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CANTORIAN WELL-ORDERINGS IN QUINE'S NEW FOUNDATIONS WITH AN APPLICATION TO THE RELATIVE CONSISTENCY PROBLEM

by

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Cantorian Well-orderings in Quine's New Foundations with an Application to the Relative Consistency Problem

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NF is the system of set theory of Rosser's Logic for Mathematicians, excluding the Axioms 14 ("Counting") and 15 ("Denumerable Choice"), usually known as "New Foundations." For ordinals θ in NF, we write "$C(\theta)$" for "the initial segment of ordinals less than θ has ordinal θ."

It has previously been shown that the Gödel construction of constructible sets can be modeled in the ordinals of NF. If the resulting relation is restricted to those ordinals for which $C(\theta)$ holds, a relative interpretation in NF of a weakened form of Zermelo set theory is obtained (namely, in which the Axiom of Subsets is weakened to require that the defining formula contain only quantifiers which are bounded by sets). In two natural extensions of NF this relation determines a relative interpretation of the full Zermelo-Fraenkel set theory (including the Axioms of Choice and Constructibility). One extension is obtained by adding the statement interpreted as "every Cantorian, well-ordered set has only Cantorian subsets." (A set x is "Cantorian" if it has the same cardinality as USC(x).) The second extension of NF is obtained by adding all statements of the form "$t_1 < t_2$.&. $C(t_2)$. $\Longrightarrow C(t_1)$", and all statements expressing "the ordering of the initial segment of ordinals less than t_1 has ordinal at most t_1", for every pair of constant, stratified terms t_1, t_2.

The proofs depend on a close examination of certain symmetries relative to the operation of passing from an ordinal No(R) to No(RUSC(R)). Using the same method of proof one can prove in NF that (i) no well-ordered set has the same cardinality as its subset class, and (ii) the cardinal of NO is not less than the cardinal of its subset class. (NO denotes the set of ordinals in NF.)

Using a model-theoretic argument based on a technique of E. Specker one can show that if NF is consistent then (i) the Counting Axiom is not a theorem of NF, nor of any consistent extension of NF by stratified sentences, and (ii) it is consistent with NF to assume the existence of a finite set x such that the cardinal of x is less than the cardinal of USC(x). It follows from these results that if NF is consistent then it does not provide for every schema of induction over Nn.

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The author was born in Worcester, Massachusetts on September 25, 1940 and attended public elementary and secondary school. During the period September, 1958 to June, 1962 he attended Harvard College, receiving a Bachelor of Arts degree (in Mathematics) *magna cum laude* in June, 1962. Since September, 1962 he has attended the Massachusetts Institute of Technology as a graduate student in the Department of Mathematics, and will receive the degree of Doctor of Philosophy in June, 1967. During the academic years 1965-66 and 1966-67 he has been a part-time Instructor in the Department of Mathematics. He holds membership in the Association for Symbolic Logic and the American Mathematical Society.

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Introduction:

In this paper we consider the system of set theory presented in Rosser's Logic for Mathematicians (excluding Axioms 14 and 15). [All references to this text will be in the form [R, p.--].] This system is a slight variant of the one introduced by Quine in [18] and is usually known as "New Foundations" (NF). The underlying system of logic is a system of first order predicate calculus (with identity) and the only non-logical symbol is a binary predicate symbol $\in$. [In addition to the equality symbol, the logical symbols consist of parentheses, brackets, a list of variables, and symbols for negation ($\neg$) and disjunction (v). In addition we use '$(\exists x)$' as the symbol for the existential quantifier. In terms of the above, symbols for the universal quantifier '(x)', conjunction (&), conditional ($\Longrightarrow$) and biconditional ($\Longleftrightarrow$) are introduced by definition.] The additional axioms of NF are: (i) the axiom of extensionality

$$(x)(y)[(z)(z \in x \Longleftrightarrow z \in y) \Longrightarrow x = y]$$

and (ii) the axiom schema of comprehension

$$(\exists y)(x)[x \in y \Longleftrightarrow P]$$

whenever P is a stratified formula which does not contain y. [For a precise definition of the concept of stratification see [R, p.209f].] We adopt the notation of [R], and thus introduce notation for descriptive terms. In particular this allows for the use of terms of class abstraction. For any formula $\emptyset$ (in the language of NF) we use the symbol $\vdash \emptyset$ to mean "$\emptyset$ is a theorem of NF." For any theory S which is an extension of NF we will write $\vdash_S \emptyset$

to mean "the formula $\emptyset$ is a theorem of S." Familiarity with the results of [R] is assumed throughout the present work.

In [13] Specker introduced the term $T(n)$ defined by

$$T(n) =_{df} \{x \mid (\exists y)[y \in n \& USC(y) \text{ sm } x\} \qquad (1)$$

and used the fact that the ordering $\leq_c$ of cardinal numbers possesses certain symmetries relative to $T(n)$ in showing that the Axiom of Choice is refutable in NF. We define the term x^* by the analogous definition

$$x^* =_{df} \{R \mid (\exists S)(S \in x \& R \text{ smor } RUSC(S))]\}. \qquad (2)$$

Our main interest in (2) is in the case where x is an ordinal, and we introduce a list of special variables $\theta, \delta, \lambda, \theta_o, \delta_o, \lambda_o, \ldots$ (ie. lower case Greek letters with or without subscripts) which are restricted to represent elements of NO. The first two sections of this paper are devoted to showing that the ordering $\leq$ of the set of ordinals possesses certain important symmetries relative to the operation of passing from an ordinal θ to the ordinal θ^*. The course of this paper is essentially an exposition of such symmetries and their applications.

The main application of this technique is in giving relative interpretations of some standard systems of set theory in NF and two of its natural extensions. [To give a <u>relative interpretation</u> of a system of set theory S in a theory T is to give two formulas D(x) and E(x,y) in the language of T (each having the free variables exhibited) such that if any formula in the language of S is translated by replacing every atomic formula $x \in y$ by $E(x,y)$ and relativizing quantifiers by $D(x)$, then the axioms of S pass to theorems of T.] In doing this we make use of the modeling of the Gödel construction of constructible sets in the set of ordinals of NF which was carried out by Orey in [7]. By showing that the relation $E(x,y)$ defined by that process has a symmetry relative to the *-operation, we obtain (in Section 3) a relative interpretation in NF of a set theory B_o which is a weakened form of Zermelo set theory. [B_o consists of the Axioms of Extensionality, Foundation, Power set, Pairing, Union, and Infinity, and the Axiom of Subsets when the defining formula is required to contain only quantifiers which are relativized to some set. Zermelo set theory (Z) consists of B_o plus the full Axiom of Subsets.]

In Section 2 we introduce an extension NF_1 of NF obtained by adding as an additional axiom the statement expressing "every Cantorian, well-ordered set has only Cantorian subsets." In Section 3 we show that in NF_1 the above technique provides a relative interpretation for the Zermelo-Fraenkel system of set theory (Z-F), which consists of Z plus the Axioms of Replacement. In Section 4 we introduce a second extension of NF, NF_2 for which the same is true. The additional axioms of NF_2 similarly express a certain regularity among ordinals relative to the operations $\theta \rightarrow \theta^*$ and $n \rightarrow T(n)$, and have the (apparent) advantage that all are stratified sentences. [Of course the additional axiom of NF_1 is not stratified.]

In Section 5 we apply some results obtained in the course of earlier work to the generalization of some results of [13] concerning the ordering $\leq_c$ of cardinals. Also it is shown that in both NF_1 and NF_2 the set of cardinals of well-ordered sets has cardinality at least as great as that of USC(NO).

In Section 6 we use a model-theoretic argument to settle some questions of relative consistency for NF and its extensions. In particular, it is shown that if NF is consistent then neither the Counting Axiom (Axiom 14 of [R]) nor the sentence

$$(n)[n \in Nn \Longrightarrow T(n) \leq_c n]$$

are theorems of NF (nor in fact of any consistent extension of NF by only stratified sentences). Thus in particular the full system of [R] is a non-trivial extension of NF (unless of course NF is already inconsistent). Moreover, this result shows that if NF is consistent then it does not provide for every schema of induction over Nn.

Section 1:

1.1. Definition: $\delta^* =_{df} \{R \mid (\exists S)(S \in \delta \ \& \ R \text{ smor } \mathrm{RUSC}(S)\}$.

1.2. Lemma: (i) $\vdash R \in \mathrm{Word} \Longrightarrow \mathrm{No}(R)^* = \mathrm{No}(\mathrm{RUSC}(R))$

(ii) $\vdash \delta^* \in \mathrm{NO}$.

Proof: (i) By 1.1, and the fact that the defining condition of δ^* is stratified, we have

$$S \in \mathrm{No}(R)^* \Longleftrightarrow (\exists S')(S' \in \mathrm{No}(R) \ \& \ S \text{ smor } \mathrm{RUSC}(S')).$$

That is, $S \in \mathrm{No}(R)^* \Longleftrightarrow S \text{ smor } \mathrm{RUSC}(R)$

or $S \in \mathrm{No}(R)^* \Longleftrightarrow S \in \mathrm{No}(\mathrm{RUSC}(R))$.

(ii) Since $\delta \in \mathrm{NO}$, there is an $R \in \mathrm{Word}$ with $\delta = \mathrm{No}(R)$; then $\delta^* = \mathrm{No}(\mathrm{RUSC}(R)) \in \mathrm{NO}$ by [R, p. 338].

Thus 'δ^*' is a stratified term (assigned one higher type than δ) and takes on as values only elements of NO. In the remainder of this section we prove some elementary results which follow from results proved in [R], and which are essential in later work. In general, they can be viewed as showing that the operation of passing from δ to δ^* (which is not a function in NF) preserves order, is 1-1 and maps onto an initial segment of NO, and under which certain important properties are invariant.

1.3. Lemma: (i) $\vdash \delta = \theta \Longleftrightarrow \delta^* = \theta^*$

(ii) $\vdash \delta < \theta \Longleftrightarrow \delta^* < \theta^*$

(iii) $\vdash \delta \leq \theta \Longleftrightarrow \delta^* \leq \theta^*$

Proof: (i) Using [R, p. 457], and 1.2, we see that if $R,S \in \text{Word}$

$$\begin{aligned} \text{No}(R) = \text{No}(S) &\Longleftrightarrow \text{No}(\text{RUSC}(R)) = \text{No}(\text{RUSC}(S)) \\ &\Longleftrightarrow \text{No}(R)^* = \text{No}(S)^*. \end{aligned}$$

(ii) For $R,S \in \text{Word}$, we have:

$$\begin{aligned} \text{No}(R) < \text{No}(S) &\Longleftrightarrow (\exists x)(R \text{ smor } \text{seg}_x S) && \text{[R, p. 471]} \\ &\Longleftrightarrow (\exists x)(\text{RUSC}(R) \text{ smor } \text{RUSC}(\text{seg}_x S)) && \text{[R, p.457]} \\ &\Longleftrightarrow (\exists x)(\text{RUSC}(R) \text{ smor } \text{seg}_{\{x\}} \text{RUSC}(S)) && \text{[R, p. 338]} \\ &\Longleftrightarrow \text{No}(\text{RUSC}(R)) < \text{No}(\text{RUSC}(S)) && \text{[R, p.471]} \\ &\Longleftrightarrow \text{No}(R)^* < \text{No}(S)^*, \text{ by 1.2.} \end{aligned}$$

(iii) Follows from (i) and (ii), since

$$\delta \leq \theta \Longleftrightarrow \delta < \theta \vee \delta = \theta.$$

1.4. Lemma: $\vdash \delta \leq \theta^* \Longleftrightarrow (\exists \delta_1)(\delta_1 \leq \theta \ \& \ \delta = \delta_1^*)$

Proof: If $\delta = \delta_1^*$ and $\delta_1 \leq \theta$, then by 1.2: $\delta \leq \theta^*$.

Conversely, if $\delta \leq \theta^*$, and $\delta = \text{No}(R)$, $\theta = \text{No}(S)$, then by 1.1, $\text{No}(R) \leq \text{No}(\text{RUSC}(S))$. If $\delta = \theta^*$, then $\theta \leq \theta$ and $\delta = \theta^*$ so the conclusion holds. Thus we may suppose that

$$\text{No}(R) < \text{No}(\text{RUSC}(S)).$$

But then by [R, p.471], R smor seg_x RUSC(S) for some $x \in AV(RUSC(S))$. Then since,

$$AV(RUSC(S)) = USC\,(AV(S)) \text{ by [R, p.302]}$$

we have $x = \{y\}$, for some $y \in AV(S)$. But then

$$R \text{ smor } (\mathrm{seg}_{\{y\}} RUSC(S)) = RUSC\,(\mathrm{seg}_y S)$$

by [R, p.338]. Hence $No(R) = No(RUSC(\mathrm{seg}_y S)) = No(\mathrm{seg}_y S)^*$. Then, since $No(\mathrm{seg}_y S) < No(S)$, [R, p.471], we are done.

In the following, we define A^* for sets $A \subseteq NO$ and $A \subseteq NO \times NO$.

[We use the same *-notation for convenience, and make clear when the context does not, whether the domain of the variable is to be NO, SC(NO) or SC(NO×NO).]

1.5. <u>Definition</u>:

(i) for $A \subseteq NO$, $A^* =_{df} \{\delta^* \mid \delta \in A\}$

(ii) for $R \subseteq NO \times NO$, $R^* =_{df} \{<\delta^*, \theta^*> \mid <\delta, \theta> \in R\}$.

Each term is stratified, and has type one higher than its argument.

1.6. <u>Lemma</u>:

(i) $\vdash A \subseteq NO \Longrightarrow [\delta \in A^* \Longleftrightarrow (\exists\theta)(\delta = \theta^* \;\&\; \theta \in A)]$

(ii) $\vdash R \subseteq NO \times NO \Longrightarrow$

$$[<\delta, \theta> \in R^* \Longleftrightarrow (\exists\delta_1)(\exists\theta_1)(\delta = \delta_1^* \;\&\; \theta = \theta_1^* \;\&\; <\delta_1, \theta_1> \in R)].$$

<u>Proof</u>: Both follow trivially from the fact that the term $\langle\delta^*,\theta^*\rangle$ and the defining conditions are stratified.

1.7. <u>Lemma</u>:

(i) $\vdash (A,B \subseteq NO) \Longrightarrow (A = B \Longleftrightarrow A^* = B^*)$

(ii) $\vdash (A,B \subseteq NO) \Longrightarrow (A \subseteq B \Longleftrightarrow A^* \subseteq B^*)$

(iii) $\vdash (R,S \subseteq NO{\times}NO) \Longrightarrow (R = S \Longleftrightarrow R^* = S^*)$

(iv) $\vdash (R,S \subseteq NO{\times}NO) \Longrightarrow (R \subseteq S \Longleftrightarrow R^* \subseteq S^*)$

<u>Proof</u>: The proofs all follow directly from 1.3.i and 1.6. We prove (i) as an example:

If $A = B$, then

$$\begin{aligned} \delta \in A^* &\Longleftrightarrow (\exists\theta)(\delta = \theta^* \;\&\; \theta \in A) \\ &\Longleftrightarrow (\exists\theta)(\delta = \theta^* \;\&\; \theta \in B) \\ &\Longleftrightarrow \delta \in B^* \end{aligned}$$

so $A^* = B^*$.

