, ordinals, relations, and functions will be used without proof or statement.

The following theorems are easily proved:

T101. $\vdash(P):P\varepsilon\Omega.\supset.rCan(P)\varepsilon\Omega.$

T102. $\vdash(P):P\varepsilon 1\text{-}1.\supset.rCan(P)\varepsilon 1\text{-}1.$

T103. $\vdash(P):Arg(rCan(P))=Can(Arg(P)).$

T104. $\vdash(P):Val(rCan(P))=Can(Val(P)).$

T105. $\vdash(P,Q):P\ smor\ Q.\equiv.(rCan(P))smor(rCan(Q)).$

T106. $\vdash(x,y):x\ sm\ y.\equiv.Can(x)\ sm\ Can(y).$

Our first concern is with the problem of definition by induction. It is no more difficult to prove the theorems which follow than it is to prove the weaker theorems which we actually need. And the theorems which follow are, with respect to restrictions on free and bound variables, somewhat weaker than could be proved if we wanted.

T107. If

1) the following variables are distinct and are of the types indicated beneath them, where m and n are arbitrary positive integers:

$\alpha,\ z,\ R,\ u,\ y,\ v,\ w$

$m,\ m,\ m{+}1,\ n,\ n,\ n,\ n{+}1;$

2) ξ_0 is a variable or noun of type n, and none of the above variables occurs free in ξ_0;

3) q is a proposition which contains no bound occurrences of α, v, w, or R and no free or bound occurrences of u, y, or z;

then there is a proposition p such that:

$$\vdash (R) :: R \varepsilon \Omega :. \supset :. (\alpha) : \alpha \varepsilon C(R) . \supset . (E_1 u) p :.$$
$$\iota u \{S\alpha p\} (\iota z(z = \min_R C(R))) = \xi_0 :. (\alpha) : \alpha \varepsilon C(R) .$$
$$\alpha \neq \iota z(z = \min_R C(R)) . \supset . \iota up = \iota v \{Swq\} (\hat{y}((Ez) . zR\alpha . z \neq \alpha .$$
$$y = \iota u \{S\alpha p\} (z))) .$$

We shall abbreviate $\iota v \{Swq\}(t)$ by $\theta(t)$.

Suppose n+1=m+j, and let $k > |j|$. Assume $R \varepsilon \Omega$, and let $\mu = \iota z(z = \min_R C(R))$. Let $\beta, \gamma, \delta, \xi, x$ be new variables of types m, k+n+2, k+n+2, n, k+n+1 respectively. Let

$$Pres_\gamma(\alpha) = \hat{y}((Ez) . zR\alpha . z \neq \alpha . \langle U^{k+j}(z), U^{k+1}(y) \rangle \varepsilon \gamma)$$ and

$$A = \hat{x}((\gamma) :: \langle U^{k+j}(\mu), U^{k+1}(\xi) \rangle \varepsilon \gamma :. (\alpha, \delta) :. \alpha \varepsilon C(R) .$$
$$\alpha \neq \mu . (\beta) : \beta R \alpha . \beta \neq \alpha . \supset . (Ey) . \langle U^{k+j}(\beta), U^{k+1}(y) \rangle \varepsilon \gamma \cap \delta :$$
$$\supset : \langle U^{k+j}(\alpha), U^{k+1}(\theta(Pres_{\gamma \cap \delta}(\alpha))) \rangle \varepsilon \gamma :. \supset :. x \varepsilon \gamma) .$$

From the hypotheses 1), 2), 3) and the choice of new variables above, it can be seen that the formulas occurring in $Pres_{\gamma \cap \delta}(\alpha)$ and A are propositions and that there is no confusion of bound variables in $\iota v \{Swq\} (Pres_{\gamma \cap \delta}(\alpha))$. Also, $Pres_{\gamma \cap \delta}(\alpha)$ is of type n+1, the same type as that of w.

We now give a sequence of lemmas from which the theorem follows.

Lemma 1. $(v, \xi) : \langle U^{k+j}(\mu), U^{k+1}(v) \rangle \varepsilon A . \equiv . v = \xi .$

Proof. Suppose $\langle U^{k+j}(\mu),U^{k+1}(v)\rangle \varepsilon A$ and $v \neq \xi$. Take γ to be $V-U(\langle U^{k+j}(\mu),U^{k+1}(v)\rangle)$. This leads to a contradiction.

Lemma 2. $(\xi,\alpha):.\alpha\varepsilon C(R).\alpha\neq\mu.(\beta):\beta R\alpha.\beta\neq\alpha.\supset.(Ey).\langle U^{k+j}(\beta),U^{k+1}(y)\rangle\varepsilon A:\supset:\langle U^{k+j}(\alpha),U^{k+1}(\theta(Pres_A(\alpha)))\rangle\varepsilon A.$

Lemma 3. $(\alpha,\xi):\alpha\varepsilon C(R).\alpha\neq\mu.\supset.\langle U^{k+j}(\alpha),U^{k+1}(\theta(Pres_A(\alpha)))\rangle\varepsilon A.$

Proof by induction on α, using Lemma 1 and Lemma 2.

Lemma 4. $(\alpha,\xi):.\alpha\varepsilon C(R).\alpha\neq\mu:\supset:(y):\langle U^{k+j}(\alpha),U^{k+1}(y)\rangle\varepsilon A.\equiv.y=\theta(Pres_A(\alpha)).$

Proof. By Lemma 3, we have $(\alpha,\xi):.\alpha\varepsilon C(R).\alpha\neq\mu:\supset:(y):y=\theta(Pres_A(\alpha)).\supset.\langle U^{k+j}(\alpha),U^{k+1}(y)\rangle\varepsilon A.$

To prove the implication from right to left, suppose α is the least member of $C(R)$ for which the theorem is false; so suppose

(1) $\alpha\varepsilon C(R).\alpha\neq\mu.$

(2) $(\nu):.\nu R\alpha.\nu\neq\alpha.\nu\neq\mu:\supset:(y):\langle U^{k+j}(\nu),U^{k+1}(y)\rangle\varepsilon A.\equiv.y=\theta(Pres_A(\nu)).$

(3) $\langle U^{k+j}(\alpha),U^{k+1}(y)\rangle\varepsilon A.$

(4) $y\neq\theta(Pres_A(\alpha)).$

From (3) we get

(5) $(\gamma)::\langle U^{k+j}(\mu),U^{k+1}(\xi)\rangle\varepsilon\gamma:.(\varphi,\delta):.\varphi\varepsilon C(R).\varphi\neq\mu.(\beta):\beta R\varphi.\beta\neq\varphi.\supset.(Ey).\langle U^{k+j}(\beta),U^{k+1}(y)\rangle\varepsilon\gamma\cap\delta:\supset:\langle U^{k+j}(\varphi),U^{k+1}(\theta(Pres_{\gamma\cap\delta}(\varphi)))\rangle\varepsilon\gamma:.\supset:.\langle U^{k+j}(\alpha),U^{k+1}(y)\rangle\varepsilon\gamma.$

Now let $\gamma_o = (V - \hat{\lambda}((E\eta,\rho): \eta R\alpha . \lambda = \langle U^{k+j}(\eta), U^{k+1}(\rho)\rangle)) \cup \hat{\lambda}((E\eta,\rho): \eta R\ \alpha . \eta \neq \alpha . \lambda = \langle U^{k+j}(\eta), U^{k+1}(\rho)\rangle . \lambda \varepsilon A) \cup U(\langle U^{k+j}(\alpha), U^{k+1}(\theta(Pres_A(\alpha)))\rangle)$.

By (1), we have

(6) $\langle U^{k+j}(\mu), U^{k+1}(\xi)\rangle \varepsilon\ \gamma_o$.

Suppose

(7) $\varphi \varepsilon C(R) . \varphi \neq \mu . (\beta): \beta R \varphi . \beta \neq \varphi . \supset . (Ey) . \langle U^{k+j}(\beta), U^{k+1}(y)\rangle \varepsilon\ \gamma_o \cap \delta$.

We then have three cases:

Case I. $\sim \varphi R \alpha$.

Case II. $\varphi R \alpha . \varphi \neq \alpha$.

Case III. $\varphi = \alpha$.

In each case it is found that

(8) $\langle U^{k+j}(\varphi), U^{k+1}(\theta(Pres_{\gamma_o \cap \delta}(\varphi)))\rangle \varepsilon\ \gamma_o$.

From (8), (6), (7), (5) we get

(9) $\langle U^{k+j}(\alpha), U^{k+1}(y)\rangle \varepsilon\ \gamma_o$.

By (9) and (4) we have a contradiction.

Lemma 5. $(\alpha,\xi): \alpha \varepsilon C(R) . \supset . (E_1 u) . \langle U^{k+j}(\alpha), U^{k+1}(u)\rangle \varepsilon A$.

Proof. Use Lemma 1, 3, 4.

Now define $p = \{S\ \xi(\langle U^{k+j}(\alpha), U^{k+1}(u)\rangle \varepsilon A)\}(\xi_o)$. The theorem now follows from replacing ξ by ξ_o in Lemmas 1, 3, 5.