If $A^* = B^*$, then

$$\delta \in A \Longleftrightarrow \theta^* \in A^* \Longleftrightarrow \delta^* \in B^* \Longleftrightarrow \delta \in B,$$

so $A = B$.

1.8. <u>Lemma</u>:

(i) $\vdash A \subseteq NO^* \Longleftrightarrow (\exists B)(A = B^* \;\&\; B \subseteq NO)$.

(ii) $\vdash R \subseteq NO^*{\times}NO^* \Longleftrightarrow (\exists S)(S \subseteq NO{\times}NO \;\&\; R = S^*)$

<u>Proof</u>: (i) If $A = B^* \;\&\; B \subseteq NO$, then by 1.7.ii, $B^* \subseteq NO^*$, or $A \subseteq NO^*$.

If $A \subseteq NO^*$, let $B = \{\delta \mid \delta^* \in A\}$;

then $B \subseteq NO$, and

$$\begin{aligned} \delta \in B^* &\iff (\exists\theta)(\delta = \theta^* \ \&\ \theta \in B) \quad \text{by 1.6.i.} \\ &\iff (\exists\theta)(\delta = \theta^* \ \&\ \theta^* \in A) \\ &\iff \delta \in A, \end{aligned}$$

so $A = B^*$.

(ii) is proved similarly.

1.9. Lemma:

$$\vdash (R \subseteq NO \times NO) \implies [(Arg(R^*) = (Arg(R))^*) \\ \&\ (Val(R^*) = Val(R)^*) \ \&\ (AV(R^*) = AV(R)^*)].$$

Proof: The proofs of the three conclusions are similar, so we prove only the first.

$$\begin{aligned} \delta \in Arg(R^*) &\iff (\exists\theta)(<\delta,\theta> \in R^*) \\ &\iff (\exists\theta)(\exists\delta_1)(\exists\theta_1)(\delta = \delta_1{}^* \ \&\ \theta = \theta_1{}^* \ \&\ <\delta_1,\theta_1> \in R) \\ &\iff (\exists\delta_1)(\delta = \delta_1{}^* \ \&\ \delta_1 \in Arg(R)) \\ &\iff \delta \in (Arg(R))^*, \end{aligned}$$

using 1.6. Thus $Arg(R^*) = (Arg(R))^*$.

1.10. Lemma:

(i) $\vdash R \subseteq NO \times NO \implies [R \in Funct \iff R^* \in Funct]$

(ii) $\vdash R \subseteq NO \times NO \implies [R \in 1\text{-}1 \iff R^* \in 1\text{-}1]$.

Proof: (i) Suppose $R \in Funct$, and $\delta \in Arg(R^*)$. Then, by 1.9 $\delta = \delta_1{}^*$ for some $\delta_1 \in Arg(R)$. If $<\delta,\theta>$

and $<\delta,\lambda>$ are in R^*, then $\theta,\lambda \in Val(R^*)$, so that $\theta = \theta_1{}^*$ and $\lambda = \lambda_1{}^*$ for some $\theta_1,\lambda_1 \in Val(R)$ (1.9). But then, $<\delta_1,\theta_1>$ and $<\delta_1,\lambda_1>$ are elements of R, (by 1.5), so that $\theta_1 = \lambda_1$, or $\theta = \lambda$ (by 1.3). Therefore $R^* \in Funct$.

Conversely, if $R^* \in Funct$, and $<\delta,\theta>,<\delta,\lambda> \in R$, then $<\delta^*,\theta^*>,<\delta^*,\lambda^*> \in R^*$; thus $\theta^* = \lambda^*$, and $\theta = \lambda$ (by 1.3). Hence $R \in Funct$ as well.

(ii) Follows from (i) upon noting that

$$R \in 1\text{-}1 \iff (R \in Funct \;\&\; Conv(R) \in Funct). \quad [R, p.324]$$

and that evidently $Conv(R^*) = Conv(R)^*$.

1.11. <u>Lemma</u>: $\vdash [R \subseteq NO \times NO \;\&\; R \in Funct] \Longrightarrow$
$$[\delta^* \in Arg(R^*) \Longrightarrow R^*(\delta^*) = R(\delta)^*]$$

<u>Proof</u>: This is simply a restatement of 1.6.ii in light of 1.10.

1.12. <u>Definition</u>: $lubA =_{df} min_{\leq}\left\{\theta \mid (\delta)(\delta \in A \Longrightarrow \delta < \theta)\right\}$.

1.13. <u>Lemma</u>: $\vdash [A \subseteq NO \;\&\; (\exists\theta)(\delta)(\delta \in A \Longrightarrow \delta < \theta)] \Longrightarrow$
$$[lubA \in NO \;\&\; (\delta)(\delta \in A \Longrightarrow \delta < lubA)$$
$$\& \;(\theta)((\delta)(\delta \in A \Longrightarrow \delta < \theta) \Longrightarrow lubA \leq \theta)].$$

<u>Proof</u>: Suppose the assumptions hold and set

$$B = \left\{\theta \mid (\delta)(\delta \in A \Longrightarrow \delta < \theta)\right\}.$$

Then $B \neq \Lambda$, and hence $\text{lub}A \in NO$. Moreover, the last two conclusions follow since $\text{lub}A \in B$.

1.14. Lemma:

(i) $\vdash \; A \subseteq NO \implies [\min_{\leq}(A^*) = (\min_{\leq}A)^*]$

(ii) $\vdash \; A \subseteq NO \implies [\max_{\leq}(A^*) = (\max_{\leq}A)^*]$

(iii) $\vdash \; [A \subseteq NO \;\&\; \text{lub}A \in NO] \implies [\text{lub}(A^*) = (\text{lub}A)^*]$.

Proof: (i) If $\delta = \min_{\leq}(A^*)$, then $\delta = \theta^*$ for some $\theta \in A$ (by 1.6). Moreover, if $\theta_1 \in A$, then $\delta \leq \theta_1{}^*$ (since $\theta_1{}^* \in A^*$), or $\theta \leq \theta_1$. Thus $\theta = \min_{\leq}A$, and $(\min_{\leq}A)^* = \theta^* = \min_{\leq}(A^*)$.

(ii) If there is a $\leq$-maximum $\delta \in A^*$, then $\delta = \theta^*$ for some $\theta \in A$. Again, if $\theta_1 \in A$, then $\theta_1{}^* \in A^*$ so that $\theta_1{}^* \leq \theta^*$. That is $\theta_1 \leq \theta$; hence $\theta = \max_{\leq}A$. Otherwise, if $\max_{\leq}(A^*) = \Lambda$ (ie., if A^* has no maximum) and $\theta \in A$, (so that $\theta^* \in A^*$) then there is a $\delta \in A^*$ with $\theta^* < \theta$. But then $\delta = \delta_1{}^*$ for some $\delta_1 \in A$, and $\theta < \delta_1$. Thus $\max_{\leq}A = \Lambda$ as well (and $\Lambda^* = \Lambda$).

(iii) Suppose $\text{lub}A \in NO$, so that

$$\delta \in A \implies \delta < \text{lub}A$$

But then $\delta \in A^*$ implies $\delta = \theta^*$, for some $\theta \in A$, which in turn implies $\delta < (\text{lub}A)^*$. Thus $\text{lub}(A^*) \leq (\text{lub}A)^*$.

Moreover, if $\theta < (\text{lub}A)^*$, then $\theta = \theta_1{}^*$, for some θ_1, (by 1.4) with $\theta_1 < \text{lub}A$. But then (by 1.13) there is a

$\delta \in A$ with $\theta_1 \leq \delta$, and hence a $\delta^* \in A^*$ with $\theta = \theta_1{}^* \leq \delta^*$. Thus $(\mathrm{lub}A)^* \leq \mathrm{lub}(A^*)$ and hence $(\mathrm{lub}A)^* = \mathrm{lub}(A^*)$.

1.15. Lemma: (i) $\vdash \{\delta \mid \delta < \theta^*\} = \{\delta \mid \delta < \theta\}^*$

(ii) $\vdash (\mathrm{seg}_{\theta^*}\leq) = (\mathrm{seg}_{\theta}\leq)^*$.

Proof: (i) is just a restatement of 1.3 and 1.4.

(ii) $(\langle\delta_1,\delta_2\rangle \in \mathrm{seg}_{\theta^*}\leq) \Longleftrightarrow (\delta_1 \leq \delta_2 < \theta^*)$ by definition. Thus (by 1.3, 1.4)

$$\begin{aligned} &\langle\delta_1,\delta_2\rangle \in \mathrm{seg}_{\theta^*}\leq \\ &\quad\Longleftrightarrow (\exists\lambda_1)(\exists\lambda_2)(\lambda_1 \leq \lambda_2 < \theta \;\&\; \lambda_1{}^* = \delta_1 \;\&\; \lambda_2{}^* = \delta_2) \\ &\quad\Longleftrightarrow (\exists\lambda_1)(\exists\lambda_2)(\langle\lambda_1,\lambda_2\rangle \in \mathrm{seg}_{\theta}\leq \;\&\; \lambda_1{}^* = \delta_1 \;\&\; \lambda_2{}^* = \delta_2) \\ &\quad\Longleftrightarrow \langle\delta_1,\delta_2\rangle \in (\mathrm{seg}_{\theta}\leq)^*. \quad \text{(by 1.6)} \end{aligned}$$

1.16. Lemma:

$$\vdash A,B \subseteq NO \Longrightarrow [A^* \;\mathrm{sm}_s\; B^* \Longleftrightarrow (\exists t)(A \;\mathrm{sm}_t\; B \;\&\; s = t^*)]$$

Proof: (i) If $A \;\mathrm{sm}_t\; B$, then $A = \mathrm{Arg}(t)$, $B = \mathrm{Val}(t)$ and $t \in 1\text{-}1$. Then, by 1.9 and 1.10 (since $t \subseteq A\times B \subseteq NO\times NO$) $\mathrm{Arg}(t^*) = A^*$, $\mathrm{Val}(t^*) = B^*$, and $t^* \in 1\text{-}1$; that is $A^* \;\mathrm{sm}_{t^*}\; B^*$.

(ii) If $A^* \;\mathrm{sm}_s\; B^*$, (so that $s \subseteq A^*\times B^*$ and hence $s \subseteq NO^*\times NO^*$) then there is a $t \subseteq NO\times NO$ such that $t^* = s$ (by 1.8). Also, $A^* = \mathrm{Arg}(s)$, $B^* = \mathrm{Val}(s)$ and $s \in 1\text{-}1$, so that by 1.9 and 1.10, $A = \mathrm{Arg}(t)$, $B = \mathrm{Val}(t)$, and $t \in 1\text{-}1$; that is, $A \;\mathrm{sm}_t\; B$.

1.17. <u>Lemma</u>: $\vdash R,S \subseteq NO \times NO \Longrightarrow$

$$(R^* \text{ smor}_s S^* \Longleftrightarrow (\exists t)(s = t^* \ \& \ R \text{ smor}_t S))$$

<u>Proof</u>: (i) If R smor$_t$ S, then $AV(R)$ sm$_t$ $AV(S)$. Since $AV(R^*) = AV(R)^*$ and $AV(S^*) = AV(S)^*$, we have $AV(R^*)$ sm$_{t^*}$ $AV(S^*)$ (by 1.16).

Then assume that $<\delta,\theta> \in R \Longleftrightarrow <t(\delta),t(\theta)> \in S$. If $<\omega,\lambda> \in R^*$ there are δ,θ such that $\delta^* = \omega$, $\theta^* = \lambda$, and $<\delta,\theta> \in R$ (by 1.6). Thus $<t(\delta),t(\theta)> \in S$, and hence $<t(\delta)^*,t(\theta)^*> \in S^*$, so that by 1.11, $<t^*(\delta^*),t^*(\theta^*)> \in S^*$. That is, $<\omega,\lambda> \in R^*$ implies $<t^*(\omega),t^*(\lambda)> \in S^*$. The converse is proved similarly. Therefore, R^* smor$_{t^*}$ S^*.

(ii) Suppose R^* smor$_s$ S^*; then by 1.9, $AV(R)^*$ sm$_s$ $AV(S)^*$. By 1.16, there is a t such that $AV(R)$ sm$_t$ $AV(S)$ and $s = t^*$. Furthermore, if $<\delta,\theta> \in R$ (so $<\delta^*,\theta^*> \in R^*$ by 1.6) then $<t^*(\delta^*), t^*(\theta^*)> \in S^*$ (by assumption). Hence $<t(\delta)^*,t(\theta)^*> \in S^*$ by 1.11. That is, $<t(\delta),t(\theta)> \in S$, so that

$$<\delta,\theta> \in R \Longrightarrow <t(\delta),t(\theta)> \in S.$$

The converse proceeds similarly, and completes the proof that R smor$_t$ S.

If $A \subseteq NO$, there is a relation

$$t = \{<\delta^*,\{\delta\}> \mid \delta \in A\}$$

since the term $<\delta^*,\{\delta\}>$ and the condition $\delta \in A$ are stratified. Furthermore, it is clear that $\mathrm{Arg}(t) = A^*$ and $\mathrm{Val}(t) = \mathrm{USC}(A)$. Since $\delta^* = \theta^* \iff \delta = \theta \iff \{\delta\} = \{\theta\}$, it follows that $t \in 1\text{-}1$. Thus we have proved:

1.18. Lemma: $\vdash A \subseteq \mathrm{NO} \Longrightarrow A^* \mathrm{\ sm\ } \mathrm{USC}(A)$.

1.19. Lemma: $\vdash R \subseteq \mathrm{NO}\times\mathrm{NO} \Longrightarrow R^* \mathrm{\ smor\ } \mathrm{RUSC}(R)$.

Proof: Let t be the relation

$$t = \{<\delta^*, \{\delta\}> \mid \delta \in \mathrm{AV}(R)\}.$$

Then, as above, $\mathrm{AV}(R)^* \mathrm{\ sm}_t \mathrm{\ USC}(\mathrm{AV}(R))$, and thus by 1.9 and the fact that $\mathrm{AV}(\mathrm{RUSC}(R)) = \mathrm{USC}(\mathrm{AV}(R))$, [R, p.302] one has

$$\mathrm{AV}(R^*) \mathrm{\ sm}_t \mathrm{\ AV}(\mathrm{RUSC}(R)).$$

Furthermore, if $<\delta_1,\theta_1> \in R^*$, then there are δ,θ with $\theta^* = \theta_1$, $\delta^* = \delta_1$ and $<\delta,\theta> \in R$, by 1.6. Thus, $<\{\delta\},\{\theta\}> \in \mathrm{RUSC}(R)$, so that $<t(\delta_1),t(\theta_1)> \in \mathrm{RUSC}(R)$. The converse is proved similarly; hence $R^* \mathrm{\ smor\ } \mathrm{RUSC}(R)$.

As in [R, p.477], we make the definition:

1.20. Definition: $\mathrm{Card}(\delta) =_{df} \mathrm{Nc}(\{\theta \mid \theta < \delta\})$.

Then, from 1.15 and 1.18, we obtain immediately:

1.21. <u>Lemma</u>: $\vdash \mathrm{Card}(\delta^*) = T(\mathrm{Card}(\delta))$.

<u>Proof</u>:
$$\begin{aligned}\mathrm{Card}(\delta^*) &= \mathrm{Nc}(\{\theta \mid \theta < \delta^*\}) \\ &= \mathrm{Nc}(\{\theta \mid \theta < \delta\}^*) \\ &= \mathrm{Nc}(\mathrm{USC}(\{\theta \mid \theta < \delta\})) \\ &= T(\mathrm{Card}(\delta)).\end{aligned}$$

1.22. <u>Lemma</u>: (i) $\vdash \mathrm{seg}_\theta \leq \, \in \theta^{**}$.

(ii) $\vdash (\mathrm{seg}_\theta \leq \, \in \theta) \iff \theta = \theta^*$

<u>Proof</u>: (i) By [R, p.474], we know that

$$P \in \theta \iff (\mathrm{RUSC}^2(P) \text{ smor } \mathrm{seg}_\theta \leq).$$

Thus if $\theta = \mathrm{No}(P)$, then

$$\begin{aligned}\mathrm{seg}_\theta \leq \, \in \mathrm{No}(\mathrm{RUSC}^2(P)) &= \mathrm{No}(\mathrm{RUSC}(P))^* \\ &= \mathrm{No}(P)^{**} \\ &= \theta^{**}.\end{aligned}$$

(ii) If $\theta = \theta^*$, so that (by 1.3) $\theta = \theta^* = \theta^{**}$, then $\mathrm{seg}_\theta \leq \, \in \theta$ by (i).

Conversely, if $\mathrm{seg}_\theta \leq \, \in \theta$, then by (i) $\theta = \theta^{**}$. Hence if either $\theta < \theta^*$ (whence by 1.3, $\theta < \theta^* < \theta^{**}$) or $\theta^* < \theta$ (whence $\theta^{**} < \theta^* < \theta$) we obtain a contradiction. Thus $\theta = \theta^*$, as claimed.

1.23: <u>Lemma</u>: $\vdash$ stCan($\{\delta \mid \delta < \theta\}$) $\Longleftrightarrow$ $(\delta)(\delta < \theta \Longrightarrow \delta = \delta^*)$.

<u>Proof</u>: If stCan($\{\delta \mid \delta < \theta\}$), then there is a t such that $t \in$ Funct, Arg(t) = $\{\delta \mid \delta < \theta\}$ and for all δ, if $\delta < \theta$ then $t(\delta) = \{\delta\}$. [R, p.486].

Thus: $$(\mathrm{seg}_\theta \leq) \ \mathrm{smor}_t \ \mathrm{RUSC}(\mathrm{seg}_\theta \leq).$$

Also, by 1.15 and 1.19,

$$(\mathrm{seg}_{\theta *} \leq) \ \mathrm{smor}_s \ \mathrm{RUSC}(\mathrm{seg}_\theta \leq),$$

where $\delta^* < \theta^* \Longrightarrow s(\delta^*) = \{\delta\}$. Also $\mathrm{seg}_\theta \leq \ \in$ Word, so that by [R, p.457,460] we have $\theta = \theta^*$. But the isomorphism between two well-orderings is unique, so $t = s$. [R, p.461]. Thus, if $\delta < \theta$, then $t(\delta) = \{\delta\} = s(\delta^*) = t(\delta^*)$. And since $t,s \in$ 1-1 this implies $\delta = \delta^*$. Then since δ was arbitrary, we have

$$(\delta)(\delta < \theta \Longrightarrow \delta = \delta^*). \qquad (1)$$

Conversely, suppose (1) holds and let s be the function (given by 1.19) for which

$$\mathrm{seg}_{\theta *} \leq \ \mathrm{smor}_s \ \mathrm{RUSC}(\mathrm{seg}_\theta \leq).$$

Then $\mathrm{Arg}(s) = \{\delta \mid \delta < \theta^*\} = \{\delta \mid \delta < \theta\}^*$ (by 1.15)
$= \{\delta \mid \delta < \theta\}$ by (1).

Thus $\theta = \theta^*$, and for any $\delta < \theta$ we have $s(\delta) = s(\delta^*) = \{\delta\}$. But then we have $\{\delta \mid \delta < \theta\} \ \mathrm{sm}_s \ \mathrm{USC}(\{\delta \mid \delta < \theta\})$, so that finally we conclude stCan($\{\delta \mid \delta < \theta\}$).

Section 2:

In this section we consider some symmetry properties relative to the *-operation, especially in connection with functions which have their arguments and their values in NO. As a useful example, we consider the set W of ω-numbers in NO, showing that $\theta \in W$ if and only if $\theta^* \in W$. We then use this symmetry of W to prove the main result of this section: if every Cantorian, well-ordered set has only Cantorian subsets, then every Cantorian, well-ordered set is in fact strictly Cantorian.

2.1. Definition: $\omega_o =_{df} \min_{\leq} \{ \delta \mid \neg\{\theta \mid \theta < \delta\} \in \mathrm{Fin} \}$.

Clearly ω_o is a stratified term, and may be assigned any type.

2.2. Lemma: (i) $\vdash \omega_o = \mathrm{No}(\mathrm{Nn}\upharpoonright \leq_c \upharpoonright \mathrm{Nn})$.

(ii) $\vdash \omega_o = \omega_o{}^*$.

Proof: For (i) see [R, p.478], where the equivalence is stated as an exercise. The proof is an easy consequence of the fact that $\delta < \omega_o \iff \mathrm{Card}(\delta) \in \mathrm{Nn}$. Once we have (i), (ii) follows by [R, p.477].

We can now define the ω-numbers of NO:

2.3. Definition:

$$W =_{df} \left\{\theta \mid \omega_o \leq \theta \ \& \ (\delta)(\delta < \theta \Longrightarrow \mathrm{Card}(\delta) <_c \mathrm{Card}(\theta))\right\}.$$

W is a stratified term, and can be assigned any type.

Remark: If $\theta_1, \theta_2 \in W$, then

$$\theta_1 < \theta_2 \Longleftrightarrow (\mathrm{Card}(\theta_1) <_c \mathrm{Card}(\theta_2)).$$

Indeed, if $\theta_1 < \theta_2$, then $\mathrm{Card}(\theta_1) \leq_c \mathrm{Card}(\theta_2)$ by [R, p.478], and $\mathrm{Card}(\theta_1) \neq \mathrm{Card}(\theta_2)$ by definition. The converse follows from the fact that $\leq$ is a linear ordering.

Note that the same holds true if θ_1, θ_2 are taken $< \omega_o$. This follows since any well-ordering R of a finite set is order-isomorphic to an initial segment of the well-ordering $\leq_c$ of Nn [Lemma 2.2 and [R, p.473]], and the length of that segment depends only on the cardinality of AV(R).

2.4. Lemma: (i) $\vdash \omega_o \in W$.

(ii) $\vdash W \subseteq NO$

(iii) $\vdash \delta \in W \Longleftrightarrow \delta^* \in W$.

Proof: (i) follows from 2.2 and [R, p.477]. (That is, $\mathrm{Card}(\omega_o) = \mathrm{Nc}(\mathrm{Nn})$.)

(ii) trivial

(iii) By 1.3 and 2.2, we see that $\omega_o \leq \delta \Longleftrightarrow \omega_o \leq \delta^*$. Then, if $\delta \in W$, and $\theta_1 < \delta^*$, there is a $\theta < \delta$ such that

$\theta^* = \theta_1$, by 1.4. By the assumption that $\delta \in W$, we have $\mathrm{Card}(\theta) <_c \mathrm{Card}(\delta)$, so that by [R, p.368],

$$T(\mathrm{Card}(\theta)) <_c T(\mathrm{Card}(\delta)).$$

However, by 1.21 this is just

$$\mathrm{Card}(\theta^*) <_c \mathrm{Card}(\delta^*).$$

That is, for any $\theta_1 < \delta^*$ we have $\mathrm{Card}(\theta_1) <_c \mathrm{Card}(\delta^*)$, so that $\delta^* \in W$.

Conversely, if $\delta^* \in W$ and $\theta < \delta$, we have $\theta^* < \delta^*$ and hence $\mathrm{Card}(\theta^*) <_c \mathrm{Card}(\delta^*)$. Thus by 1.21 and [R, p.368], $\mathrm{Card}(\theta) <_c \mathrm{Card}(\delta)$. Since this is true for any $\theta < \delta$, we have $\delta \in W$, completing the proof of (iii).

Thus the property of being an ω-number is invariant with respect to the *-operation. We show below that for any set x with the same property (ie. the analogue of 2.4.iii) the function which enumerates x has a similar symmetry. First we must show that such an enumerating function exists.

2.5. <u>Lemma</u>: $\vdash A \subseteq NO \Longrightarrow [\mathrm{No}(A \upharpoonleft \leq \upharpoonright A) \leq \mathrm{No}(\leq)]$.

<u>Proof</u>: Let R denote $A \upharpoonleft \leq \upharpoonright A$. Then by [R, p.337] $R \in \mathrm{Word}$. If the conclusion is false, so that $\mathrm{No}(\leq) < \mathrm{No}(R)$, then for some $\theta \in A$, $\leq$ smor $\mathrm{seg}_\theta R$. It follows (since evidently $AV(\mathrm{seg}_\theta R) = A \cap \{\delta \mid \delta < \theta\}$) that

$$\mathrm{Nc}(NO) \leq_c \mathrm{Nc}(\{\delta \mid \delta < \theta\}),$$

which contradicts T2.1 of [7, p.288], and completes the proof.

It follows from 2.5 that for any $A \subseteq NO$, either $\leq \text{smor}_t\ (A \rceil \leq \lceil A)$ or, for a unique $\theta \in NO$, $(\text{seg}_\theta \leq)\ \text{smor}_t\ (A \rceil \leq \lceil A)$. In either case t is unique, by [R, p.461].

2.6. <u>Definition</u>:

$$\Delta_A =_{df} (\iota t)[(\exists R)(R\ \text{smor}_t\ (A \rceil \leq \lceil A) \\ \& ((R = \leq) v (\exists \theta)(\theta \in NO \ \& \ R = \text{seg}_\theta \leq)]$$

Δ_A is a stratified term, and is assigned the same type as A.

From the above remark, we have immediately:

2.7. <u>Lemma</u>:

$$\vdash A \subseteq NO \implies (\exists_1 R)[R\ \text{smor}_{\Delta_A}\ (A \rceil \leq \lceil A) \\ \& ((R = \leq)\ v\ (\exists \theta)(\theta \in NO \ \& \ R = \text{seg}_\theta \leq))].$$

2.8. <u>Lemma</u>: $\vdash NO^* \neq NO$

<u>Proof</u>: Otherwise, if $\delta = No(\leq)$, then (since $NO^* = NO$ implies $NO^{**} = NO$, by 1.7) there is a θ with $\delta = \theta^{**}$. But then

$$\text{seg}_\theta \leq \ \in \theta^{**} = No(\leq) \qquad \text{(by 1.22)}$$

so that $\text{seg}_\theta \leq$ smor $\leq$, contradicting [R, p.459].

2.9. Definition: $\Omega_o =_{df} \min_{\leq}(NO - NO^*)$

$$\Omega_{k+1} =_{df} \Omega_k^*$$

for each integer k.

2.10. Lemma: (i) $\vdash \; NO^* = \{\delta \mid \delta < \Omega_o\}$

(ii) $\vdash \; (\leq)^* = seg_{\Omega_o}\leq .$

Proof: (i) is simply a restatement of 1.4 and Definition 2.9.

(ii) follows immediately from (i) and 1.3.

2.11. Theorem: $\vdash \; A \subseteq NO \Longrightarrow (\Delta_{(A^*)} = (\Delta_A)^*)$.

Proof: Evidently $(A \upharpoonleft \leq \upharpoonright A)^* = (A^*) \upharpoonleft \leq \upharpoonright (A^*)$, so that by 1.17, if $R \; smor_{\Delta_A} \; (A \upharpoonleft \leq \upharpoonright A)$, then

$$R^* \; smor_{(\Delta_A)^*} \; (A^*) \upharpoonleft \leq \upharpoonright (A^*).$$

But if $R = seg_\theta\leq$, then $R^* = seg_{\theta^*}\leq$ (by 1.15) and if $R = \leq$, then $R^* = (\leq)^* = seg_{\Omega_o}\leq$, by 2.10. Thus $(\Delta_A)^* = \Delta_{(A^*)}$ by 2.7.

2.12. Corollary:

(i) $\vdash \; \delta \in Arg(\Delta_W) \Longrightarrow \delta^* \in Arg(\Delta_W)$

(ii) $\vdash \; \delta \in Arg(\Delta_W) \Longrightarrow (\Delta_W(\delta^*) = \Delta_W(\delta)^*)$.

<u>Proof</u>: By 2.4.iii, $W^* \subseteq W$, and in fact W^* is an initial segment of W under $\leq$. [Since $\delta, \theta^* \in W$ and $\delta < \theta^*$ imply $\delta = \lambda^*$, for some $\lambda < \theta$, and by 2.4 $\lambda \in W$. That is, $\delta \in W^*$]. Thus $\mathrm{Arg}(\Delta_{W^*}) \subseteq \mathrm{Arg}(\Delta_W)$, which by 1.9 and 2.11 implies $(\mathrm{Arg}(\Delta_W))^* \subseteq \mathrm{Arg}(\Delta_W)$. Thus we have proved (i).

Furthermore, if $\delta \in \mathrm{Arg}(\Delta_W)$, then $\delta^* \in \mathrm{Arg}(\Delta_{W^*})$ (again by 1.9 and 2.11) and in fact:

$$\Delta_{W^*}(\delta^*) = \Delta_W(\delta^*)$$

[using [R, p. 461], and the fact that W^* is an initial segment of W]. Thus by 2.11,

$$\Delta_W(\delta^*) = (\Delta_W)^*(\delta^*).$$

so that
$$\Delta_W(\delta^*) = \Delta_W(\delta)^*$$

by 1.11. This completes the proof of 2.12.