T108. If

1) R, α, z are distinct variables of types m+1, m, m respectively;

2) j and k are non-negative integers, and w is a variable of type n, distinct from R, α, z, where $n=m+j+1$ and $n>k+1$;

3) F is a variable of type n, and v is a variable of type $n-k-1$;

4) ξ is a variable or noun of type $m+j$, and none of the above variables occurs free in ξ;

5) q is a proposition such that there is no confusion of bound variables in $\iota v\{Swq\}((Can^j(C(seg_\alpha R)))\upharpoonright F)$;

then:

$\vdash (R):.R\varepsilon\Omega:\supset:(EF):F\varepsilon Fnc.Arg(F)=Can^j(C(R)).$
$F{}^{\backprime}U^j(\iota z(z=min_R C(R)))=\xi:(\alpha):\alpha\varepsilon C(R).\alpha\neq\iota z(z=min_R C(R)).$
$\supset.F{}^{\backprime}U^j(\alpha)=U^k(\iota v\{Swq\}((Can^j(C(seg_\alpha R)))\upharpoonright F).$

Proof. Start with $R\varepsilon\Omega$. By T107, there is a θ such that
$\theta(\iota z(z=min_R C(R)))=\langle U^j(\iota z(z=min_R C(R))),\xi\rangle:(\alpha):\alpha\varepsilon C(R).$
$\alpha\neq\iota z(z=min_R C(R)).\supset.\theta(\alpha)=\langle U^j(\alpha),U^k(\iota v\{Swq\}(x((E\beta).$
$\beta R\alpha.\beta\neq\alpha.x=\theta(\beta))))\rangle.$

Let $F=\hat{x}((E\gamma).\gamma\varepsilon C(R).x=\theta(\gamma))$. It is easily seen that $F\varepsilon Fnc$, $Arg(F)=Can^j(C(R))$, and $F{}^{\backprime}U^j(\iota z(z=min_R C(R)))=\xi$. Suppose $\alpha\varepsilon C(R)$ and $\alpha\neq\iota z(z=min_R C(R))$. Then $F{}^{\backprime}U^j(\alpha)=U^k(\iota v\{Swq\}(\hat{x}((E\beta).\beta R\alpha.\beta\neq\alpha.x=\theta(\beta))))$. But $\hat{x}((E\beta).\beta R\alpha.\beta\neq\alpha.x=\theta(\beta))=(Can^j(C(seg_\alpha R)))\upharpoonright F$. Hence $F{}^{\backprime}U^j(\alpha)=U^k(\iota v\{Swq\}((Can^j(C(seg_\alpha R)))\upharpoonright F))$.

T37 of §3 now follows from T108.

T109. $\vdash(R):R\varepsilon\Omega.\supset.Cn(Nr(R))=Nc(C(R)).$

T110. $\vdash (x,y):Nc(x)\leq_c Nc(y).Nc(y)\leq_c Nc(x).\supset.Nc(x)=Nc(y).$

T111. $\vdash (\alpha,\beta):\alpha,\beta\varepsilon NO.Cn(\alpha)\leq_c Cn(\beta).Cn(\beta)\leq_c Cn(\alpha).$ $\supset.Cn(\alpha)=Cn(\beta).$

T112. $\vdash (\alpha,\beta):\alpha,\beta\varepsilon NO.\supset.Cn(\alpha)<_c Cn(\beta).\vee.$ $Cn(\alpha)=Cn(\beta).\vee.Cn(\beta)<Cn(\alpha).$

T110, T111, and T112 follow from D110-D113 by means of the Schröder-Bernstein theorem. A proof of the Schröder-Bernstein theorem may be found in Principia Mathematica, *73.88.

We assume that a well-ordering relation, $\leq n$, for the non-nègative integers has been defined; since there are integers of type 3, $\leq n$ can be of type 4. We define

D119. $\omega_o=Nr(\leq n)$

and assume the following theorems:

T113. $\vdash\omega_o\varepsilon NO.$

T114. $\vdash (\alpha):\alpha<\omega_o.\supset.\alpha+1<\omega_o.$

T115. $\vdash (\alpha):0<\alpha<\omega_o.\supset.(E_1\beta).\beta<\omega_o.\alpha=\beta+1.$

T113-T115 may be proved by methods similar to those used in Principia Mathematica, *262, *263.

T116. There is a noun, $\theta(\beta)$, such that

$\vdash \theta(0)=\omega_o$

$\vdash (\beta):\beta\varepsilon NO.\beta\neq 0.\supset.\theta(\beta)=min_{\leq}\hat{\alpha}(\alpha\varepsilon NO.(\gamma):\gamma<\beta.$ $\supset.Cn(\theta(\gamma))<_c Cn(\alpha)).$

Proof. Use T107.

Note that $\theta(\beta)$ is of the same type as β.

D120. Denote the $\theta(\beta)$ of T116 by ω_β.

Note that if $\sim(E\alpha).\alpha\varepsilon NO.(\gamma):\gamma<\beta.\supset.Cn(\omega_\gamma)<_c Cn(\alpha)$, then $\omega_\beta=V$. It is easily seen that $\vdash(\alpha,\beta):\alpha<\beta.\omega_\alpha=V.\supset.\omega_\beta=V$.

D121. $\beth_\beta =df\ Nc(\alpha(\alpha<\omega_\beta))$.

D122. $\aleph_\beta =df\ Cn(\omega_\beta)$.

T117. $\vdash(\alpha,\beta):\alpha\leq\beta.\supset.Cn(\alpha)\leq_c Cn(\beta)$.

T118. $\vdash(\alpha,\beta):\alpha<\beta.\omega_\beta\neq V.\supset.\omega_\alpha<\omega_\beta$.

Proof. Use D120.

T119. $\vdash(\alpha,\beta):\alpha<\beta.\omega_\beta\neq V.\supset.\omega_\alpha<_c\omega_\beta$.

Proof. Use D122, D120.

One of the most useful theorems of classical ordinal theory[7] is "$\vdash(P):P\varepsilon\Omega.\supset.P\ smor(seg_{Nr(P)}\leq)$". Unfortunately, this theorem leads at once to the Burali-Forti paradox. The theorem which follows can frequently be used for the purposes for which the above incorrect theorem is used in the classical theory.

T121. $\vdash(P,\alpha):P\varepsilon\Omega.\alpha=Nr(P).\supset.(rCan^2(P))\ smor(seg_\alpha\leq)$.

Proof by induction on α. If $\alpha=0$ the theorem is obvious. Assume $(\beta):\beta<\alpha.\beta=Nr(R).\supset.(rCan^2(R))\ smor(seg_\beta\leq)$, and suppose $x\varepsilon C(rCan^2(P))$. Then $x=U^2(y),y\varepsilon C(P)$, so that $Nr(seg_y P)<\alpha$. Hence, by hypothesis, $(rCan^2(seg_y P))\ smor(seg_{Nr(seg_y P)}\leq)$. But $rCan^2(seg_y P)=seg_y rCan^2(P)$,

7. See [2], p. 187; [12], p. 171.

and $(\mathrm{seg}_{\mathrm{Nr}(\mathrm{seg}_y P)}\leq)=(\mathrm{seg}_{\mathrm{Nr}(\mathrm{seg}_y P)}(\mathrm{seg}_{\alpha}\leq))$. Hence $(\mathrm{seg}_x(\mathrm{rCan}^2(P)))\,\mathrm{smor}(\mathrm{seg}_{\mathrm{Nr}(\mathrm{seg}_y P)}(\mathrm{seg}_{\alpha}\leq))$. So

(1) $(\mathrm{rCan}^2(P))\mathrm{LE}(\mathrm{seg}_{\alpha}\leq)$.

Now suppose $\beta\varepsilon C(\mathrm{seg}_{\alpha}\leq)$. Then $\beta<\alpha$, so $(Ey).\beta=\mathrm{Nr}(\mathrm{seg}_y P)$. By hypothesis, $(\mathrm{rCan}^2(\mathrm{seg}_y P))\,\mathrm{smor}(\mathrm{seg}_{\beta}\leq)$. But $\mathrm{rCan}^2(\mathrm{seg}_y P)=\mathrm{seg}_{U^2(y)}(\mathrm{rCan}^2(P))$, and $(\mathrm{seg}_{\beta}\leq)=(\mathrm{seg}_{\beta}(\mathrm{seg}_{\alpha}\leq))$. Hence $(\mathrm{seg}_{\beta}(\mathrm{seg}_{\alpha}\leq))\,\mathrm{smor}(\mathrm{seg}_{U^2(y)}(\mathrm{rCan}^2(P)))$. So

(2) $(\mathrm{seg}_{\alpha}\leq)\mathrm{LE}(\mathrm{rCan}^2(P))$.

From (1) and (2) we have $(\mathrm{rCan}^2(P))\,\mathrm{smor}(\mathrm{seg}_{\alpha}\leq)$.

T122. $\vdash(\alpha):\alpha\varepsilon\mathrm{NO}.\supset.\sim(\hat{\beta}(\beta<\alpha)\,\mathrm{sm}\ \mathrm{NO})$.

Proof. Suppose $\alpha\varepsilon\mathrm{NO}$ and $\hat{\beta}(\beta<\alpha)\,\mathrm{sm}_R\mathrm{NO}$. Let $\alpha=\mathrm{Nr}(P)$. Then by T121, $\mathrm{rCan}^2(P)\,\mathrm{smor}_S(\mathrm{seg}_{\alpha}\leq)$.

Let $Q=\hat{x}\hat{y}(x,y\varepsilon C(P).R`S`U^2(x)\leq R`S`U^2(y))$. Then $Q\varepsilon\Omega$, $C(Q)=C(P)$, and $\mathrm{rCan}^2(Q)\,\mathrm{smor}_{S|R}\leq$. But $\mathrm{rCan}^2(Q)\,\mathrm{smor}(\mathrm{seg}_{\mathrm{Nr}(Q)}\leq)$. Hence $(\mathrm{seg}_{\mathrm{Nr}(Q)}\leq)\,\mathrm{smor}\leq$, a contradiction.

T123. (T28.) $\vdash(x):.x\subseteq\mathrm{NO}.(E\beta).\beta\varepsilon\mathrm{NO}.(\alpha).\alpha\varepsilon x\supset\alpha<\beta:\supset:\sim(x\ \mathrm{sm}\ \mathrm{NO})$.

Proof. Use T122.