<u>Remark</u>: 2.12 could have been proved immediately following 2.4, using an induction over the ordinals in $\mathrm{Arg}(\Delta_W)$, since both statements are stratified. (W is stratified, thus Δ_W and $\mathrm{Arg}(\Delta_W)$ are also and can be assigned any type.) However, the argument given above applies to any set A such that $\theta \in A \iff \theta^* \in A$ whether defined by a stratified formula or not. The technique of delaying as long as possible the use of unstratified conditions such as this, and rather proving theorems such as 2.11 which demonstrate some more general property of symmetry which can then be

applied to special cases, is important in later sections.

We can, for example, use Theorem 2.11 to prove the following:

2.13. Corollary:

$$\vdash A \subseteq NO \;\&\; \theta^* \leq \theta \;\&\; (\delta)(\delta < \theta \Longrightarrow (\delta \in A \Longleftrightarrow \delta^* \in A)) \Longrightarrow$$

$$(\delta)[(\delta \in \mathrm{Arg}(\Delta_A) \;\&\; \Delta_A(\delta) < \theta) \Longrightarrow$$

$$(\delta^* \in \mathrm{Arg}(\Delta_A) \;\&\; \Delta_A(\delta^*) = \Delta_A(\delta)^*)].$$

Proof: Let θ be as in the antecedent and $B = A \cap \{\delta \mid \delta < \theta\}$. Then B is an initial part of A (under $\leq$) and by assumption B* is an initial part of B. Furthermore, if $\delta \in \mathrm{Arg}(\Delta_A)$ then $\delta \in \mathrm{Arg}(\Delta_B) \Longleftrightarrow \Delta_A(\delta) < \theta$, so that it suffices to consider B. The analogue of 2.12 for B is the statement

$$\delta \in \mathrm{Arg}(\Delta_B) \Longrightarrow (\delta^* \in \mathrm{Arg}(\Delta_B) \;\&\; \Delta_B(\delta^*) = \Delta_B(\delta)^*),$$

which is proved exactly in the proof of 2.12. Moreover, it is equivalent to the desired conclusion.

2.14. Lemma: (i) $\vdash \mathrm{No}(\leq) = \Omega_1$

(ii) $\vdash \Omega_o^* < \Omega_o$

(iii) $\vdash \Omega_o \in W$.

Proof: (i) By 2.10 $(\leq)^* = \mathrm{seg}_{\Omega_o}\leq$, so that by 1.15 $(\leq)^{**} = \mathrm{seg}_{\Omega_1}\leq$. But then by 1.19 iterated twice, $(\leq^{**})$ smor $\mathrm{RUSC}^2(\leq)$, and hence $\mathrm{RUSC}^2(\leq)$ smor $\mathrm{seg}_{\Omega_1}\leq$. (i) then follows by [R, p.473].

(ii) Immediate from 2.10.

(iii) If otherwise, then by 2.10.i, there is a θ with $\mathrm{Card}(\Omega_o) = \mathrm{Card}(\theta^*)$. But then, by 2.10.i, 1.15 and 1.18,

$$T(Nc(NO)) = T(Nc(\{\delta \mid \delta < \theta\}))$$

or

$$Nc(NO) = Nc(\{\delta \mid \delta < \theta\}).$$

However, this contradicts T2.1 of [7].

Note that (iii) above and 2.3 imply that $\vdash \Omega_k \in W$ for every integer k.

2.15. <u>Definition</u>: $\omega_\delta =_{df} \Delta_W(\delta)$

2.16. <u>Theorem</u>:

$$\vdash (\theta)(\theta < \delta \Longrightarrow \theta = \theta^*) \Longrightarrow [\omega_\delta \in NO \;\&\; \omega_\delta = (\omega_\delta)^*].$$

<u>Proof</u>: Let $B = \{\theta \mid \theta < \delta\}$, and suppose the assumption holds for δ. Then clearly $B = B^*$, and since $\delta = \mathrm{lub}B$, we have $\delta = \delta^*$ by 1.14. Let $\theta \in NO$ be such that $\omega_\theta = \Omega_o$ (θ exists by 2.14). Then $\Omega_o^* < \Omega_o$, so that

$$\omega_{\theta^*} = (\omega_\theta)^* < \omega_\theta \qquad \text{(by 2.12)}$$

and hence $\theta^* < \theta$. Thus $\delta < \theta$, and hence $\omega_\delta \in NO$. Furthermore, since $\delta = \delta^*$, $\omega_\delta = (\omega_\delta)^*$, by 2.12.

2.17. <u>Corollary</u>: $\vdash \neg \mathrm{stCan}(W)$

<u>Proof</u>: If $\mathrm{stCan}(W)$, then by [R, p.486], $\mathrm{stCan}(\mathrm{Arg}(\Delta_W))$. If $\omega_\theta = \Omega_o$ (so $\theta^* < \theta$ as proved in 2.16), then $\{\delta \mid \delta < \theta\} \subseteq \mathrm{Arg}(\Delta_W)$ so that by [R, p.486], $\mathrm{stCan}(\{\delta \mid \delta < \theta\})$. However, $\theta^{**} < \theta^*$ and $\theta^* \in \{\delta \mid \delta < \theta\}$ (by 1.3) which contradicts 1.23.

<u>Remark</u>: Since for each (ordinal) numeral $\underline{n}_o$ (cf [R, p.475])

$$\vdash (\delta)(\delta < \underline{n}_o \Longrightarrow \delta = \delta^*),$$

by 2.16 we have: $\vdash \omega_{\underline{n}_o} \in NO \;\&\; \omega_{\underline{n}_o} = (\omega_{\underline{n}_o})^*$.

However we see no way to prove in NF that $\omega_{\omega_o} \in NO$. Using 2.16, we can show that

$$\vdash (\theta)(\theta < \omega_o \Longrightarrow \theta = \theta^*) \Longrightarrow \omega_{\omega_o} \in NO.$$

Since the antecedent of this statement is equivalent in NF to Rosser's <u>Counting axiom</u> (cf. [R, p.485]), $\omega_{\omega_o} \in NO$ is provable in the system of [R]. It is a consequence of the next theorem that in that system, the statement $(\theta)(\theta < \omega_{\underline{n}_o} \Longrightarrow \theta = \theta^*)$ is provable, for each (ordinal) numeral $\underline{n}_o$. However it does not seem possible to prove the counterpart for ω_{ω_o} even in the system of [R].

It is a consequence of the symmetry of W, however, that

$$\vdash W \in \mathrm{Infin} \iff \omega_{\omega_0} \in NO .$$

Proof: Clearly if $\omega_{\omega_0} \in NO$, then $W \in \mathrm{Infin}$. Conversely, if $W \in \mathrm{Infin}$, then by 1.18 and [R, p.420,421] $W^* \in \mathrm{Infin}$. But then, if θ is such that $\omega_\theta = \Omega_0$, (so that by 2.11 and 2.10, $W^* = \{\omega_\delta \mid \delta < \theta\}$) then $\{\delta \mid \delta < \theta\} \in \mathrm{Infin}$ as well. But this implies $\omega_0 \leq \theta$, so that $\omega_{\omega_0} \in NO$.

2.18. Theorem: $\vdash (\omega_\theta \in NO \ \&\ \mathrm{stCan}(\{\delta \mid \delta < \omega_\theta\})$
$\implies (\omega_{\theta+1} \in NO \ \&\ \mathrm{stCan}(\{\delta \mid \delta < \omega_{\theta+1}\}))$

Proof: Assume the antecedent holds for θ. If $\delta < \theta$, then $\omega_\delta < \omega_\theta$, so that $\omega_\delta = (\omega_\delta)^*$. By 2.12, this implies $\delta = \delta^*$; hence we have $\mathrm{stCan}(\{\delta \mid \delta < \theta\})$ by 1.23. Then $\theta = \theta^*$ (since $\theta = \mathrm{lub}\{\delta \mid \delta < \theta\}$), so that by 2.16, $\omega_{\theta+1} \in NO$. Moreover, if $\delta < \omega_{\theta+1}$, then $\mathrm{Card}(\delta) = \mathrm{Card}(\omega_\theta)$, so that by assumption and [R, p.435] $\mathrm{stCan}(\{\lambda \mid \lambda < \delta\})$. Thus, as above, $\delta = \delta^*$ (and this holds for any $\delta < \omega_{\theta+1}$). Then by 1.23, $\mathrm{stCan}(\{\delta \mid \delta < \omega_{\theta+1}\})$, which completes the proof.

In 2.19 we define the set of well-ordered sets, WS. The remainder of this section will be devoted to demonstrating that if every Cantorian element of WS has only Cantorian subsets, then every Cantorian element of WS is actually strictly Cantorian.

2.19. <u>Definition</u>: $WS =_{df} \{x \mid (\exists R)(R \in Word \ \& \ x = AV(R)\}$.

2.20. <u>Lemma</u>:

(i) $\vdash (y \text{ sm } x \ \& \ y \in WS) \Longrightarrow x \in WS$

(ii) $\vdash x \in WS \Longrightarrow (\exists\theta)(USC^2(x) \text{ sm } \{\delta \mid \delta < \theta\})$.

<u>Proof</u>: (i) Let $y \text{ sm}_t \ x$ and $y = AV(R)$ for an $R \in Word$. Then let $S = \{<t(u),t(v)> \mid <u,v> \in R\}$. Evidently $AV(S) = x$ and $R \text{ smor}_t \ S$, so that $S \in Word$. Hence $x \in WS$.

(ii) Let $R \in Word$ be such that $x = AV(R)$. Then if $\theta = No(R)$,

$$RUSC^2(R) \text{ smor } seg_\theta\leq$$

by [R, p.473]. Also $USC^2(x) = AV(RUSC^2(R))$ by [R, p.302], so that

$$USC^2(x) \text{ sm } \{\delta \mid \delta < \theta\}$$

which completes the proof.

It is useful to make the following definition:

2.21. <u>Definition</u>:

$$Cd(x) =_{df} \min_{\leq} (\{\theta \mid Nc(USC^2(x)) = Card(\theta)\}).$$

$Cd(x)$ is a stratified term, and is assigned a type one higher than that assigned to x.

2.22. Lemma: (i) $\vdash x \in WS \Longrightarrow (Cd(x) \in W \;.v.\; Cd(x) < \omega_o)$

(ii) $\vdash (x \in WS \;\&\; y \text{ sm } x) \Longrightarrow (Cd(x) = Cd(y))$

(iii) $\vdash x \in WS \Longrightarrow (Cd(USC(x)) = Cd(x)^*)$

Proof: (i) By 2.20, if $x \in WS$, then $Cd(x) \in NO$. Furthermore, if $Cd(x) \notin W$ and $\omega_o \leq Cd(x)$, then there is a $\theta < Cd(x)$ with $Card(\theta) = Card(Cd(x))$. However, by definition, $Card(Cd(x)) = Nc(USC^2(x))$, so that $Card(\theta) = Nc(USC^2(x))$, which is a contradiction of Definition 2.21. Thus either $Cd(x) \in W$ or $Cd(x) < \omega_o$.

(ii) Say $y \text{ sm } x \;\&\; x \in WS$. Then by 2.20 $y \in WS$, and by [R, p.368], $USC^2(y) \text{ sm } USC^2(x)$. Then it is immediate that $Cd(x) = Cd(y)$.

(iii) Let $x \in WS$, so that $Nc(USC^2(x) = Card(Cd(x))$. Then by 1.21,

$$\begin{aligned} Card(Cd(x)^*) &= T(Card(Cd(x))) \\ &= T(Nc(USC^2(x)) \\ &= Nc(USC^3(x)). \end{aligned}$$

Since $USC^3(x) = USC^2(USC(x))$, we have

$$Cd(USC(x) \leq Cd(x)^*.$$

Also, if $\theta < Cd(x)^*$, then by 1.4, $\theta = \delta^*$ for some $\delta < Cd(x)$. Thus $Card(\delta) \neq Nc(USC^2(x))$, implying (by 1.21 and [R, p.368]

$$Card(\delta^*) \neq Nc(USC^3(x)).$$

Thus $Cd(USC(x)) = Cd(x)^*$, completing the proof.

2.23. Lemma:

(i) $\vdash x \in WS \Longrightarrow [(Nc(y) <_c Nc(x)) \Longleftrightarrow (Cd(y) < Cd(x))]$

(ii) $\vdash x \in WS \Longrightarrow [(Nc(y) = Nc(x)) \Longleftrightarrow (Cd(x) = Cd(y))]$

Proof: (i) Let $x \in WS$; then $Nc(y) <_c Nc(x)$ implies $y \in WS$, by 2.20, so $Cd(y) \in NO$. Moreover,

$$\begin{aligned} Nc(y) <_c Nc(x) &\Longleftrightarrow Nc(USC^2(y)) <_c Nc(USC^2(x)) \\ &\Longleftrightarrow Card(Cd(y)) <_c Card(Cd(x)) \\ &\Longleftrightarrow Cd(y) <_c Cd(x) \end{aligned}$$

(The last equivalence follows from 2.22 via the remark following Definition 2.3.)

(ii) follows immediately from Definition 2.21 and [R, p.368].

Remark: We define a relation $R = \{\langle Nc(x), Cd(x)\rangle \mid x \in WS\}$. Then, by 2.22, $R \in Funct$, and $Val(R) \subseteq W \cup \{\theta \mid \theta < \omega_o\}$. Furthermore, if $\theta \in W \cup \{\theta \mid \theta < \omega_o\}$ and $S \in \theta$, then $USC^2(AV(S))$ sm $\{\delta \mid \delta < \theta\}$ and $USC^2(AV(S)) \in WS$. Then by the remark following 2.3, $Cd(AV(S)) = \theta$ which implies $Val(R) = W \cup \{\theta \mid \theta < \omega_o\}$. Furthermore $R \in 1\text{-}1$ (by 2.23). Thus if we define

$$WC =_{df} \{n \mid n \in NC \ \& \ (\exists x)(x \in n \ \& \ x \in WS)\}$$

then $Arg(R) = WC$ and R is a similarity mapping between $\leq_c$ on WC and $\leq$ on $W \cup \{\delta \mid \delta < \omega_o\}$. Note that $Nn \subseteq WC$, and $R(n) < \omega_o \iff n \in Nn$, for all $n \in WC$. Moreover, if $n \in WC$ (so $T(n) \in WC$ also) then by 2.22

$$R(T(n)) = R(n)^*.$$

That is, the ordering of cardinals in WC is order-isomorphic to the ordering of finite ordinals and ω-numbers in a natural way, which commutes with the "type-raising" operations $T(n)$ and θ^*. This relation considerably simplifies the proof of the main result of this section, which follows:

2.24. Theorem:

$$\vdash (x)[(x \in WS \ \& \ Can(x)) \Longrightarrow (y)(y \subseteq x \Longrightarrow Can(y))] \Longrightarrow$$
$$(x)[(x \in WS \ \& \ Can(x)) \Longrightarrow stCan(x))]$$

For simplification, let A_1 be the sentence

$$(A_1) \qquad (x)[(x \in WS \ \& \ Can(x)) \Longrightarrow (y)(y \subseteq x \Longrightarrow Can(y)]$$

and A_2 the sentence

$$(A_2) \qquad (x)[(x \in WS \ \& \ Can(x)) \Longrightarrow stCan(x)].$$

The proof of 2.24 is accomplished by means of two lemmas.

2.25. Lemma:

(i) $\vdash A_1 \Longrightarrow [\mathrm{stCan}(\{\theta \mid \theta < \omega_o\})\ \&$
$(\theta)[(\omega_\theta \in NO\ \&\ \theta = \theta^*) \Longrightarrow \mathrm{stCan}(\{\delta \mid \delta < \theta\})]]$

(ii) $\vdash (\theta)[\theta = \theta^* \Longrightarrow \mathrm{stCan}(\{\delta \mid \delta < \theta\})] \Longrightarrow A_2.$

Proof: (i) Assume A_1 and suppose $\omega_\theta \in NO$ and $\theta = \theta^*$. Then by 2.12 $\omega_\theta = (\omega_\theta)^*$, so that

$$\{\delta \mid \delta < \omega_\theta\} = \{\delta \mid \delta < \omega_\theta\}^* \qquad \text{(by 1.15)}.$$

Thus by 1.18 we have $\mathrm{Can}(\{\delta \mid \delta < \omega_\theta\})$. Then if $\delta < \theta$, so that $\omega_\delta \in NO$ and $\omega_\delta < \omega_\theta$, we have (by A_1)

$$\mathrm{Can}(\{\lambda \mid \lambda < \omega_\delta\}).$$

We wish to show that $\omega_\delta = (\omega_\delta)^*$. If not, then since $\omega_\delta{}^* \in W$

$$\mathrm{Card}(\omega_\delta) \neq \mathrm{Card}((\omega_\delta)^*).$$

But this contradicts $\mathrm{Can}(\{\lambda \mid \lambda < \omega_\delta\})$. That is, for any $\delta < \theta$ we have $\omega_\delta \in NO$ and $\omega_\delta = (\omega_\delta)^*$. Hence by 2.12 $\delta < \theta \Longrightarrow \delta = \delta^*$, so that by 1.23 we have $\mathrm{stCan}(\{\delta \mid \delta < \theta\})$.

Thus we have proved the second clause of the conclusion of (i). To prove $\mathrm{stCan}(\{\theta \mid \theta < \omega_o\})$ we need only note that $\{\theta \mid \theta < \omega_o\}$ is Cantorian, so that $\theta < \omega_o$ implies $\theta = \theta^*$ by A_1 and the remark following 2.3; then use 1.23.

(ii) Assume the antecedent, and suppose that $x \in WS\ \&\ \mathrm{Can}(x)$. That is $x \mathrel{sm} USC(x)$, so that by [R p.368] $x \mathrel{sm} USC^2(x)$. Then by 2.22 $\mathrm{Cd}(x)^* = \mathrm{Cd}(USC(x)) = \mathrm{Cd}(x)$, so that by assumption $\mathrm{stCan}(\{\theta \mid \theta < \mathrm{Cd}(x)\})$. However

$USC^2(x)$ sm $\{\theta \mid \theta < Cd(x)\}$ so that $stCan(USC^2(x))$, and hence $stCan(x)$.

2.26. Lemma:

$$\vdash [(\theta)(\theta = \theta^* \ \&\ \omega_\theta \in NO. \Longrightarrow\ stCan(\{\delta \mid \delta < \theta\}) \ \&\ stCan(\{\theta \mid \theta < \omega_o\})] \Longrightarrow (\theta)[\theta = \theta^* \Longrightarrow stCan(\{\delta \mid \delta < \theta\})].$$

Proof: Assume the antecedent. Then if every θ with $\theta = \theta^*$ also satifies $\omega_\theta \in NO$, we are done. Thus suppose $\theta_o = \theta_o{}^*$ and $\omega_{\theta_o} \notin NO$. By 1.23 and 2.16 $\theta_o \geq \omega_o$. We let

$$\delta_o = \min_{\leq}\{\delta \mid \delta \in W \ \&\ \theta_o \leq \delta\}. \tag{1}$$

Then $\delta_o \in W$. [There is at least one $\delta \in W$ which is greater than θ_o, namely Ω_o.] Also we evidently have $\delta_o = \delta_o{}^*$ [By 2.4 $\delta \in W \Longleftrightarrow \delta^* \in W$, and $\theta_o \leq \delta \Longleftrightarrow \theta_o \leq \delta^*$ by the choice of θ_o. Then use 1.14 to conclude

$$\delta_o = \delta_o{}^*. \tag{2}$$

Next we let δ_1 be the ordinal for which $\omega_{\delta_1} = \Omega_o$. Since $\Omega_o{}^* < \Omega_o$ we have $\delta_1{}^* < \delta_1$ by 2.12. [And thus $\omega_o < \delta_1$ by 1.23 and the assumption.] But then $\omega_{\theta_o} \notin NO$ and by (1) $\theta_o \leq \delta_o$, so that

$$\delta_1 < \delta_o. \tag{3}$$

Then let

$$\delta_2 = \min_{\leq}\{\lambda \mid Card(\lambda) = Card(\delta_1)\}.$$

Evidently $\delta_2 \in W$ and $\delta_2 \leq \delta_1$. Moreover, since $\delta_2 < \delta_o$, by (2) and the assumption, $\delta_2 = \delta_2*$. [If $\delta_2 = \omega_\lambda$, $\delta_o = \omega_\theta$, then $\theta = \theta*$ by (2) and 2.12; so $\lambda < \theta$, and $\lambda = \lambda*$, by the assumption. Hence $\omega_\lambda = (\omega_\lambda)*$, or $\delta_2 = \delta_2*$]. But $\omega_{\delta_2} \in NO$ (since $\delta_2 \leq \delta_1$), so that by the assumption, we have:

$$\mathrm{stCan}(\{\lambda \mid \lambda < \delta_2\}).$$

However, $\mathrm{Card}(\delta_2) = \mathrm{Card}(\delta_1)$, (by definition of δ_2) so that by [R, p.486], $\mathrm{stCan}(\{\lambda \mid \lambda < \delta_1\})$. Thus, by 1.23, and since $\delta_1* < \delta_1$, we have $\delta_1* = \delta_1**$. That is, by 1.3, we have $\delta_1 = \delta_1*$, which is a contradiction. Thus the proof of 2.26 is complete.

The proof of Theorem 2.24 follows from 2.25 and 2.26 immediately. Moreover $\vdash A_2 \Longrightarrow A_1$ is obviously true (cf [R, p.486]), and thus the four statements A_1, A_2, $\omega_{\omega_o} \in NO$ & $(\theta)[(\theta = \theta* \ \& \ \omega_\theta \in NO) \Longrightarrow \mathrm{stCan}(\{\delta \mid \delta < \theta\})$ and $(\theta)[\theta = \theta* \Longrightarrow \mathrm{stCan}(\{\delta \mid \delta < \theta\})]$ are equivalent in NF.

We let NF_1 denote the extension of NF obtained by adding A_1 to the list of axioms: for any statement S we use the symbol $\vdash_{NF_1} S$ to denote the fact that S is a theorem of NF_1 (or equivalently, as a synonym for $\vdash A_1 \Longrightarrow S$).

NF_1 is in some sense a "natural" extension of NF, since its additional axiom A_1 provides for a certain regularity among Cantorian, well-ordered sets. As Rosser has noted in [R], many of the sets used in classical mathematics (for example the sets of integers and real numbers, and the set of functions taking real numbers as arguments and values) can be shown to be Cantorian in NF. [Or more precisely, the sets of NF which are taken to represent these classical objects are Cantorian.] Moreover, the Cantorian sets are "small" in some sense, since for example V and NO are both non-Cantorian. It thus seems striking that the assumption of regularity represented by A_1, in which we restrict attention to the well-ordered sets, should provide for an apparently much stronger property of regularity.

Indeed, it seems proper to regard only the strictly Cantorian sets of NF as those corresponding to classical mathematical objects. Rosser has argued in [11] that by taking this view and by adding to NF statements which postulate the existence of certain strictly Cantorian sets, one can obtain extensions of NF which correspond in strength to the more standard set theories. In fact NF_1 is already strong enough to possess a relative interpretation of Zermelo-Fraenkel set theory, as we prove in Section 3. Moreover, as follows from Theorem 2.27, NF_1 provides increasing sequences of cardinals of strictly Cantorian sets of arbitrary, Cantorian length.

Of course NF_1 may not contain sets which correspond to the sets of Z-F in terms of the $\in$-relation of NF. Also, there is the danger of contradiction, which can never be ignored even for NF itself. Indeed, as we prove in Section 6, by adding A_1 to NF we have extended it in a non-trivial way. (Unless of course NF is already inconsistent.)

2.27. Theorem: $\vdash_{NF_1} (\theta)[\theta = \theta^* \Longrightarrow (\omega_\theta \in NO \ \&\ \omega_\theta = (\omega_\theta)^*)]$.

Proof: By 2.25, if $\theta = \theta^*$ then $\mathrm{stCan}(\{\delta \mid \delta < \theta\})$, so the Theorem follows by 1.23 and 2.16.

Note that by Theorem 2.27 and Lemma 2.25, if θ is an ordinal with $\theta = \theta^*$, then in NF_1 the ω-function provides a sequence of ω-numbers of length θ, all of which are the ordinals of well-orderings of strictly Cantorian sets. Thus, using the correspondence R of the remark following 2.23, the same can be said of θ-length increasing sequences of cardinals of stCan sets in NF_1.

We complete this section with a result needed in later work, but which has a proof similar to ones given above. First it is shown that there is a stratifiably definable function f such that $NO \times NO \ \mathrm{sm}_f \ NO$ and such that $f(<\theta,\delta>)^* = f(<\theta^*,\delta^*>)$. Then, by iteration, a similar result holds for NO^k, for each integer k (where $NO^{n+1} = NO \times NO^n$ is the recursion formula defining each NO^n, according to the standard technique for obtaining an ordered n-tuple from an ordered pair).

2.28. Lemma: For each integer $n > 0$, there is a constant, stratified term f such that

$$\vdash [f \in 1\text{-}1 \,\&\, \mathrm{Arg}(f) = NO^n \,\&\, \mathrm{Val}(f) = NO \,\&\, (\theta_1)\ldots(\theta_n)(f(<\theta_1,\ldots,\theta_n>)^* = f(<\theta_1^*,\ldots,\theta_n^*>))]$$

Proof: We first consider the case $n = 2$. We define a relation R on $NO^2 \times NO^2$ by the condition:

$$R = \{<<\theta_1,\theta_2>,<\delta_1,\delta_2>> \mid [\max(\theta_1,\theta_2) < \max(\delta_1,\delta_2)] \vee [\max(\theta_1,\theta_2) = \max(\delta_1,\delta_2) \,\&\, \theta_1 < \delta_1] \vee [\max(\theta_1,\theta_2) = \max(\delta_1,\delta_2) \,\&\, \theta_1 = \theta_2 \,\&\, \delta_1 \leq \delta_2]\}.$$

Evidently R is a constant, stratified term. We will show R is order-isomorphic to $\leq$, and that the function providing this isomorphism is the one desired. Then if R smor $\leq$ we have R $\mathrm{smor}_f \leq$ for a unique f, which is thus given by a constant, stratified term.

The proof that $R \in$ Word is precisely the same as in the familar set theories. For example, if $A \subseteq NO^2$, let A' be the set of $<\theta,\delta> \in A$ such that $\max(\theta,\delta)$ is minimal for elements of A. Then if $<\theta_1,\delta_1>$ is any element of A-A', it is greater (under R) than any element of A'. Let A" be the set of all elements $<\theta,\delta> \in A'$ with a minimal θ (among members of A'). Then any element of A-A" is R-greater than any element of A". Finally, let $<\theta_0,\delta_0>$ be the element of A" which has a minimal second entry (among members of A").

Then $<\theta_o,\delta_o>$ is the R-least element of A.

Next we show

$$No(R) = No(\leq).$$

If $No(R) < No(\leq)$, then R smor $seg_\theta\leq$, for some $\theta \in NO$. Thus

$$Nc(NO^2) = Nc(AV(R)) = Card(\theta) <_c Nc(NO)$$

as in [7, p.288]). However, evidently $Nc(NO) \leq_c Nc(NO^2)$, so that we have $Nc(NO) <_c Nc(NO)$, a contradiction.

If, on the other hand, $No(\leq) < No(R)$, then for some $<\theta,\delta> \in NO^2$

$$\leq \text{ smor } (seg_{<\theta,\delta>} R).$$

Now, evidently $<<\theta',\delta'>,<\theta,\delta>> \in R$ implies $\max(\theta',\delta') \leq \max(\theta,\delta)$. Thus, if $\max(\theta,\delta) = \theta_o$, we have

$$Nc(NO) \leq_c Nc(\{<\theta',\delta'> \mid \theta',\delta' \leq \theta_o\})$$

or

$$Nc(NO) \leq_c Nc(\{\theta' \mid \theta' < \theta_o + 1\}^2).$$

However $Nc(\{\theta' \mid \theta' < \theta_o + 1\}) \leq_c Nc(USC^2(V))$. [Since, if $P \in \theta_o+1$, then

$$\{\theta' \mid \theta' < \theta_o+1\} \text{ sm } USC^2(AV(P)) \subseteq USC^2(V);$$

see [R, p.473].] Thus

$$Nc(NO) \leq_c Nc(USC^2(V))$$

(since $(USC^2(V) \times USC^2(V))$ sm $USC^2(V)$). However, this is a contradiction (see [R, p.476]). Thus (1) is established.