T124. $\vdash(\alpha,\beta):\alpha,\beta\varepsilon\mathrm{NO}.\omega_{\beta}\leq\alpha<\omega_{\beta+1}.\ \ \mathrm{Cn}(\alpha)=\aleph_{\beta}$.

Proof. Use D120.

T125. $\vdash(\alpha,\beta):\alpha,\beta\varepsilon\mathrm{NO}.\omega_{\beta+1}=V.\omega_{\beta}\leq\alpha.\supset.\mathrm{Cn}(\alpha)=\aleph_{\beta}$.

Proof. If $\omega_{\beta}=\alpha$ the theorem is obvious, so suppose $\omega_{\beta}<\alpha$. Let $\omega_{\beta}=\mathrm{Nr}(P)$, $\alpha=\mathrm{Nr}(Q)$. If $\mathrm{Cn}(\omega_{\beta})=\mathrm{Cn}(\alpha)$ the theorem is proved, so suppose $\mathrm{Cn}(\omega_{\beta})<_c\mathrm{Cn}(\alpha)$. Let

$\gamma = \min_{\leq} \hat{\alpha}(Cn(\omega_\beta) <_c Cn(\alpha))$. Then by D120, $\gamma = \omega_{\beta+1}$ contradicting the hypothesis of the theorem.

T126. $\vdash (\alpha) :: \omega_0 \leq \alpha : . \supset : . (E_1 \beta) : . \beta \varepsilon NO . \omega_\beta \leq \alpha . (\gamma) : \beta < \gamma . \supset . \alpha < \omega_\gamma . \vee . \omega_\gamma = V$.

Proof by induction on α: Let α be the least ordinal such that

(1) $\omega_0 \leq \alpha . (\beta) : \omega_\beta \leq \alpha . \supset . (E\gamma) . \beta < \gamma . \omega_\gamma \leq \alpha$, and let $A = \hat{\delta}(\omega_\delta \neq V . \omega_\delta \leq \alpha)$.

Case 1. $A = NO$. We have $(\delta) . \delta \varepsilon \hat{\delta}(\omega_\delta \neq V . \omega_\delta \leq \alpha) \supset \delta < \alpha$. Hence by T123, $\sim (A \text{ sm } NO)$, a contradiction.

Case 2. $A \neq NO$. Let $\mu = \min_{\leq} \bar{A}$, and suppose $\beta < \mu$. Then $\omega_\beta \neq V$ and $\omega_\beta \leq \alpha$, so that $Cn(\omega_\beta) \leq Cn(\alpha)$. But by (1), $(E\gamma_0) . \beta < \gamma_0 . \omega_{\gamma_0} \leq \alpha$. So by T119, $Cn(\omega_\beta) <_c Cn(\omega_{\gamma_0}) \leq_c Cn(\alpha)$.

Thus $(\beta) . \beta < \mu \supset Cn(\omega_\beta) <_c Cn(\alpha)$. Then $\omega_\mu = \min_{\leq} \hat{\alpha}(\alpha \varepsilon NO . (\beta) . \beta < \mu \supset Cn(\omega_\beta) <_c Cn(\alpha))$, so $\omega_\mu \neq V . \omega_\mu \leq \alpha$ contradicting $\mu = \min_{\leq} \bar{A}$.

Thus (1) leads to a contradiction, so $(E\beta) : . \beta \varepsilon NO . \omega_\beta \leq \alpha . (\gamma) : \beta < \gamma . \supset . \alpha < \omega_\gamma . \vee . \omega_\gamma = V$. The uniqueness of β is obvious.

T127. $\vdash (\alpha) : \omega_0 \leq \alpha . \supset . (E\beta) . \beta \varepsilon NO . \omega_\beta \neq V . Cn(\alpha) = \aleph_\beta$.

Proof. Using T126, let $\beta = \max_{\leq} \hat{\beta}(\omega_\beta \leq \alpha)$. Then by T124 or T125, $Cn(\alpha) = \aleph_\beta$.

T129. $\vdash (\alpha, \beta) : . \alpha, \beta \varepsilon NO . \omega_\beta \neq V : \supset : Cn(\alpha) = \aleph_\beta . \equiv . Nc(\hat{\gamma}(\gamma < \alpha)) = \beth_\beta$.

Proof. Assume $\alpha,\beta\varepsilon NO, \omega_\beta \neq V$, and let $\alpha = Nr(P)$, $\omega_\beta = Nr(Q)$. Then by T121, $rCan^2(P)\, smor(seg_\alpha \leq)$ and $rCan^2(Q)\, smor(seg\, \omega_\beta \leq)$.

Suppose $Cn(\alpha) = \aleph_\beta = Cn(\omega_\beta)$. Then $Nc(C(P)) = Nc(C(Q))$, so $Nc(C(rCan^2(P))) = Nc(C(rCan^2(Q)))$. Hence $Nc(\hat{\gamma}(\gamma < \alpha)) = \beth_\beta$.

Suppose $Nc(\hat{\gamma}(\gamma<\alpha)) = \beth_\beta = Nc(\hat{\gamma}(\gamma < \omega_\beta))$. Then $Nc(C(rCan^2(P))) = Nc(C(rCan^2(Q)))$, so $Nc(C(P)) = Nc(C(Q))$. Hence, $Cn(\alpha) = \aleph_\beta$.

T130. $\vdash (\alpha,\beta) : \alpha,\beta\varepsilon NO . \omega_\beta \leq \alpha < \omega_{\beta+1} . \supset . Nc(\hat{\gamma}(\gamma<\alpha)) = \beth_\beta$.

T131. $\vdash (\alpha,\beta) : \alpha,\beta\varepsilon NO . \omega_{\beta+1} = V . \omega_\beta \leq \alpha . \supset . Nc(\hat{\gamma}(\gamma<\alpha)) = \beth_\beta$.

Proofs of T130 and T131. Under either hypothesis we have, by T124, T125, $Cn(\alpha) = \aleph_\beta$. So by T129, $Nc(\hat{\gamma}(\gamma<\alpha)) = \beth_\beta$.

T131.1. $\vdash (\alpha) : \omega_0 \leq \alpha . \supset . (E\beta) . \beta\varepsilon NO . \omega_\beta \neq V . Nc(\hat{\gamma}(\gamma<\alpha)) = \beth_\beta$.

Proof. From T126, it follows that there is a β such that the hypothesis of either T130 or T131 is satisfied.

T132. $\vdash (x,y) : Nc(x) <_c Nc(y) . \equiv . Nc(Can^2(x)) <_c Nc(Can^2(y))$.

Proof. Obvious.

T133. $\vdash (\alpha,\beta) : \alpha < \beta . \omega_\beta \neq V . \supset . \beth_\alpha <_c \beth_\beta$.

Proof. Let $\omega_\alpha = Nr(P)$, $\omega_\beta = Nr(Q)$. Then $rCan^2(P)\, smor(seg\, \omega_\alpha \leq)$, so $\beth_\alpha = Nc(Can^2(C(P)))$. Similarly, $\beth_\beta = Nc(Can^2(C(Q)))$. By T109, $Nc(C(P)) <_c Nc(C(Q))$. Hence by T132, $\beth_\alpha <_c \beth_\beta$.

In order to prove T16-T19, which concern an ordering relation, $\leq_t$, for ordered triples of ordinals, $\langle\mu\alpha\beta\rangle$, with $\mu \leq \beta$, we first define ordering relations $\leq_p$, $\leq_d$ for ordered pairs of ordinals. $\leq_p$ will be seen to well-order ordered pairs of ordinals, $\langle\alpha\beta\rangle$, first according to the maximum of α and β, then according to β, and then according to α. $\leq_d$ well-orders ordered pairs of ordinals, $\langle\mu\alpha\rangle$, with $\mu\leq 8$, first according to α and then according to μ. $\leq_t$ is then defined so as to well-order ordered triples, $\langle\mu\alpha\beta\rangle$, $\mu\leq 8$, first according to the ordering by $\leq_p$ of $\langle\alpha\beta\rangle$, and then according to the ordering by $\leq$ of μ. It is easily shown that $\leq_p \varepsilon\,\Omega$ and $\leq_d \varepsilon\,\Omega$, and it then follows that $\leq_t \varepsilon\,\Omega$.

D121. $\leq_p =_{df} \hat{u}\hat{w}((E\alpha,\beta,\gamma,\delta):.\alpha,\beta,\gamma,\delta\varepsilon NO.u=\langle\alpha\beta\rangle.$ $w=\langle\gamma\delta\rangle:.(\max_{\leq}(U(\alpha)\cup U(\beta)))<(\max_{\leq}(U(\gamma)\cup U(\delta))):$ $\vee :(\max_{\leq}(U(\alpha)\cup U(\beta)))=(\max_{\leq}(U(\gamma)\cup U(\delta))):\beta<\delta.\vee.\beta=\delta.\alpha\leq\gamma).$

D122. $<_p =_{df} \hat{u}\hat{w}(u\leq_p w.u\neq w).$

D123. $\leq_d =_{df} \hat{u}\hat{w}((E\alpha,\beta,\mu,\nu):\alpha,\beta\varepsilon NO.\mu,\nu\leq 8.$ $u=\langle\mu\alpha\rangle.w=\langle\nu\beta\rangle:\alpha<\beta.\vee.\alpha=\beta.\mu\leq\nu).$

D124. $\leq_t =_{df} \hat{u}\hat{w}((E\alpha,\beta,\gamma,\delta,\mu,\nu):\alpha,\beta,\gamma,\delta\varepsilon NO.$ $\mu,\nu\leq 8.u=\langle\mu\alpha\beta\rangle.w=\langle\nu\gamma\delta\rangle:\langle\alpha\beta\rangle<_p\langle\gamma\delta\rangle.\vee.\langle\alpha\beta\rangle=\langle\gamma\delta\rangle.\mu\leq\nu).$

D125. $<_t =_{df} \hat{u}\hat{w}(u\leq_t w.u\neq w).$

T134. $\vdash \leq_p\varepsilon\,\Omega.\leq_d\varepsilon\,\Omega.\leq_t\varepsilon\,\Omega.$

Proof. It is easily proved (by cases) that $\leq_p\varepsilon\,\Omega$ and $\leq_d\varepsilon\,\Omega$. A similar proof then shows that $\leq_t\varepsilon\,\Omega$.