Now let f be the unique function such that

$$R \ \mathrm{smor}_f \leq$$

We must show that f has the required symmetry property. First, we note that by 1.3 and 1.14,

$$\langle\langle\theta_1,\theta_2\rangle,\langle\delta_1,\delta_2\rangle\rangle \in R \iff \langle\langle\theta_1^*,\theta_2^*\rangle,\langle\delta_1^*,\delta_2^*\rangle\rangle \in R \qquad (2)$$

[For example, by 1.14 $\max(\theta_1,\theta_2)^* = \max(\theta_1^*,\theta_2^*)$, and hence by 1.3 $\max(\theta_1,\theta_2) < \max(\delta_1,\delta_2)$ if and only if $\max(\theta_1^*,\theta_2^*) < \max(\delta_1^*,\delta_2^*)$. The equivalence for the other clauses of the definition of R follow similarly.]
Moreover,

$$\langle\langle\theta_1,\theta_2\rangle,\langle 0,\Omega_o\rangle\rangle \in R \iff (\theta_1,\theta_2 \in NO^*) \vee (\theta_1 = 0 \ \& \ \theta_2 = \Omega_o) \qquad (3)$$

For if the left side holds, either $\max(\theta_1,\theta_2) < \Omega_o$ (whence $\theta_1,\theta_2 \in NO^*$, by 2.10) or $\theta_1 = 0$ and $\theta_2 = \Omega_o$. Conversely, if $\theta_1,\theta_2 \in NO^*$, then $\max(\theta_1,\theta_2) < \Omega_o$. Therefore, if we define

$$S = \left\{\langle\langle\theta_1^*,\theta_2^*\rangle,\langle\delta_1^*,\delta_2^*\rangle\rangle \ \middle|\ \langle\langle\theta_1,\theta_2\rangle,\langle\delta_1,\delta_2\rangle\rangle \in R\right\} \qquad (4)$$

we have (by (3)) $S = \mathrm{seg}_{\langle 0,\Omega_o\rangle}R$. But by (2), if we define $g(\langle\theta_1^*,\theta_2^*\rangle) = f(\langle\theta_1,\theta_2\rangle)^*$ (for each $\theta_1,\theta_2 \in NO$), then

$$S \ \mathrm{smor}_g \ (\leq)^*. \tag{5}$$

However, by 2.10, $(\leq)^* = \mathrm{seg}_{\Omega_o}\leq$, so that

$$(\mathrm{seg}_{<0,\Omega_o>}R) \ \mathrm{smor}_g \ (\mathrm{seg}_{\Omega_o}\leq) \tag{6}$$

But evidently

$$(\mathrm{seg}_{<0,\Omega_o>}R) \ \mathrm{smor} \ (\mathrm{seg}_{f(<0,\Omega_o>)}\leq)$$

so, by [R, p.460], $f(<0,\Omega_o>) = \Omega_o$, and g is just f restricted to $AV(S)$.

That is,

$$g(<\theta_1{}^*,\theta_2{}^*> = f(<\theta_1{}^*,\theta_2{}^*>) \tag{7}$$

for any $\theta_1,\theta_2 \in NO$. Thus, by the definition of g,

$$f(<\theta_1{}^*,\theta_2{}^*>) = f(<\theta_1,\theta_2>)^* \tag{8}$$

for all $\theta_1,\theta_2 \in NO$, completing the proof of 2.28 in the case $n = 2$.

For $n > 2$, we simply iterate the use of f: for example, if $n = 3$, then $NO^3 = NO \times NO^2$ and if we define $f_3(<\theta_1,\theta_2,\theta_3>) = f(<\theta_1,f(<\theta_2,\theta_3>))$ for all $\theta_1,\theta_2,\theta_3 \in NO$, the f_3 is evidently the desired function.

Section 3:

In [7] Orey used a modification of Gödel's construction of constructible sets to show that Zermelo-Fraenkel set theory has a relative interpretation in the extension of NF obtained by adding the statements

(i) $(\theta)(\exists\delta)(\theta \leq \delta \ \& \ \delta \in W)$

(ii) $(x)[(x \subseteq NO \ \& \neg x \text{ sm } NO) \Longrightarrow (\text{lub}(x) \in NO)]$

as axioms. The similarity between these propositions and the definition of a (weakly) inaccessible ordinal is clear, and as Orey notes, the extension of NF obtained by adding the statement

$$(\exists\theta)[\omega_\theta = \theta \ \& \ (x)[(x \subseteq \{\delta \mid \delta < \theta\} \ \& \neg x \text{ sm } \{\delta \mid \delta < \theta\}) \Longrightarrow (\text{lub}(x) < \theta)]]$$

(which expresses the existence of an inaccessible ordinal) is sufficiently strong to permit a relative interpretation of Z-F. Although the first extension of NF appears weaker than the second, the opposite is in fact true, as we now show.

3.1. Definition:

$$\text{In} =_{df} \{\theta \mid \omega_\theta = \theta \ \& \ (x)[(x \subseteq \{\delta \mid \delta < \theta\} \ \& \neg x \text{ sm } \{\delta \mid \delta < \theta\}) \Longrightarrow \text{lub}(x) < \theta]\}.$$

3.2. Lemma: $\vdash \theta \in \mathrm{In} \iff \theta^* \in \mathrm{In}$

Proof: By 2.12, $\omega_\theta = \theta \iff \omega_{\theta^*} = \theta^*$. Moreover, $x \subseteq \{\delta \mid \delta < \theta^*\} \iff (\exists y)(x = y^* \ \&\ y \subseteq \{\delta \mid \delta < \theta\})$ and $x \text{ sm } \{\delta \mid \delta < \theta\} \iff x^* \text{ sm } \{\delta \mid \delta < \theta^*\}$ (by 1.15). Finally, by 1.14 and 1.3, $\mathrm{lub}(x) < \theta \iff \mathrm{lub}(x^*) < \theta^*$ if $x \subseteq \mathrm{NO}$. The desired equivalence follows immediately.

3.3. Theorem: $\vdash$ (i) & (ii) $\Longrightarrow \Omega_o \in \mathrm{In}$

Proof: We note first that if $A \subseteq W$ is non empty and has no maximum element, and $\mathrm{lub}(A) \in \mathrm{NO}$, then $\mathrm{lub}(A) \in W$. Indeed, if this holds for A, and $\theta < \mathrm{lub}(A)$, then for some $\delta \in A$, $\theta < \delta < \mathrm{lub}(A)$. But then

$$\mathrm{Card}(\theta) <_c \mathrm{Card}(\delta) \leq_c \mathrm{Card}(\mathrm{lub}(A))$$

Since this holds for every $\theta < \mathrm{lub}(A)$, and since evidently $\omega_o \leq \mathrm{lub}(A)$, we have $\mathrm{lub}(A) \in W$.

Thus from (i) and (ii) it follows that W sm NO, and hence $\omega_\theta \in \mathrm{NO}$ for all θ. This implies $\omega_{\Omega_o} = \Omega_o$; [since otherwise $\omega_{\theta^*} = \Omega_o$ for some $\theta^* < \Omega_o$. Then for any δ, $\omega_\delta \in \mathrm{NO}$ implies $\omega_{\delta^*} = (\omega_\delta)^*$, so $\omega_{\delta^*} < \Omega_o$ and $\delta^* < \theta^*$. That is $\delta \in \mathrm{NO} \Longrightarrow \delta < \theta$, which is a contradiction.] Moreover, if $x \subseteq \{\theta \mid \theta < \Omega_o\}$, then for some $y \subseteq \mathrm{NO}, x = y^*$. Moreover, if $\neg x \text{ sm } \{\theta \mid \theta < \Omega_o\}$ then $\neg y$ sm NO, so by (ii), $\mathrm{lub}(y) \in \mathrm{NO}$. Then by 1.14 $\mathrm{lub}(x) = \mathrm{lub}(y^*) = \mathrm{lub}(y)^* < \Omega_o$. Hence $\Omega_o \in \mathrm{In}$, completing the proof.

Thus the extension of NF by adding the statement $In \neq \Lambda$ is only apparently stronger than the extension by adding (i) and (ii), and in fact is somewhat weaker. From 3.2 and 1.14 we see immediately that if In is non-empty, then it contains an element θ with $\theta = \theta^*$. Moreover, by 2.11 (as in the proof of 2.12), (i) and (ii) imply that the cardinal of In is greater than $\underline{n}$, for every numeral $\underline{n} = 0,1,\ldots$.

In the remainder of this section we will show that the relation obtained in [7] by modeling the Gödel construction in NO possesses a symmetry of the sort considered in Section 2. We then use this fact to show that by relativizing quantifiers to sets with order θ where $\theta = \theta^*$, we obtain (in NF_1) a relative interpretation of Z-F, and (in NF) a relative interpretation of a weakened form of Zermelo set theory. (Namely, in which the axiom of subsets is weakened to require that the defining condition contain <u>only</u> quantifiers which are relativized to a free variable.)

It is useful to define the property of being a <u>Cantorian ordinal</u> (ie., a θ for which $\theta = \theta^*$), by analogy with the similar notion for sets, and of being a <u>strictly Cantorian ordinal</u> (ie., every smaller ordinal is Cantorian). Since the only ordinal which is Cantorian as a set is $0_o = No(\Lambda)$ (and this is Cantorian as an ordinal) no confusion between the two senses of "Cantorian" or of "strictly Cantorian" will arise.

3.4. <u>Definition</u>: (i) $C(\theta) =_{df} \theta = \theta^*$

(ii) $stC(\theta) =_{df} (\delta)(\delta < \theta \Longrightarrow C(\delta))$

Note that neither $C(\theta)$ nor $stC(\theta)$ are stratified formulas.

We turn now to Orey's modification of the Gödel construction. Although the presentation of [3] is not the most intuitively clear of the various approaches to the concept of "constructible set" which are possible, by taking it as our basis we can make use of the machinery developed in [7] for NF. Orey defines [7, p.281] a well-ordering relation S with field the set of all ordered triples $<\theta,\delta,\lambda>$ of ordinals with $\theta \leq 8_{\bar{o}}$. Specifically, if $<\theta_1,\delta_1,\lambda_1>$ and $<\theta_2,\delta_2,\lambda_2>$ are two such ordered triples,

$$<<\theta_1,\delta_1,\lambda_1>,<\theta_2,\delta_2,\lambda_2>> \in S \iff$$

$$[\max_{\leq}(\delta_1,\lambda_1) < \max_{\leq}(\delta_2,\lambda_2)]$$

$$v[(\max_{\leq}(\delta_1,\lambda_1) = \max_{\leq}(\delta_2,\lambda_2)) \ \&$$

$$[(\lambda_1 < \lambda_2) \ v \ (\lambda_1 = \lambda_2 \ \& \ \delta_1 < \delta_2)$$

$$v \ (\lambda_1 = \lambda_2 \ \& \ \delta_1 = \delta_2 \ \& \ \theta_1 \leq \theta_2)]].$$

Then it is shown that S smor $\leq$, and J is defined as the unique function for which S smor_J $\leq$. Evidently S and J are stratified terms, and can thus be assigned any type. K_1 and K_2 are defined as the unique functions such that for each δ,

$$(\exists\theta)(J(<\theta,K_1(\delta),K_2(\delta)>) = \delta),$$

and are also constant, stratified terms.

Important for our purposes is the following:

3.5. Lemma: (i) $\vdash \theta \le 8_o \Longrightarrow (J(<\theta,\delta,\lambda>)^* = J(<\theta,\delta^*,\lambda^*>))$

(ii) $\vdash K_1(\theta)^* = K_1(\theta^*)$

(iii) $\vdash K_2(\theta)^* = K_2(\theta^*)$

Proof: One first shows, by a proof similar to the proof of 2.28, that if $\theta_1, \theta_2 \le 8_o$ then

$$<<\theta_1,\delta_1,\lambda_1>,<\theta_2,\delta_2,\lambda_2>> \in S \Longleftrightarrow$$

$$<<\theta_1^*,\delta_1^*,\lambda_1^*>,<\theta_2^*,\delta_2^*,\lambda_2^*>> \in S.$$

Then, as in the proof of 2.28 (and since $\theta \le 8_o$ implies $\theta = \theta^*$), (i) follows directly. Hence, if $J(<\theta,K_1(\delta),K_2(\delta)>)=\delta$, we have $J(<\theta,K_1(\delta)^*, K_2(\delta)^*>) = \delta^*$, and (since $J \in 1\text{-}1$) (ii) and (iii) follow.

The construction in NF of the analogue of the function F of [3] which enumerates the constructible sets is based on Orey's observation that any function f whose argument values lie in USC(NO) and whose range is a subset of SC(NO) determines a relation E (a subset of SC(NO)×SC(NO)) by the definition

$$<x,y> \in E \Longleftrightarrow [x,y \subseteq NO \;\&\; \min_{\le}\{\theta \mid x = f(\{\theta\})\} \in y].$$

3.6. Definition:

$$G =_{df} \{f \mid f \in \text{Funct} \ \& \ \text{Val}(f) \subseteq SC(NO) \ \& \ [\text{Arg}(f) = USC(NO) \vee (\exists\theta)(\text{Arg}(f) = \{\{\delta\} \mid \delta < \theta\}]$$

G is a constant, stratified term, and can be assigned any type. It consists of exactly those functions f which are defined on an initial segment of RUSC($\leq$), and for which $\text{Val}(f) \subseteq SC(NO)$.

For any $f \in G$, we define a function f' by letting $\text{Arg}(f') = \{\{\theta*\} \mid \{\theta\} \in \text{Arg}(f)\}$, and setting $f'(\{\theta*\}) = f(\{\theta\})*$. (Note that the right side of the defining equation involves the *-operation on subsets of NO).

3.7. Definition: $f' =_{df} \{<\{\theta*\}, x*> \mid x \subseteq NO \ \& \ <\{\theta\}, x> \in f\}$.

Then f' is a stratified term, and is assigned a type one higher than that of f.

The next Lemma shows that for f in G, the terms defined in [7, p.282] to facilitate the definition of the function corresponding to Gödel's F (which enumerates the constructible sets) possess a symmetry relative to the operation of passing from f to f'. [Note that each such term is stratifiable in such a way that it has the same type as each of its arguments except f, which has one higher type.]

3.8. <u>Lemma</u>:

(i) $\vdash$ $f \subseteq USC(NO) \times SC(NO) \Longrightarrow (f \in G \Longleftrightarrow f' \in G)$

(ii) $\vdash$ $(f \in G \;\&\; x \subseteq NO) \Longrightarrow (Od_{f'}(x) = Od_f(x)^*)$

(iii) $\vdash$ $f \in G \Longrightarrow (NO_{f'} = (NO_f)^*)$

(iv) $\vdash$ $(f \in G \;\&\; x,y \subseteq NO) \Longrightarrow (x \in^f y \Longleftrightarrow x^* \in^{f'} y^*)$

(v) $\vdash$ $(f \in G \;\&\; x,y \in Val(f)) \Longrightarrow [\{x^*,y^*\}^{f'} = (\{x,y\}^{f'})^*]$

(vi) $\vdash$ $(f \in G \;\&\; x,y \in Val(f)) \Longrightarrow [<x^*,y^*>^{f'} = (<x,y>^f)^*]$

(vii) $\vdash$ $f \in G \Longrightarrow (E^{f'} = (E^f)^*)$

(viii) $\vdash$ $(f \in G \;\&\; y \subseteq NO) \Longrightarrow [(V \times (y^*))^{f'} = ((V \times y)^f)^*]$

(ix) $\vdash$ $(f \in G \;\&\; y \subseteq NO) \Longrightarrow (D^{f'}(y^*) = D^f(y)^*)$

(x) $\vdash$ $(f \in G \;\&\; y \subseteq NO) \Longrightarrow (Cnv_1{}^{f'}(y^*) = Cnv_1{}^f(y)^*)$

(xi) $\vdash$ $(f \in G \;\&\; y \in Val(f)) \Longrightarrow (Cnv_2{}^{f'}(y^*) = Cnv_2{}^f(y)^*)$

(xii) $\vdash$ $(f \in G \;\&\; y \in Val(f)) \Longrightarrow (Cnv_3{}^{f'}(y^*) = Cnv_3{}^f(y)^*)$

<u>Proof</u>: The proof is elementary, and in most cases we give only a sketch.