T16-T18 now follow from D134 and T134.

In order to prove T19 we first prove some auxiliary theorems.

T135. $\vdash (\alpha):\alpha\varepsilon NO.\supset.(seg_{\alpha}\leq)LE(seg_{\langle\alpha 0\rangle}\leq_p).$

T135.1. $\vdash (\alpha):\alpha\varepsilon NO.\supset.(seg_{\alpha}\leq)LE(seg_{<0\alpha>}\leq_{d})$.

Proof of T135, by induction: T135 is obvious when $\alpha=0$. Suppose $(\beta):\beta<\alpha.\supset.(seg_{\beta}\leq)LE(seg_{<\beta 0>}\leq_{p})$, and

$\sim((seg_{\alpha}\leq)LE(seg_{<\alpha 0>}\leq_{p}))$. Then $(seg_{<\alpha 0>}\leq_{p})smor(seg_{\gamma}\leq)$, where $\gamma<\alpha$. But by hypothesis, $(seg_{\gamma}\leq)LE(seg_{<\gamma 0>}\leq_{p})$. Hence $(seg_{<\alpha 0>}\leq_{p})LE(seg_{<\gamma 0>}\leq_{p})$. So $\alpha\leq\gamma$, a contradiction.

T135.1 is proved similarly.

T136. $\vdash (seg\,\omega_{o}\leq)smor(seg_{<\omega_{o},0>}\leq_{p})$.

T136.1. $\vdash (seg\,\omega_{o}\leq)smor(seg_{<0,\omega_{o}>}\leq_{d})$.

Proof of T136. By T135, $(seg\,\omega_{o}\leq)LE(seg_{<\omega_{o},0>}\leq_{p})$.

Suppose $x\varepsilon C(\leq_{p})$ and $(seg\,\omega_{o}\leq)smor_{F}seg_{x}(seg_{<\omega_{o},0>}\leq_{p})$.

Then $x=<\alpha\beta>$, $\alpha<\omega_{o}$, $\beta<\omega_{o}$, and $(seg\,\omega_{o}\leq)smor_{F}(seg_{<\alpha\beta>}\leq_{p})$.

Obviously, $\sim(\alpha=0.\beta=0)$.

Let $\beta-1=df\ \iota\alpha(\alpha\varepsilon NO.\alpha+1=\beta)$, and let

$y=\iota z(\beta<\alpha.\beta\neq 0.z=<\alpha,\beta-1>.\vee.\beta=0.z=<\alpha-1,\alpha-1>.\vee.$
$\alpha\leq\beta.\alpha\neq 0.z=<\alpha-1,\beta>.\vee.\alpha=0.z=<\beta,\beta-1>)$.

Then $y=max_{\leq_{p}}\hat{z}(z<_{p}<\alpha\beta>)$. Let $\breve{F}'y=m<\omega_{o}$. Then $y<_{p}F'm+1$, so $F'm+1\sim\varepsilon C(seg_{<\alpha\beta>}\leq_{p})$, which contradicts $m+1\ \varepsilon\ C(seg\,\omega_{o}\leq)$.

Thus $(seg_{<\omega_{o},0>}\leq_{p})LE(seg\,\omega_{o}\leq)$. So

$(seg\,\omega_{o}\leq)smor(seg_{<\omega_{o},0>}\leq_{p})$.

Thus 136.1 is proved similarly.

T137. $\vdash (\alpha):\alpha\varepsilon NO.\omega_\alpha\neq V.\supset.(seg\,\omega_\alpha\leq)\, smor(seg_{<\omega_\alpha,0>}\leq_p)$.

T137.1. $\vdash (\alpha):\alpha\varepsilon NO.\omega_\alpha\neq V.\supset.(seg\,\omega_\alpha\leq)\, smor(seg_{<0,\omega_\alpha>}\leq_d)$.

Proof of T137, by induction. When $\alpha=0$ the theorem follows from T136. Assume $(\beta):\beta<\alpha.\ {}_\beta\neq V.$ $\supset.(seg\,\omega_\beta\leq)\, smor(seg_{<\omega_\beta,0>}\leq_p)$.

By T135,

(1) $(seg\,\omega_\alpha\leq) LE(seg_{<\omega_\alpha,0>}\leq_p)$.

Suppose $x<_p<\omega_\alpha,0>$, and $(seg\,\omega_\alpha\leq)\, smor(seg_x\leq_p)$. Then $x=<\gamma\,\delta>$, $\gamma<\omega_\alpha$, $\delta<\omega_\alpha$. Let $\xi=max_\leq(U(\gamma)\cup U(\delta))+1$. Then $\xi<\omega_\alpha$. Since $\gamma<\xi$, $\delta<\xi$, we have

(2) $Nc(\hat{x}(x<_p<\gamma\,\delta>))\leq_c Nc(\hat{x}(x<_p<\xi,0>))$.

Using T126, let $\varphi=max_\leq\hat{\varphi}(\varphi\varepsilon NO.\omega_\varphi\leq\xi)$. Then by T118, $\varphi<\alpha$, so by hypothesis of induction, $(seg\,\omega_\varphi\leq)\, smor(seg_{<\omega_\varphi,0>}\leq_p)$. Hence,

(3) $\aleph_\varphi=Nc(\hat{x}(x<_p<\omega_\varphi,0>))$. By T124,

(4) $Nc(\hat{\beta}(\beta<\xi))=\aleph_\varphi$.

So from (3) and (4),

(5) $Nc(\hat{x}(x<_p<\xi,0>))=Nc(\hat{x}(x<_p<\omega_\varphi,0>))=\aleph_\varphi$. But since $(seg\,\omega_\alpha\leq)\, smor(seg_{<\gamma\,\delta>}\leq_p)$, we have

(6) $Nc(\hat{x}(x<_p<\gamma\,\delta>))=\aleph_\alpha$.

Then from (2), (5), and (6) we get $\aleph_\alpha\leq_c\aleph_\varphi$, contradicting $\varphi<\alpha$.

T137.1 is proved similarly.

T138. $\vdash (\leq_p)\, smor(\leq)$.

T138.1. $\vdash (\leq_d) \mathrm{smor} (\leq)$.

Proof of T138. By T135, $(\leq) \mathrm{LE} (\leq_p)$. Suppose $x = \langle \alpha\beta \rangle$, and $(\leq) \mathrm{smor} (\mathrm{seg}_{\langle \alpha\beta \rangle} \leq_p)$.

Let $\gamma = \max_{\leq} (U(\alpha) \cup U(\beta)) + 1$, so that $\langle \alpha\beta \rangle <_p \langle \gamma 0 \rangle$. Let $\varphi = \max_{\leq} \hat{\varphi} (\varphi \varepsilon \mathrm{NO} . \omega_\varphi \leq \gamma)$. Then by T124 or T125, $\mathrm{Nc}(\hat{\delta}(\delta < \omega_\varphi)) = \mathrm{Nc}(\hat{\delta}(\delta < \gamma))$. Hence, $\mathrm{Nc}(\hat{x}(x <_p \langle \omega_\varphi, 0 \rangle)) = \mathrm{Nc}(\hat{x} <_p \langle \gamma 0 \rangle))$. But by T137, $\aleph_\varphi = \mathrm{Nc}(\hat{x}(x <_p \langle \omega_\varphi, 0 \rangle))$, and by hypothesis, $\mathrm{Nc}(\mathrm{NO}) = \mathrm{Nc}(\hat{x}(x <_p \langle \alpha\beta \rangle)) \leq_c \mathrm{Nc}(\hat{x}(x <_p \langle \gamma, 0 \rangle))$. Hence $\mathrm{Nc}(\mathrm{NO}) \leq_c \aleph_\varphi$, contradicting T123.

T138.1 is proved similarly.

T139. $\vdash (\leq_t) \mathrm{smor} (\leq)$.

T139.1. $\vdash (\alpha) : \alpha \varepsilon \mathrm{NO} . \omega_\alpha \neq V . \supset . (\mathrm{seg}\, \omega_\alpha \leq) \mathrm{smor} (\mathrm{seg}_{\langle 0, \omega_\alpha, 0 \rangle} \leq_t)$.

Proof of T139 and T139.1. Suppose $(\leq_p) \mathrm{smor}_R (\leq)$ and $(\leq_d) \mathrm{smor}_S (\leq)$. Let

$H = \hat{u}\hat{w}((E \mu, \alpha, \beta) : \mu \leq 8 . \alpha, \beta \varepsilon \mathrm{NO} . u = \langle \mu \alpha\beta \rangle . w = \langle \mu, R`\langle \alpha\beta \rangle \rangle)$.

Schematically, when $\mu \leq 8$, $\alpha, \beta \varepsilon \mathrm{NO}$, and '$\xleftrightarrow{P}$' means 'is carried into by P', we have

$$\langle \mu \alpha\beta \rangle \xleftrightarrow{H} \langle \mu, R`\langle \alpha\beta \rangle \rangle \xleftrightarrow{S} S`\langle \mu, R`\langle \alpha\beta \rangle \rangle.$$

Thus it is clear that from T138 and T138.1 one can derive $(\leq_t) \mathrm{smor}_{H|S} (\leq)$. T139.1 follows from T137 and T137.1.