(i) If $f \subseteq USC(NO) \times SC(NO)$, we have

$$<\{\theta\},x> \in f' \Longleftrightarrow (\exists\delta)(\exists y)[\theta = \delta^* \;\&\; x = y^* \;\&\; <\{\delta\},y> \in f] \quad (1)$$

Thus we see that $f \in Funct \Longleftrightarrow f' \in Funct$, and

$$Arg(f') = \{\{\delta^*\} \mid \{\delta\} \in Arg(f)\} \quad (2)$$

and $$Val(f') = \{y^* \mid y \in Val(f)\}. \quad (3)$$

It follows immediately that $f \in G \Longleftrightarrow f' \in G$.

(ii) This follows directly from the definition of f', since if $f \in G$.

$$f(\{\theta\}) = y \iff f'(\{\theta*\}) = y* \qquad (4)$$

by (1). Then use 1.3 and 1.4.

(iii) Use (ii) and (3)

(iv) By (i), if $x \subseteq NO$ then $Od_{f'}(x*) \in NO \iff Od_f(x) \in NO$; also if $y \subseteq NO$, $Od_{f'}(x*) \in y* \iff Od_f(x) \in y$.

(v) follows immediately from (i) and the fact that $\{\theta,\delta\}* = \{\theta*,\delta*\}$.

(vi) Immediate from (v).

(vii) E^f is

$$\{\theta \mid (\exists x)(\exists y)(x,y \in Val(f) \ \& \ x \in^f y \ \& \ \theta = Od_f(\langle x,y\rangle^f)\}.$$

Then (vii) follows, using (3), (i), (iv) and (vi). That is, for example, if $\theta = Od_f(\langle x,y\rangle^f)$ (where $x,y \in Val(f)$ and $x \in^f y$), then $x*,y* \in Val(f')$ by (3), $x* \in^{f'} y*$ by (iv), and $\theta* = Od_{f'}(\langle x*,y*\rangle^{f'})$ by (vi) and (i). Hence $(E^f)* \subseteq (E^{f'})$. The converse follows by a similar argument.

(viii) If $y \subseteq NO$ $(V \times y)^f$ is

$$\{\theta \mid (\exists x)(\exists z)(x,z \in Val(f) \ \& \ z \in^f y \ \& \ \theta = Od_f(\langle x,z\rangle^f)\}.$$

The desired equality follows by (3), (i), (iv) and (vi).

(ix) If $y \subseteq NO$, $D^f(y)$ is

$$\{\theta \mid \theta \in NO_f \ \& \ (\exists z)(z \in Val(f) \ \& \ <z, f(\{\theta\})>^f \in^f y\}.$$

The equality then follows by (3), (iii), (iv) and (v).

(x) If $y \subseteq NO$, then $Cnv_1{}^f(y)$ is the set

$$\{\theta \mid (\exists x)(\exists z)(x, z \in Val(f) \ \& \ \theta = Od_f(<x,z>^f) \ \& \ <z,x>^f \in^f y\}$$

The equality then follows, using (3), (i), (iv) and (v).

(xi) If $y \in Val(f)$, $Cnv_2{}^f(y)$ is the set

$$\{\theta \mid (\exists x)(\exists x)(\exists v)(x, z, v \in Val(f) \ \& \ \theta = Od_f(<x,<z,v>^f>^f)$$
$$\& \ <z,<v,x>^f>^f \in^f y\}.$$

If the condition of membership in this set holds for θ [so that for some $x, z, v \in Val(f)$ we have $\theta = Od_f(<x,<z,v>^f>^f)$ and $<z,<v,x>^f>^f \in^f y$] then evidently both $<z,v>^f$ and $<v,x>^f$ are in $Val(f)$. (Otherwise, for example, $Od_f(<z,v>^f) = \Lambda$, and

$$\begin{aligned} <x,<z,v>^f>^f &= \{\{x,x\}^f, \{x,<z,f>^f\}^f\}^f \\ &= \{\{Od_f(x)\}, \{Od(x),\Lambda\}\}^f \\ &= \{Od_f(\{Od_f(x)\}),\Lambda\} \neq f(\{\theta\}) \end{aligned}$$ which is

a contradiction (Note: since $Val(f) \subseteq SC(NO)$, if $x \notin NO$ then $Od_f(x) = \Lambda$). But then the equality follows by (3), (i), (iv) and (v).

(xii) Proof is as in (xi).

3.9. <u>Definition</u>: (i) $f_\theta =_{df} \{\{\delta\} \mid \delta < \theta\} \upharpoonright f$

Let $G_1(f)$ be the formula

$$(\theta)[\{\theta\} \in \mathrm{Arg}(f) \Longrightarrow$$

$$\theta \in \mathrm{Val}(J_o) \mathbin{\&} f(\{\theta\}) = \{\delta \mid \delta \in NO_{f_\theta} \mathbin{\&} \delta < \theta\}$$

$$\vee\ \theta \in \mathrm{Val}(J_1) \mathbin{\&} f(\{\theta\}) = \{f_\theta(\{K_1(\theta)\}), f_\theta(\{K_2(\theta)\})\}^{f_\theta}$$

$$\vee\ \theta \in \mathrm{Val}(J_2) \mathbin{\&} f(\{\theta\}) = f_\theta(\{K_1(\theta)\}) \cap E^{f_\theta}$$

$$\vee\ \theta \in \mathrm{Val}(J_3) \mathbin{\&} f(\{\theta\}) = f_\theta(\{K_1(\theta)\}) \cap (NO - f_\theta(\{K_2(\theta)\}))$$

$$\vee\ \theta \in \mathrm{Val}(J_4) \mathbin{\&} f(\{\theta\}) = f_\theta(\{K_1(\theta)\}) \cap (V \times f_\theta(\{K_2(\theta)\}))^{f_\theta}$$

$$\vee\ \theta \in \mathrm{Val}(J_5) \mathbin{\&} f(\{\theta\}) = f_\theta(\{K_1(\theta)\}) \cap D^{f_\theta}(f_\theta(\{K_2(\theta)\}))$$

$$\vee\ \theta \in \mathrm{Val}(J_6) \mathbin{\&} f(\{\theta\}) = f_\theta(\{K_1(\theta)\}) \cap \mathrm{Cnv}_1{}^{f_\theta}(f_\theta(\{K_2(\theta)\}))$$

$$\vee\ \theta \in \mathrm{Val}(J_7) \mathbin{\&} f(\{\theta\}) = f_\theta(\{K_1(\theta)\}) \cap \mathrm{Cnv}_2{}^{f_\theta}(f_\theta(\{K_2(\theta)\}))$$

$$\vee\ \theta \in \mathrm{Val}(J_8) \mathbin{\&} f(\{\theta\}) = f_\theta(\{K_1(\theta)\}) \cap \mathrm{Cnv}_3{}^{f_\theta}(f_\theta(\{K_2(\theta)\}))].$$

Then we define

(ii) $\qquad G_1 =_{df} \{f \mid G_1(f) \mathbin{\&} f \in G\}$

G_1 is a constant, stratified term, and can be assigned any type.

One then proves by induction over $\leq$ (as in [7], using Theorems XII.2.9-11 of [R]) that for each θ, there is a unique $f \in G_1$ having $Arg(f) = USC\{\delta \mid \delta < \theta\}$, and a unique $f \in G_1$ with $Arg(f) = USC(NO)$. The object in displaying Definition 3.9 is that, with Lemma 3.8, one obtains

3.10. <u>Lemma</u>: $\vdash f \in G_1 \Longrightarrow f' \in G_1$.

<u>Proof</u>: One has evidently, if $\{\theta\} \in Arg(f)$

$$(f')_{\theta*} = (f_\theta)' \tag{1}$$

Moreover, by 3.5, for each $\lambda \leq 8_o$ and each δ,

$$\delta \in Val(J_\lambda) \Longleftrightarrow \delta* \in Val(J_\lambda) \tag{2}$$

(since $\lambda = \lambda*$ in that case). Thus one proves $G_1(f')$ by examining each of the nine cases $\{\theta\} \in Arg(f')$ & $\theta \in Val(J_\lambda)$, for $\lambda \leq 8_o$. For example, if $\{\theta\} \in Arg(f')$ and $\theta \in Val(J_o)$, then for some δ: $\theta = \delta*$, $\delta \in Arg(f)$ and $\delta \in Val(J_o)$. Then one has

$$f(\{\delta\}) = \{\lambda \mid \lambda \in NO_{f_\delta} \ \&\ \lambda < \delta\}$$

or $$f(\{\delta\})* = \{\lambda \mid \lambda \in NO_{(f_\delta)'} \ \&\ \lambda < \delta*\} \text{ (by 3.8.iii)}$$

or $$f(\{\delta\})* = \{\lambda \mid \lambda \in NO_{(f')_\theta} \ \&\ \lambda < \theta\}, \text{ (by (1))}$$

That is, $$f'(\{\theta\}) = \{\lambda \mid \lambda \in NO_{(f')_\theta} \ \&\ \lambda < \theta\},$$

so f' satisfies the first clause, as well.

The other clauses are treated similarly, using the appropriate statement from 3.8. Thus we establish $G_1(f')$. Also, by 3.8.i, $f' \in G$; together these imply $f' \in G_1$ and we are done.

3.11. Definition: $s = (\iota f)(f \in G_1 \ \& \ Arg(f) = USC(NO))$

The uniqueness of s (as the member of G_1 with USC(NO) as set of argument values) is proved as sketched in the remark following 3.9. This is just the function s of [7, p.283], and is a stratified term which can thus be assigned any type. From 3.10, and the uniqueness of the member of G_1 whose argument values lie in $USC(\{\theta \mid \theta < \Omega_o\})$ we have immediately the desired symmetry of s:

3.12. Theorem: $\vdash s(\{\theta^*\}) = s(\{\theta\})^*$.

Proof: s' is in G_1, by 3.10, and

$$Arg(s') = \{\{\theta^*\} \mid \{\theta\} \in Arg(s)\} = USC(\{\theta \mid \theta < \Omega_o\}).$$

But one also has

$$(USC(\{\theta \mid \theta < \Omega_o\}) \upharpoonleft s) \in G_1.$$

By the uniqueness property of G_1, this implies

$$s' = (USC(\{\theta \mid \theta < \Omega_o\}) \upharpoonleft s);$$

that is, for any $\theta < \Omega_o$, $s'(\{\theta\}) = s(\{\theta\})$. Hence, for all $\theta \in NO$,

$$s(\{\theta*\}) = s'(\{\theta*\}) = s(\{\theta\})*,$$

completing the proof of 3.12. [Note that $s(\{\theta\})$ is a subset of NO, so $s(\{\theta\})* = \{\delta* \mid \delta \in s(\{\theta\})\}$].

Remark: Once we had proved 3.8, Theorem 3.12 could have been proved by induction over $\leq$, since s is a stratified term. At any rate, the argument is straightforward and elementary, although checking the details of each step results in a lengthy discourse. However, familiarity with [7] and with the type of argument used in Sections 1 and 2 to demonstrate other forms of symmetry should suffice to make the above argument for Theorem 3.12 convincing.

Following [7] we make the definition:

3.13. Definition: $L = \{x \mid Od_s(x) \in NO\}$

L is a constant, stratified term (since s is) and may be assigned any type.

3.14. Lemma: $\vdash \theta \in s(\{\delta\}) \Longrightarrow (\theta \in NO_s \;\&\; \theta < \delta)$.

Proof: The proof is by induction on δ (over $\leq$), and is exactly as in the proof of T11 of [7].

Note that it follows from 3.14 that if $x \in^{s} y$ and $y \in L$ (or even $y \subseteq NO_s$) then $x \in L$.

Let Π denote the class of formulas constructed from $x \in^{s} y$ and $x = y$ by using quantifiers relativized to L and truth-functional connectives. That is, Π is the smallest class of formulas satisfying

(i) $x \in^{s} y$ and $x = y$ are in Π for each pair of variables x and y;

(ii) if A and B are in Π, so are $\neg A$, $(A \vee B)$ and $(A \& B)$;

(iii) if A is in Π, so are $(\exists x)(x \in L \& A)$ and $(x)(x \in L \Longrightarrow A)$, for each variable x.

For each formula A of Π, and each variable x not occuring in A, we let $A(x \mid y_o,\ldots,y_n)$ (where $y_o,\ldots,y_n$ are the free variables of A) denote the result of replacing each occurrence of $z \in L$ by $z \in^{s} x$ (where z is the variable of a quantifier introduced in the construction of A as a member of Π). Each such formula is equivalent to a formula of Π with free variables $x, y_o, \ldots y_n$. Note that any formula of Π is stratifiable in such a way that all of its free variables are assigned the same type.

We will need the fact that for any $x \in L$ and any $y_1,\ldots,y_n \in^{s} x$,

$$\{Od_s(y_o) \mid A(x \mid y_o,y_1,\ldots,y_n) \ \& \ y_o \in^{s} x\} \in L.$$

That is, that any set $y \subseteq NO_s$ which is first order definable over x, relative to $\in^s$, is again a member of L. The proof uses standard techniques for constructing sets based on the analysis of the syntactic structure of A (as a member of Π). It is first necessary to prove Lemma 3.16 (below), which follows from the analogues of T14 through T26 of [7] for s. (As noted in Section 2 of [7] these are all provable in NF.] First we define:

3.15. Definition: $(x \times y)^s = \{<u,v>^s \mid u \in^s x \;\&\; v \in^s y\}$.

3.16. Lemma: $\vdash x,y \in L \Longrightarrow$
$[(x \times y)^s \in L \;\&\; x \cup y \in L \;\&\; x \cap y \in L \;\&\; D^s(x) \in L$
$\;\&\; x \cap E^s \in L \;\&\; Cnv_1{}^s(x) \in L \;\&\; Cnv_2{}^s(x) \in L \;\&\; Cnv_3{}^s(x) \in L]$.

Proof: The proof for the last six terms depends on the fact that the set $\{Od_s(z) \mid z \in^s t\}$ is bounded in NO (where t is the term for which $t \in L$ is being proved). Once this is established, $t \in L$ follows by the analogue of T17 (from [7]) for s, and the appropriate analogue for s from T19-24. The exceptions are $x \cap y \in L$ (which follows immediately from the analogue of T15), $x \cup y \in L$ and $(x \times y)^s \in L$.