T140. $\vdash (\alpha) : \alpha \varepsilon \mathrm{NO} . \supset . (\mathrm{seg}_\alpha \leq) \mathrm{LE} (\mathrm{seg}_{\langle 0 \alpha 0 \rangle} \leq_t)$.

Proof. Use T135, T135.1.

We now proceed to the proofs of T32 through T36. For the definitions of $J, J_0, \ldots, J_8, K_1, K_2$ see p. 16.

T141. $\vdash(\alpha):\alpha\varepsilon NO.\supset.\alpha\leq J`\langle 0\alpha 0\rangle$.

Proof. By T140, $(seg_\alpha\leq)LE(seg_{\langle 0\alpha 0\rangle}\leq_t)$; i.e., $(seg_\alpha\leq)LE(seg_{J`\langle 0\alpha 0\rangle}\leq)$.

T142. (T32) $\vdash(\alpha,\beta,\mu):\alpha,\beta\varepsilon NO.\mu\leq 8.\supset.max_\leq(U(\alpha)\cup U(\beta))\leq J`\langle\mu\alpha\beta\rangle$.

Proof. Let $\gamma=max_\leq(U(\alpha)\cup U(\beta))$. Then $\langle 0\gamma 0\rangle\geq_t\langle\mu\alpha\beta\rangle$. So $J`\langle 0\gamma 0\rangle\leq J`\langle\mu\alpha\beta\rangle$. Use T141.

T143. (T33) $\vdash(\alpha,\beta,\mu):\alpha,\beta\varepsilon NO.0<\mu\leq 8.\supset.max_\leq(U(\alpha)\cup U(\beta))<J`\langle\mu\alpha\beta\rangle$.

T144. (T34) $\vdash(\alpha):\alpha\varepsilon NO.\supset.K_1`\alpha\leq\alpha.K_2`\alpha\leq\alpha$.

Proof. Let $\alpha=J`\langle\mu\beta\gamma\rangle$. Then $K_1`\alpha=\beta$, $K_2`\alpha=\gamma$. Use T142.

T145. (T35) $\vdash(\alpha):\alpha\varepsilon NO.\alpha\sim\varepsilon Val(J_0).\supset.K_1`\alpha<\alpha.K_2`\alpha<\alpha$.

T146. $\vdash(\alpha):\alpha\varepsilon NO.\omega_\alpha\neq V.\supset.\omega_\alpha\varepsilon Val(J_0)$.

Proof. By T139.1, $\omega_\alpha=J`\langle 0,\omega_\alpha,0\rangle$.

T36 is a special case of T146.

We have now proved all the propositions of §2 which are marked with an asterisk except T27 and T84. It appears unlikely that either of these propositions is provable in the system of type theory so far described. Thus, from the point of view of using as weak a system for L_2 as possible, the best thing we can do is to add T27 and T84 themselves to L_2 as axioms. While this procedure has certain advantages (in particular, it reduces the length and number of necessary proofs to a minimum), it is perhaps of interest to show that T27 and T84 can be derived from more well known propositions of classical ordinal theory.

We therefore add the following axioms to L_2:

Axiom A. $(x):.x \subseteq NO.(E\alpha).\alpha\varepsilon NO.\omega_\alpha \neq V.Nc(x)=\aleph_\alpha:\supset:$ $(Ey)(\beta):\beta\varepsilon x.\supset.(E_1P).P\varepsilon\beta.\beta\varepsilon y.$

Axiom B. $(\alpha):\alpha\varepsilon NO.\supset.(E\beta).\beta\varepsilon NO.\alpha<\omega_\beta$

Axiom A is a weak form of the axiom of choice, and Axiom B is a trivial consequence of the classical theorem of ordinal theory which asserts that ω_α exists for every ordinal α. In fact, both Axiom A and Axiom B are provable in classical ordinal theory without use of the axiom of choice. The standard proofs of both, however, depend on a theorem from which the Burali-Forti paradox is an immediate consequence.[8] Thus the classical proofs of Axioms A and B are not valid in type theory, and we are forced to assume these propositions as axioms.

As pointed out previously, Axiom B contradicts strong forms of the axiom of choice; namely, any axiom from which the well-ordering theorem for arbitrary classes is provable. For let P be a well-ordering of V (V of type 3), and let $\alpha=Nr(P)$. Then obviously $(\beta).\beta\varepsilon NO\supset Cn(\beta)\leq_c Cn(\alpha)$, so if we let $\gamma=\min_{\leq}\hat{\beta}(Cn(\beta)=Cn(\alpha))$, then γ will be the largest omega, contradicting Axiom B.

In spite of the above, one can add reasonably strong forms of the axiom of choice to the logic consisting of the theory of types plus Axioms A and B without (apparently)

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8. See [2], p. 193; [12], pp. 214, 215.

introducing a contradiction. An example of such an axiom of choice is

$$(x):.(u).u\varepsilon x\supset u\neq\Lambda:(u,v,w):u,v\varepsilon x.w\varepsilon u\cap v.\supset.$$
$$u=v:(E\alpha).\alpha\varepsilon NO.\ \omega_\alpha\neq V.Nc(x)=\aleph_\alpha:\supset:(Ey)(u):u\varepsilon x.\supset.(E_1z).z\varepsilon u\cap y.$$

Since the hypothesis of this axiom implies that x is already well-ordered, it appears unlikely that any very significant well-ordering theorems can be proved from this axiom; in particular, it appears most unlikely that one can well-order by means of this axiom, the continuum, the universe, and various other classes about whose well-ordering there is considerable dispute. Nevertheless, such an axiom would probably be adequate for most of the uses made of the axiom of choice in classical mathematics, except for uses for the explicit purpose of well-ordering.

In any case, T27 now follows from Axiom A.

T147. (T27.) $\vdash(x):x\subseteq NO.\sim(x\ sm\ NO).\supset.$
$(E\alpha).\alpha\varepsilon NO.(\beta).\beta\varepsilon x\supset\beta<\alpha.$

Proof. Let $S=\hat{\alpha}\hat{\beta}(\alpha,\beta\varepsilon x.\alpha\leq\beta)$. Then from $(\leq)LE(S)$ we get $(x\ sm\ NO)$, so $\sim((\leq)LE(S))$. Hence $S\ smor(seg_\gamma\leq)$.

If $\gamma<\omega_0$, then the theorem is obvious. So suppose $\omega_0\leq\gamma$. Then by T131.1, $(E\delta).\delta\varepsilon NO.\ \omega_\delta\neq V.Nc(\hat{\beta}(\beta<\gamma))=\aleph_\delta$. But $x=C(S)\ sm\ \hat{\beta}(\beta<\gamma)$, so $(E\delta).\delta\varepsilon NO.\ \omega_\delta\neq V.Nc(x)=\aleph_\delta$. Thus the hypothesis of Axiom A is satisfied.

Now suppose $R\varepsilon\gamma$. Then $rCan^2(R)\ smor(seg_\gamma\leq)$ by T121. Hence $Can^2(C(R))\ sm_F x$.

Using Axiom A, let y be such that $(\beta):\beta\varepsilon x.$ $\supset.(E_1P).P\varepsilon\beta.P\varepsilon y$, and let $P_\beta=\iota P(P\varepsilon\beta.\beta\varepsilon y)$.

Now let

$A=\hat{z}((Eu,w).z=\langle uw\rangle .u\varepsilon C(R).w\varepsilon C(P_{F`U^2(u)}))$

$W=\hat{x}\hat{y}((Eu,w,u_1,w_1):x=\langle uw\rangle .y=\langle u_1w_1\rangle .$
$x,y\varepsilon A:F`U^2(u)<F`U^2(u_1).\vee .F`U^2(u)=F`U^2(u_1).wP_{F`U^2(u)}w_1).$

Then $W\varepsilon\Omega$ and $C(W)=A$.

Let $\alpha=Nr(W)$. Suppose $\beta\varepsilon x$, so that $\beta=Nr(P_\beta)$. Let

$T=\hat{x}\hat{y}((Eu,v,w).v,w\varepsilon C(P_\beta).U^2(u)=\breve{F}`\beta .x=\langle uv\rangle .y=\langle uw\rangle .xWy)$.

Then P_β smor T.

If $\sim(E\delta).\delta\varepsilon x.\beta<\delta$, then the theorem is trivial. So assume $(E\delta).\delta\varepsilon x.\beta<\delta$. Then $T\subseteq(\mathrm{seg}_z W)$. Hence $\sim(T$ smor $W)$, so $\sim(P_\beta$ smor $W)$. Similarly, $\sim(W$ smor$(\mathrm{seg}_u P_\beta))$. So P_β smor $\mathrm{seg}_v W$.

Thus $\beta<\alpha$. Hence, $(E\alpha).\alpha\varepsilon NO.(\beta).\beta\varepsilon x\supset\beta<\alpha$.

In order to prove T84, we need some auxiliary theorems.

T148. $\vdash(\alpha):\alpha\varepsilon NO.\omega_\alpha\neq V.\supset .\vartheta_\alpha\times_c\vartheta_\alpha=\vartheta_\alpha$.

Proof. By T137, $(\mathrm{seg}_{\langle\omega_\alpha,0\rangle}\leq_p)$ smor$(\mathrm{seg}_{\omega_\alpha}\leq)$.

Hence $\vartheta_\alpha=Nc(\hat{\gamma}(\gamma<\omega_\alpha))=Nc(\hat{x}((E\beta,\gamma).\beta,\gamma<\omega_\alpha .$
$x=\langle\beta\gamma\rangle))=Nc((\hat{\gamma}(\gamma<\omega_\alpha))\times(\hat{\gamma}(\gamma<\omega_\alpha)))$
$=Nc(\hat{\gamma}(\gamma<\omega_\alpha))\times_c Nc(\hat{\gamma}(\gamma<\omega_\alpha))=\vartheta_\alpha\times_c\vartheta_\alpha$.

T148.1. $\vdash(\alpha):\alpha\varepsilon NO.\omega_\alpha\neq V.\supset .\aleph_\alpha\times_c\aleph_\alpha=\aleph_\alpha$.

Proof. Obviously $\aleph_\alpha\leq_c\aleph_\alpha\times_c\aleph_\alpha$. Let $\omega_\alpha=Nr(P)$ so that $\aleph_\alpha=Nc(C(P))$, and suppose $\aleph_\alpha<_c\aleph_\alpha\times_c\aleph_\alpha$. Then $Nc(C(P))<_c Nc(C(P))\times_c Nc(C(P))=Nc((C(P))\times(C(P)))$. Hence by T132, $Nc(Can^2(C(P)))<_c Nc(Can^2((C(P))\times(C(P))))$
$=Nc(Can^2(C(P)))\times_c Nc(Can^2(C(P)))$. But $Nc(Can^2(C(P)))=$
$Cn(Nr(rCan^2(P)))=Cn(Nr(\mathrm{seg}_{\omega_\alpha}\leq))=Nc(\hat{\gamma}(\gamma<\omega_\alpha))=\vartheta_\alpha$.

Hence $\aleph_\alpha <_c \aleph_\alpha \times_c \aleph_\alpha$, contradicting T148. Thus,

$\aleph_\alpha \times_c \aleph_\alpha \leq_c \aleph_\alpha$, so $\aleph_\alpha \times_c \aleph_\alpha = \aleph_\alpha$.

T149. $\vdash (\alpha,\beta) : \alpha \leq \beta \,.\, \omega_\beta \neq V \,.\supset.\, \aleph_\alpha \times_c \aleph_\beta = \aleph_\beta$.

Proof. $\aleph_\beta \leq_c \aleph_\alpha \times_c \aleph_\beta \leq_c \aleph_\beta \times_c \aleph_\beta = \aleph_\beta$. Hence

$\aleph_\alpha \times_c \aleph_\beta = \aleph_\beta$.

T150. $\vdash (\alpha,\beta,x,y) : \alpha \leq \beta \,.\, \omega_\beta \neq V \,.\, Nc(x) = \aleph_\alpha \,.\, Nc(y) = \aleph_\beta \,.$
$\supset.\, Nc(x \times y) = \aleph_\beta \,.\, Nc(x \cup y) = \aleph_\beta$.

Proof. $\aleph_\beta = Nc(y) \leq_c Nc(x \times y) \leq_c Nc(x) \times_c Nc(y)$

$= \aleph_\alpha \times_c \aleph_\beta = \aleph_\beta$. So $Nc(x \times y) = \aleph_\beta$.

$\aleph_\beta = Nc(y) \leq_c Nc(x \cup y) \leq_c Nc(x \times y) = \aleph_\beta$. So $Nc(x \cup y) = \aleph_\beta$.

T151. $\vdash (R,x,\alpha) : \alpha \varepsilon NO \,.\, \omega_\alpha \neq V \,.\, R \varepsilon Sv \,.\, Nc(x) = \aleph_\alpha \,.$
$\supset.\, Nc(R``x) \leq_c Nc(x)$.

Proof. Suppose $\hat{\gamma}(\gamma < \omega_\alpha)\, sm_F\, x$. Let
$y = \hat{z}((Eu) . u \varepsilon Val(x \upharpoonright R) . z = F`(\min_{\leq} \hat{\beta}(\beta < \omega_\alpha . R`F`\beta = u)))$.

Then $y \subseteq x$ and $y\ sm_{y \upharpoonright R}\ R``x$. So $Nc(R``x) \leq_c Nc(x)$.

The following theorem is analogous to *8.73 of [5]. However, the proof given here does not depend on the axiom of choice.

T152. $\vdash (x,\alpha,R,S) : x \subseteq NO \,.\, \alpha \varepsilon NO \,.\, \omega_\alpha \neq V \,.\, Nc(x) = \aleph_\alpha \,.$
$R, S \varepsilon Fnc \,.\, Arg(R) = NO \,.\, Val(R) \subseteq NO \,.\, Arg(S) = NO \times NO \,.\, Val(S) \subseteq NO \,.$
$\supset.\, Nc(Closure(x, (R)_2, (S)_3)) = \aleph_\alpha$.

The hypothesis "$x \subseteq NO . Arg(R)=NO . Val(R) \subseteq NO .$ $Arg(S)=NO \times NO . Val(S) \subseteq NO$" is not actually necessary for this theorem, but it avoids certain awkward circumlocutions which would be necessary without it because of D18.

Proof. Let

$$H=\hat{u}\hat{w}(w=u \cup R``u \cup S``(u \times u)).$$

Using T108, let f be a function such that $Arg(f)=Can(\hat{\beta}(\beta < \omega_0))$, and

$$f`U(0)=x$$

$$(n):0<n<\omega_0 . \supset . f`U(n+1)=H`f`U(n).$$

Then we have

$$f`U(0)=x$$

$$f`U(1)=x \cup R``x \cup S``(x \times x)$$

$$f`U(2)=f`U(1) \cup R``(f`U(1)) \cup S``((f`U(1)) \times (f`U(1)))$$

etc.

It is easily shown that

$$Sum(f``Can(\hat{\beta}(\beta < \omega_0)))=Closure(x,(R)_2,(S)_3).$$

It is not so easily shown that $Nc(Closure(x,(R)_2,(S)_3))=\aleph_\alpha$. To do this we start over and proceed as follows.

We define ordinal multiplication, $\alpha \times_o \beta$, and ordinal subtraction, $\alpha -_o \beta$, as follows:

$$P \times_r Q =df \hat{x}\hat{y}((E\alpha,\beta,\gamma,\delta) . x=\langle\alpha\beta\rangle . y=\langle\gamma\delta\rangle . \alpha,\gamma \varepsilon C(P) . \beta,\delta \varepsilon C(Q) : \beta Q \delta . \beta \neq \delta . \vee . \beta=\delta . \alpha P \gamma).$$

$$\alpha \times_o \beta =df \hat{R}((EP,Q) . P,Q \varepsilon \Omega . \alpha=Nr(P) . B=Nr(Q) . R \text{ smor}(P \times_r Q)).$$

$$z_{Q,P} =df \iota z(Q \text{ smor}(seg_z P))$$

$$P -_r Q =df \hat{x}\hat{y}(z_{Q,P} P x . x P y)$$

$\alpha -_o \beta =df\ \iota\gamma(\alpha\leq\beta.\gamma=0.\vee.\beta<\alpha.(P,Q):\alpha=Nr(P).$
$\beta=Nr(Q).\supset.\gamma=Nr(P -_r Q)).$

One can then prove:

1) $\vdash(\alpha,m,n):\alpha\varepsilon NO.\omega_\alpha\neq V.m,n<\omega_o.m<n.\supset.\omega_\alpha\ \times_o m<\omega_\alpha\ \times_o n.$

2) $\vdash(\alpha,\gamma,n):\alpha,\gamma\varepsilon NO.\omega_\alpha\neq V.n<\omega_o.(\omega_\alpha\ \times_o n)\leq\gamma<$
$(\omega_\alpha\ \times_o(n+1)).\supset.(\gamma -_o(\omega_\alpha\ \times_o n))<\omega_\alpha.$

3) $\vdash(\alpha):\alpha\varepsilon NO.\omega_\alpha\neq V.\supset.Nc(\hat\beta(\beta<\omega_\alpha\ \times_o\ \omega_o))=\aleph_\alpha.$

1) and 2) above may be proved by using theorems similar to those in Principia Mathematica, *160, *166, *180, *184, *255. (Note, however, that $P\ \times_r Q$ and $\alpha\ \times_o\beta$ are defined by ordering by first differences in P.M., rather than by second differences as above. Thus the order of some of the products in the theorems in *255 must be reversed.)

To prove 3) we first prove two lemmas.

Lemma 1. $\vdash(\alpha,\beta):\alpha,\beta\varepsilon NO.\supset.Cn(\alpha\ \times_o\beta)=Cn(\alpha)\ \times_c Cn(\beta).$

Proof. Let $\alpha=Nr(P)$, $\beta=Nr(Q)$. Then
$Cn(\alpha\ \times_o\beta)=Nc(C(P\ \times_r Q))=Nc((C(P))\times(C(Q)))$
$=Nc(C(P))\ \times_c Nc(C(Q))=Cn(\alpha)\ \times_c Cn(\beta).$

Lemma 2. $\vdash(\alpha):\alpha\varepsilon NO.\omega_\alpha\neq V.\supset.Cn(\omega_\alpha\ \times_o\omega_\alpha)=\aleph_\alpha.$

Proof. By Lemma 1, $Cn(\omega_\alpha\ \times_o\omega_\alpha)=Cn(\omega_\alpha)\times_c Cn(\omega_\alpha)$
$=\aleph_\alpha\ \times_c\aleph_\alpha$. So by T148.1, $Cn(\omega_\alpha\ \times_o\omega_\alpha)=\aleph_\alpha$.

Proof of 3): $\aleph_\alpha=Nc(\hat\beta(\beta<\omega_\alpha))\leq_c Nc(\hat\beta(\beta<\omega_\alpha\ \times_o\ \omega_o))$
$\leq_c Nc(\hat\beta(\beta<\omega_\alpha\ \times_o\ \omega_\alpha))=\aleph_\alpha$ by T129 and Lemma 2.

Using T137, one can define a function F such that
$\hat\gamma(\gamma<\omega_\alpha)\ sm_F\ \hat z((E\delta,\beta).\delta,\beta<\omega_\alpha.z=\langle\delta\beta\rangle),$

and by proving a more general theorem similar to T137.1, one can define a function F_n such that

$$(n):0<n<\omega_o.\supset.\hat{\gamma}(\gamma<\omega_\alpha)\,sm_{F_n}\hat{z}((Em,\beta).m<n.\beta<\omega_\alpha.z=\langle m\beta\rangle).$$

We then define

$$Q_1 = df\ \hat{x}\hat{\gamma}((E\alpha,\beta).x=\langle\alpha\beta\rangle.\gamma=\alpha)$$

$$Q_2 = df\ \hat{x}\hat{\gamma}((E\alpha,\beta).x=\langle\alpha\beta\rangle.\gamma=\beta).$$

By hypothesis, there is a function K such that

$$\hat{\gamma}(\gamma<\omega_\alpha)\,sm_K x.$$

We now wish to define a function W which is a many-one mapping of $\hat{\gamma}(\gamma<\omega_\alpha\times_o\omega_o)$ onto Closure$(x,(R)_2,(S)_3)$. We shall do this approximately as follows (where "$\longrightarrow$" means "is mapped by W onto"):

$$\hat{\gamma}(\gamma<\omega_\alpha)\longrightarrow x=f`U(0)$$

$$\hat{\gamma}(\gamma<\omega_\alpha\times_o 3)\longrightarrow H`f`U(0)=x\cup R``x\cup S``(x\times x)=f`U(1)$$

$$\hat{\gamma}(\gamma<\omega_\alpha\times_o 5)\longrightarrow H`f`U(1)=f`U(1)\cup R``(f`U(1))\cup S``((f`U(1))\times(f`U(1)))$$
$$=f`U(2)$$

. . . .

$$\hat{\gamma}(\gamma<(\omega_\alpha\times_o(2n+1)))=H`f`U(n-1)=f`U(n-1)\cup R``(f`U(n-1))$$
$$\cup S``((f`U(n-1))\times(f`U(n-1)))=f`U(n)$$

. . . .

It is clear, however, that rather than map $\hat{\gamma}(\gamma<\omega_\alpha\times_o 3)$ onto $x\cup R``x\cup S``(x\times x)$ in the second step, we could as well map $\hat{\gamma}(\omega_\alpha\leq\gamma<\omega_\alpha\times_o 2)$ onto $R``x$, and $\hat{\gamma}(\omega_\alpha\times_o 2\leq\gamma<\omega_\alpha\times_o 3)$ onto $S``(x\times x)$, since $\hat{\gamma}(\gamma<\omega_\alpha)$ has previously been mapped onto x. We can accomplish this by mapping $\hat{\gamma}(\omega_\alpha \leq \gamma < \omega_\alpha \times_0 2)$ back onto $\hat{\gamma}(\gamma < \omega_\alpha)$ by subtraction, mapping $\hat{\gamma}(\gamma < \omega_\alpha)$ onto x by K, and mapping x onto $R``x$. We then map $\hat{\gamma}(\omega_\alpha \times_0 2 \leq \gamma < \omega_\alpha \times_0 3)$ onto $\hat{\gamma}(\gamma < \omega_\alpha)$ by subtraction, map $\hat{\gamma}(\gamma < \omega_\alpha)$ onto $\hat{z}((E\delta,\beta).\delta,\beta < \omega_\alpha . z = \langle \delta\beta \rangle)$ by F, map this onto $x \times x$, and map $x \times x$ onto $S``(x \times x)$.

We then continue in the same manner, although in succeeding steps it will be convenient at each stage to take R and S of all elements so far obtained, thus introducing some harmless duplication.

The above considerations suggest that we define a W which behaves as follows:

For $0 \leq \gamma < \omega_\alpha$: $W`\gamma = K`\gamma$.

For $\omega_\alpha \leq \gamma < \omega_\alpha \times_0 2$: $W`\gamma = R`K`(\gamma - \omega_\alpha)$

$\omega_\alpha \times_0 2 \leq \gamma < \omega_\alpha \times_0 3$: Let $F`(\gamma - \omega_\alpha \times_0 2) = \langle \mu\nu \rangle; \mu, \nu < \omega_\alpha$.
Then $W`\gamma = S`\langle K`\mu, K`\nu \rangle$.

For $\omega_\alpha \times_0 3 \leq \gamma < \omega_\alpha \times_0 4$: Let $F_3`(\gamma - \omega_\alpha \times_0 3) = \langle m\beta \rangle; m < 3, \beta < \omega_\alpha$.
Then $W`\gamma = R`W`((\omega_\alpha \times_0 m) + \beta)$.

$\omega_\alpha \times_0 4 \leq \gamma < \omega_\alpha \times_0 5$: Let $F`(\gamma - \omega_\alpha \times_0 4) = \langle \mu\nu \rangle; \mu, \nu < \omega_\alpha$.
Let $F_3`\mu = \langle m, \mu' \rangle; m < 3, \mu' < \omega_\alpha$
and $F_3`\nu = \langle j, \nu' \rangle; j < 3, \nu' < \omega_\alpha$.
Then $W`\gamma = S`\langle W`((\omega_\alpha \times_0 m) + \mu'),$
$W`((\omega_\alpha \times_0 j) + \nu' \rangle$

. . . .

For $\omega_\alpha \times_0 (2n+1) \leq \gamma < \omega_\alpha \times_0 (2n+2)$: Let $F_{2n+1}`(\gamma - \omega_\alpha \times_0 (2n+1))$
$= \langle m\beta \rangle; m < 2n+1, \beta < \omega_\alpha$.

Then $W`\gamma = R`W`((\omega_\alpha \times_o m)+\beta)$.

$\omega_\alpha \times_o (2n+2) \leq \gamma < \omega_\alpha \times_o (2n+3)$:

Let $F`(\gamma - \omega_\alpha \times_o (2n+2)) = \langle \mu, \nu \rangle$;

$\mu, \nu < \omega_\alpha$.

Let $F_{2n+1}`\mu = \langle m, \mu' \rangle$; $m < 2n+1$, $\mu' < \omega_\alpha$

and $F_{2n+1}`\nu = \langle j, \nu' \rangle$; $j < 2n+1$, $\nu' < \omega_\alpha$.

Then $W`\gamma = S`\langle W`((\omega_\alpha \times_o m)+\mu')$,

$W`((\omega_\alpha \times_o j)+\nu')\rangle$.

. . . .

Using T108, there is a W such that

$W \varepsilon Fnc, Arg(W) = \hat{\beta}(\beta < \omega_\alpha \times_o \omega_o), W`0 = K`0$, and when

$0 < \gamma < \omega_\alpha \times_o \omega_o$:

$$W\text{‘}\gamma = \iota z[0<\gamma<\omega_\alpha . z=K\text{‘}\gamma : \vee :$$
$$(En):0\leq n<\omega_0 : (\omega_\alpha \times_0 (2n+1)) \leq \gamma < (\omega_\alpha \times_0 (2n+2)) .$$
$$z=R\text{‘}P\text{‘}((\omega_\alpha \times_0 (Q_1\text{‘}F\text{‘}_{2n+1}(\gamma -_0 (\omega_\alpha \times_0 (2n+1)))))+$$
$$Q_2\text{‘}F\text{‘}_{2n+1}(\gamma -_0 (\omega_\alpha \times_0 (2n+1)))) . \vee . (\omega_\alpha \times_0 (2n+2)) \leq \gamma < (\omega_\alpha \times_0 (2n+3)) .$$
$$z=S\text{‘}\langle P\text{‘}((\omega_\alpha \times_0 (Q_1\text{‘}F\text{‘}_{2n+1} Q_1\text{‘}F\text{‘}(\gamma -_0 (\omega_\alpha \times_0 (2n+2)))))$$
$$+Q_2\text{‘}F\text{‘}_{2n+1} Q_1\text{‘}F\text{‘}(\gamma -_0 (\omega_\alpha \times_0 (2n+2)))),$$
$$P\text{‘}((\omega_\alpha \times_0 (Q_1\text{‘}F\text{‘}_{2n+1} Q_2\text{‘}F\text{‘}(\gamma -_0 (\omega_\alpha \times_0 (2n+2)))))$$
$$+Q_2\text{‘}F\text{‘}_{2n+1} Q_2\text{‘}F\text{‘}(\gamma -_0 (\omega_\alpha \times_0 (2n+2))))\rangle]$$

where P stands for $\hat{\beta}(\beta<\gamma) \upharpoonright W$.

It can be shown that the function defined above has the desired properties.

It follows that

$$\aleph_\alpha = Nc(x) \leq_c Nc(Closure(x,(R)_2,(S)_3)) \leq_c$$
$$Nc(\hat{\beta}(\beta<\omega_\alpha \times_0 \omega_0)) = \aleph_\alpha .$$

Hence, $Nc(Closure(x,(R)_2,(S)_3)) = \aleph_\alpha$.

T152 and its proof can be generalized to obtain an analogous theorem for any finite number of dyadic relations, R, and any finite number of triadic relations, S. The complications are mainly notational, so we omit the proof.

The two theorems which follow could have been proved in §2, but at the time were irrelevant.

T153. $\vdash (\alpha,\beta,\gamma,\mu) : \alpha,\beta,\gamma,\mu \,\varepsilon\, NO . \omega_\gamma \neq V .$
$\mu \leq 8 . \alpha,\beta < \omega_\gamma . \supset . J_\mu\text{‘}\langle\alpha\beta\rangle < \omega_\gamma .$

Proof. $\langle\mu\alpha\beta\rangle <_t \langle 0,\omega_\gamma,0\rangle$, so
$J_\mu\text{‘}\langle\alpha\beta\rangle = J\text{‘}\langle\mu\alpha\beta\rangle < J\text{‘}\langle 0,\omega_\gamma,0\rangle = \omega_\gamma$.

T154. $\vdash (x,y,\alpha):\alpha\varepsilon NO.\omega_\alpha \neq V.x,y\varepsilon L.Ind(x)<\omega_\alpha.$ $Ind(y)<\omega_\alpha.\supset.Ind(x\cap y)<\omega_\alpha.$

Proof. Let $x=G\text{‘}U(\beta)$, $\beta\varepsilon NO_G$, $\beta<\omega_\alpha$, and $y=G\text{‘}U(\gamma)$, $\gamma\varepsilon NO_G.\gamma<\omega_\alpha$. Then by T41 and T153, $x\cap\overline{y}=G\text{‘}U(\delta),\delta\varepsilon NO_G,\delta<\omega_\alpha$. So by T41 and T153 again, $x\cap y=x\cap\overline{x\cap\overline{y}}=G\text{‘}U(\nu),\nu\varepsilon NO_G,\ \nu<\omega_\alpha$; i.e., $Ind(x\cap y)<\omega_\alpha$.

The two theorems which follow are translations of Theorems 12.4 and 12.51 of [5] into our notation. The only modifications which must be made in the proofs given in [5] are those caused by the differences between the ordinals of set theory and the ordinals of the theory of types. The proofs will be omitted.

T155. $\vdash(x,P,\gamma):.\gamma\varepsilon NO.x\subseteq NO.x Closed(K_1,K_2)_2.$ $x\ Closed(J_0,\cdots,J_8)_3.\hat{\alpha}\hat{\beta}(\alpha,\beta\varepsilon x.\alpha\leq\beta)\ smor_P(seg_\gamma\ \leq):$ $\supset:\hat{\beta}(\beta<\gamma)Closed(J_0,\cdots,J_8)_3:(\alpha,\beta,\mu):\alpha,\beta\varepsilon x.\mu\leq 8.\supset.$ $J_\mu\text{‘}\langle P\text{‘}\alpha,P\text{‘}\beta\rangle=P\text{‘}J_\mu\text{‘}\langle\alpha\beta\rangle.$

T156. $\vdash(x,P,\gamma):.\gamma\varepsilon NO.x\subseteq NO.x\ Closed(K_1,K_2)\ .$ $x\ Closed(J_0,\cdots,J_8)_3.\hat{\alpha}\hat{\beta}(\alpha,\beta\varepsilon x.\alpha\leq\beta)\ smor_P(seg_\gamma\ \leq):$ $\supset:(\alpha,\mu):\alpha\varepsilon x.\mu\leq 8.\supset.\alpha\varepsilon Val(J_\mu)\supset P\text{‘}\alpha\varepsilon Val(J_\mu).$

We now define a function, Ch, which in the theorem that follows serves the same purposes for us as the function, C, serves in the proof of Theorem 12.6 of [5].

D126. $Ch=\hat{\alpha}\hat{\beta}(\beta=min_{\leq}G\text{‘}U(\alpha)).$

The following theorem can now be proved:

T157. $\vdash (x,P,\gamma) :: \gamma \varepsilon NO . x \subseteq NO . x \text{ Closed } (K_1,K_2,Ch)_2 . x \text{ Closed}(J_0,\cdots,J_8)_3 . \widehat{\alpha\beta}(\alpha,\beta \varepsilon x . \alpha \leq \beta) \text{ smor}_P(\text{seg}_\gamma \leq) :. \supset :. (\alpha,\beta) :. \alpha,\beta \varepsilon x : \supset : G`U(\alpha) \varepsilon_g G`U(\beta) . \equiv . G`U(P`\alpha) \varepsilon_g G`U(P`\beta) : G`U(\alpha) = G`U(\beta) . \equiv . G`U(P`\alpha) = G`U(P`\beta)$.

The proof can be obtained from the proof of 12.6 in [5] by replacing G by P, F by G, and, at appropriate places, ε by ε_g. The manner in which this is done can be determined by comparing the statement of T157 with that of 12.6 of [5]. With respect to Gödel's proof we note the following:

1) The use in [5] of 12.5 to prove 12.6 is not essential, and may be avoided in proving T157.

2) The following typographical errors occur in the proof of 12.6 in [5]:

p. 56, line 33, replace "β'" by "η'".

pp. 59, 60, interchange all occurrences of "I" and "II".

p. 60, lines 35, 36, should read "$\eta = J_5\langle\beta\alpha\rangle$ and $\eta' = J_5\langle\beta'\gamma'\rangle$; that is, $F`\eta = F`\beta . P_2``(F`\gamma)$ and $F`\eta' = F`\beta' . P_2``(F`\gamma') . \cdots$".

Having proved T152 and T157, we can now prove the following theorem, which is the relativization to the model in §2 of 12.2 of [5]; the proof is essentially the same as that given in [5].

T158. $\vdash (y,\alpha,\gamma) : \alpha \varepsilon NO . \omega_{\alpha+1} \neq V . y \varepsilon SC(\hat{\beta}_G(\beta < \omega_\alpha)) . \gamma \varepsilon NO_G . y = G`U(\gamma) . \supset . \gamma < \omega_{\alpha+1}$.

Proof. Assume the hypothesis, and let

$x=\text{Closure}((\hat{\beta}(\beta<\omega_\alpha)\cup\overline{U}(\gamma)),(K_1,K_2,Ch)_2,(J_0,\cdots,J_8)_3)$.

Then by T152, $Nc(x)=\aleph_\alpha$.

Let $Q=\hat{\alpha}\hat{\beta}(\alpha,\beta\varepsilon x.\alpha\leq\beta)$. Then $Q\varepsilon\Omega$, so there are three possibilities:

1) Suppose $(\leq)\,\text{smor}(\text{seg}_z Q)$. Then $Nc(C(\leq))=Nc(NO)\leq_c Nc(C(Q))=Nc(x)$. But $Nc(x)<_c \aleph_{\alpha+1}\leq_c Nc(NO)$, a contradiction.

2) Suppose Q smor $\leq$. Then $C(Q)$ sm NO, so $Nc(x)=Nc(NO)$. However, $Nc(x)<_c \aleph_{\alpha+1}\leq_c Nc(NO)$, a contradiction.

3) Suppose $Q\ \text{smor}_P(\text{seg}_\beta\leq)$. Since $\gamma\varepsilon C(Q)$, we have $P\text{‘}\gamma\,\varepsilon C(\text{seg}_\beta\leq)$. Hence, $P\text{‘}\gamma<\beta$.

$C(Q)$ sm $C(\text{seg}_\beta\leq)$, so $Nc(\hat{\delta}(\delta<\beta))=\aleph_\alpha$. Hence $\beta<\omega_{\alpha+1}$, so $P\text{‘}\gamma<\omega_{\alpha+1}$.

The hypothesis of T157 is satisfied, and $\gamma\varepsilon x$, so

$(\delta):\delta\varepsilon x.\supset.G\text{‘}U(\delta)\varepsilon_g G\text{‘}U(\gamma)\equiv G\text{‘}U(P\text{‘}\delta)\varepsilon_g G\text{‘}U(P\text{‘}\gamma)$.

Obviously $\text{seg}\,\omega_\alpha Q=\text{seg}\,\omega_\alpha\leq$, so $(\delta):\delta<\omega_\alpha.\supset.P\text{‘}\delta=\delta$.

Hence,

$(\delta):\delta<\omega_\alpha.\supset.G\text{‘}(U(\delta)\varepsilon_g G\text{‘}U(\gamma)\equiv G\text{‘}U(\delta)\varepsilon_g G\text{‘}U(P\text{‘}\gamma)$.

This may be written

$(\delta):\delta<\omega_\alpha.\supset.\text{Ind}(G\text{‘}U(\delta))\varepsilon G\text{‘}U(\gamma)\equiv\text{Ind}(G\text{‘}U(\delta))\varepsilon G\text{‘}U(P\text{‘}\gamma)$; i.e.,

$\hat{\beta}_G(\beta<\omega_\alpha)\cap G\text{‘}U(\gamma)=\hat{\beta}_G(\beta<\omega_\alpha)\cap G\text{‘}U(P\text{‘}\gamma)$.

By hypothesis, $y=G\text{‘}U(\gamma)\subseteq\hat{\beta}_G(\beta<\omega_\alpha)$, and By T146 and T41, $\hat{\beta}(\beta<\omega_\alpha)=G\text{‘}U(\omega_\alpha)$. So we have

$G`U(\gamma)=G`U(\omega_\alpha)\cap G`U(P`\gamma)$.

It follows from T154 and the fact that $P`\gamma<\omega_{\alpha+1}$, that $\gamma<\omega_{\alpha+1}$.

It should be pointed out that while 12.2 is used in [5] after the model has been constructed to prove the generalized continuum hypothesis in the model, we are using T158 in order to construct the model of L_1, and it has only an intuitive connection with the continuum hypothesis.

We can now prove T84.

††T159. (T84.) $\vdash(\gamma):.\gamma\varepsilon NO:\supset:(E\beta):\beta\varepsilon NO.(\delta).\delta\varepsilon\hat{\alpha}_G(G`U(\alpha)\subseteq G`U(\gamma))\supset\delta<\beta$.

Proof. Using Axiom B, let $\gamma<\omega_\nu$. It follows also from Axiom B, that $\omega_\nu<\omega_{\nu+1}\neq V$.

Suppose $\delta\varepsilon\hat{\alpha}_G(G`U(\alpha)\subseteq G`U(\gamma))$. Then by T38, $\delta\varepsilon NO_G$, and $G`U(\delta)\subseteq G`U(\gamma)$. But by T38 and T146, $G`U(\gamma)\subseteq G`U(\omega_\nu)$, so $G`U(\delta)\subseteq G`U(\omega_\nu)$. Hence $G`U(\delta)\varepsilon SC(\hat{\beta}_G(\beta<\omega_\nu))$. So by T158, $\delta<\omega_{\nu+1}$.

Thus $\omega_{\nu+1}$ can be used for the desired β.

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