We treat this latter term first. Suppose $x,y \in L$. Then if $\theta = \max_{\leq}(Od_s(x), Od_s(y))$, by 3.14 and the definition of s,

$$Od_s(\langle u,v\rangle^s) \le J_1(J_1(\theta,\theta),\ J_1(\theta,\theta))$$

whenever $u \in^s x$ and $v \in^s y$ or $v \in^s x$ and $u \in^s y$. Thus there is a $z \in L$ such that $(x \times y)^s \subseteq z$ and $(y \times x)^s \subseteq z$. Thus we have

$$(x \times y)^s = z \cap (V \times y)^s \cap Cnv_1{}^s(z \cap (V \times x)^s) \qquad (1)$$

and $z \cap (V \times y)^s \in L$ by the analogue of T14. Also $z \cap (V \times x)^s \in L$, and thus $z \cap Cnv_1{}^s(z \cap (V \times x)^s) \in L$ as well. Putting these together with (1) gives $(x \times y)^s \in L$.

To show $x \cup y \in L$, we again note the existence of a $z \in L$ with $x \cup y \subseteq z$. But then

$$x \cup y = \overline{(z \cap \overline{x}) \cap (z \cap \overline{y})} \cap Z$$

which is in L since $z \cap \overline{x}$ and $z \cap \overline{y}$ are.

3.17. Definition: $\langle y\rangle^s =_{df} y$

$$\langle x_1,\ldots,x_{n+1}\rangle^s = \langle x_1,\langle x_2,\ldots,x_{n+1}\rangle^s\rangle^s$$

3.18. Theorem: Let $A(y_1,\ldots,y_n x_1,\ldots,x_m)$ be a formula of Π with only the free variables exhibited, and let x,y,z be any variables not in A. Then,

$$\vdash (x)(y_1)\ldots(y_n)[y_1 \in^s x \ \&\ldots\&\ y_n \in^s x \Longrightarrow$$

$$(\exists z)[z \in L \ \&\ (y)[y \in^s z \Longleftrightarrow (\exists x_1)\ldots(\exists x_n)((x_1 \in^s x \ \&\ldots\&\ x_n \in^s x)$$

$$\&\ (y = \langle x_1,\ldots,x_n\rangle^s) \ \&\ A(x \mid y_1,\ldots,y_n,x_1,\ldots,x_n))]].$$

<u>Proof</u>: We must prove this more general result in order to proceed by induction on the construction of A as an element of Π. If A is $x_1 = x_1$, then we take $z = x$. If A is $x_1 \in^s y_1$, then we take $z = y_1 \cap x$. If A is $y_1 \in^s x_1$, we take $z = D^s((\{y_1\}^s \times x_1)^s \cap E^s)$. The other possible atomic formulas are treated similarly. If A is $\neg B$, then we take $z = (x \times (x \times (\ldots \times x)^s \ldots)^s \cap z_1$ where z_1 is the element of L given by the induction assumption for B.

If A is $(B \vee C)$, we use the cross product $(u \times v)^s$ and the iterations of $Cnv_1{}^s$, $Cnv_2{}^s$ and $Cnv_3{}^s$ in the usual way to adjust for the difference between the free variables of B and C, and then use union. If A is $(B \& C)$ we treat it as $\neg(\neg B \vee \neg C)$. If A is $(\exists x_o)(x_o \in L \& B)$, then $A(x \mid \ldots)$ is $(\exists x_o)(x_o \in^s x \& B(x \mid \ldots))$. Then if z_1 is the element of L given by the induction assumption for B, then setting $z = D^s(z_1)$ establishes the Theorem for A. Universal quantifiers are treated as negations of existential quantiifiers.

The relative interpretation of the weakened form of Zermelo set theory we will consider is obtained by restricting attention to those elements[x] of L with $x = x^*$ (equivalently, by 3.12, those x for which $Od_s(x)$ is a Cantorian ordinal).

3.19. <u>Definition</u>: $L^C(x) =_{df} x \in L \& C(Od_s(x))$.

Evidently $L^C(x)$ is <u>not</u> a stratified formula. Note that $L^C(x) \iff x \in L \;\&\; x = x^*$ by 3.12 and 1.14.

For each formula A in Π, we define A^C as the result of replacing each instance of $z \in L$ in A by $L^C(z)$, where z is the variable of a quantifier introduced in the construction of A as an element of Π. We define $A^C(x \mid \ldots)$ as the result of similarly relativizing each quantifier of $A(x \mid \ldots)$ by $L^C(z)$. [For example, $(\exists z)(z \in^s x \;\&\ldots)$ becomes $(\exists z)(L^C(z) \;\&\; z \in^s x \;\&\ldots)$.]

For each formula F in the language of set theory (ie., a formula of the first order predicate calculus whose only predicate symbols are $\in$ and $=$) the <u>C-translate</u> of F is the result of replacing every instance of $\in$ by $\in^s$ and relativizing every quantifier by L^C.

We show first that the C-translate of the extensionality axiom is a theorem of NF, following this by treatment of the axioms of foundation, pairing, union and power set and the axiom of subsets in the case where the defining condition has only relativized quantifiers.

3.20. <u>Lemma</u>:

(i) $\vdash [x,y \in L \;\&\; (z)(z \in^s x \iff z \in^s y)] \implies x = y$

(ii) $\vdash [L^C(x) \;\&\; L^C(y) \;\&\; (z)(L^C(z) \implies (z \in^s x \iff z \in^s y))]$
$\implies x = y$

<u>Proof</u>: (i) By 3.14, $x \in L \Longrightarrow x \subseteq NO_s$. If $x, y \in L$ and $x \neq y$, we may suppose without loss of generality that there is a $\theta \in NO_s$ such that $\theta \in x \cap \overline{y}$. But then $s(\{\theta\}) \in L$, and $s(\{\theta\}) \in^s x, \neg s(\{\theta\}) \in^s y$, which establishes (i).

(ii) Suppose $L^C(x)$, $L^C(y)$ and $x \neq y$. Then we may suppose $x \cap \overline{y}$ is non-empty. Let $\theta = \min_{\leq}(x \cap \overline{y})$, so by 3.14, $\theta \in NO_s$.

Since $x = x^*$ and $y = y^*$ (as subsets of NO), for any $\delta \in NO$

$$\begin{aligned}\delta \in x \cap \overline{y} &\Longleftrightarrow \delta \in x \,\&\, \neg \delta \in y\\ &\Longleftrightarrow \delta^* \in x \,\&\, \neg \delta^* \in y\\ &\Longleftrightarrow \delta^* \in x \cap \overline{y}.\end{aligned}$$

Then by 1.14 $\theta = \theta^*$. so that $L^C(s(\{\theta\}))$. But we also have $s(\{\theta\}) \in^s x$ and $\neg s(\{\theta\}) \in^s y$. Thus the proof of (ii) is complete.

Next we prove the C-translate of the axiom of foundation.

3.21. <u>Lemma</u>:

$$\vdash L^C(x) \Longrightarrow [x = \Lambda \vee (\exists y)(L^C(y) \,\&\, y \in^s x \,\&\, y \cap x = \Lambda)]$$

<u>Proof</u>: Since $L^C(x)$, we have $x = x^*$. Then if $x \neq \Lambda$, set $\theta = \min_{\leq}(x)$; thus by 1.14 $\theta = \theta^*$, and hence $L^C(s(\{\theta\}))$ and $s(\{\theta\}) \in^s x$. Furthermore, if $y \in^s s(\{\theta\})$, we have $Od_s(y) < \theta$ by 3.14, and hence $\neg y \in^s x$ by the

definition of θ. Thus $x \cap s(\{\theta\}) = \Lambda$ and the proof is complete.

The C-translate of the axiom of pairing is evidently provable in NF: If $L^C(x)$ and $L^C(y)$, then

$$\{x,y\}^s = \{Od_s(x), Od_s(y)\} = \{Od_s(x), Od_s(y)\}^*.$$

3.22. Definition:

$$(\bigcup x)^s =_{df} \{Od_s(y) \mid (\exists z)(y \in^s z \ \& \ z \in^s y)\}$$

3.23. Lemma: (i) $\vdash x \in L \implies (\bigcup x)^s \in L$

(ii) $\vdash L^C(x) \implies L^C((\bigcup x)^s)$

Proof: (i) By 3.14, if $y \in^s (\bigcup x)^s$, then $Od_s(y) < Od_s(x)$. Thus there is an element z of L such that $(\bigcup x)^s \subseteq L$. But then $(\bigcup x)^s = D^s(Cnv_1{}^s(E^s \cap (z \times x)^s))$ which is in L by 3.16.

(ii) Suppose $L^C(x)$. Then $x = x^*$, so by 3.8 and 3.12, if $y \in L$

$$y \in^s x \iff y^* \in^s x.$$

Moreover, if $y \in^s x$ we have $Od_s(y) < Od_s(x)$ by 3.14, so that there is a $\theta < Od_s(x)$ with $\theta^* = Od_s(y)$. (By 1.3, since $C(Od_s(x))$. Then by 3.12, $y = s(\{\theta\})^*$, and by (1) $s(\{\theta\}) \in^s x$.

Thus if $u \in^s (\bigcup x)^s$, so that for some z, $u \in^s z$ and $z \in^s x$, we have $u^* \in^s z^*$ and $z^* \in^s x$, or $u^* \in^s (\bigcup x)^s$. Moreover, if $u^* \in^s z$ and $z \in^s x$, then for some $y \in L$,

$z = y^*$ and $u \in^s y$, $y \in^s x$, that is $u \in^s (\bigcup x)^s$. Thus if $u \in L$, $u \in^s (\bigcup x)^s \iff u^* \in^s (\bigcup x)^s$. Finally, since for every $u \in^s (\bigcup x)^s$ we have $Od_s(u) < Od_s(x)$ and $C(Od_s(x))$, we must have $L^C((\bigcup x)^s)$.

3.23 does not quite prove that the C-translate of the axiom of union is a theorem of NF, since the quantifier in the definition of $(\bigcup x)^s$ ranges over all of L. Evidently if $L^C(x)$, $L^C(u)$, and $L^C(z)$, and if $u \in^s z$ $z \in^s x$, we obtain $u \in^s (\bigcup x)^s$. To prove the converse, suppose $L^C(u)$, $L^C(x)$ and $u \in^s (\bigcup x)^s$. Then let

$$A = \{\theta \mid u \in^s s(\{\theta\}) \ \&\ s(\{\theta\}) \in^s x\},$$

and set $\theta_o = \min_{\leq} A$.

If $\theta \in A$, then by 3.12 and 3.8, (and the fact that $u = u^*$, $x = x^*$).

$$u \in^s s(\{\theta^*\}) \quad \text{and} \quad s(\{\theta^*\}) \in^s x$$

so $\theta^* \in A$. Moreover, if $\theta^* \in A$, we have

$$u \in^s s(\{\theta\}) \quad \text{and} \quad s(\{\theta\}) \in^s x$$

again by 3.12 and 3.8. That is $\theta \in A$. Then since A is bounded by $Od_s(x)$ (by 3.14) we have $A = A^*$, and hence $\theta_o = \theta_o^*$ by 1.14. Hence $L^C(s(\{\theta_o\}))$, and

$$u \in^s s(\{\theta_o\}), \ s(\{\theta_o\}) \in^s x$$

by (1).

Thus the C-translate of the axiom of union is a theorem of NF.

3.24. Lemma:

$$\vdash L^C(x) \mathbin{\&} L^C(y) \Longrightarrow [(z)[(L^C(z) \mathbin{\&} z \in^s x) \Longrightarrow z \in^s y] \Longrightarrow x \subseteq y]$$

Proof: Suppose $L^C(x)$, $L^C(y)$ and $(z)[(L^C(z) \mathbin{\&} z \in^s x) \Longrightarrow z \in^s y]$. If $x \not\subseteq y$, then $x \cap \overline{y}$ is non-empty. If we set $\theta_o = \min_{\leq}(x \cap \overline{y})$ then $\theta_o = \theta_o{}^*$ (as in the proof of 3.20.ii), so that $L^C(s(\{\theta_o\}))$; also $s(\{\theta_o\}) \in^s x$ while $\neg s(\{\theta_o\}) \in^s y$, which completes the proof of 3.24.

3.25. Definition: $P^s(x) =_{df} \{Od_s(y) \mid y \leq x\} \cap NO$.

Note that if $x \in L$, $P^s(x) \subseteq NO_s$. To prove that the C-translate of the axiom of power-set is a theorem of NF it is sufficient to show $L^C(P^s(x))$ if $L^C(x)$. [For by 3.24, if $L^C(x)$ then $x \subseteq y$ if and only if x and y satisfy the appropriate translated formula -- namely

$$(z)[(L^C(z) \mathbin{\&} z \in^s x) \Longrightarrow z \in^s y].]$$

3.26. Lemma: $\vdash L^C(x) \Longrightarrow L^C(P^s(x))$.

<u>Proof</u>: (i) We first prove $P^s(x) \in L$ whenever $L^c(x)$, by using the analogue of T28 of [7] for s. (As Orey notes in Section 2 of that paper, this analogue is a theorem of NF.) Let $\theta = Od_s(x)$, so $\theta = \theta^*$ by assumption. Thus if ω_δ is the least element of W greater than θ (which exists since $\theta < \Omega_o$) we evidently have $\omega_\delta = (\omega_\delta)^*$, and hence $\delta = \delta^*$. Then one also has $\omega_{\delta+1} \in NO$ (since $\omega_\delta < \Omega_o$). Then since ω_δ, $\omega_{\delta+1} \in Val(J_o)$, if $u \subseteq x$ and $u \in L$ then $u \subseteq s(\{\omega_\delta\})$; hence by the analogue of T28 for s, $u \in^s s(\{\omega_{\delta+1}\})$. That is,

$$P^s(x) \subseteq s(\{\omega_{\delta+1}\}). \qquad (1)$$

But we also have $x \subseteq s(\{\omega_{\delta+1}\})$, so that $P^s(x)$ has exactly those $\in^s$-members z which satisfy

$$(y)[(y \in^s s(\{\omega_{\delta+1}\}) \ \&\ y \in^s z) \Longrightarrow y \in^s x].$$

That is, $P^s(x)$ is first order definable over $s(\{\omega_{\delta+1}\})$, and thus $P^s(x) \in L$ by 3.18.

(ii) It remains to show $P^s(x) = P^s(x)^*$. By (1), and the fact that $\omega_{\delta+1} = (\omega_{\delta+1})^*$, it is only necessary to show

$$\theta \in P^s(x) \Longleftrightarrow \theta^* \in P^s(x) \qquad (2)$$

But we have

$$\begin{aligned} \theta \in P^s(x) &\Longleftrightarrow s(\{\theta\}) \subseteq x \\ &\Longleftrightarrow s(\{\theta\})^* \subseteq x^* = x \\ &\Longleftrightarrow s(\{\theta^*\}) \subseteq x \\ &\Longleftrightarrow \theta^* \in P^s(x) \end{aligned}$$

which completes the proof.

3.27. <u>Lemma</u>: Let $A(y_1,\ldots,y_n)$ be a formula in Π with only the free variables exhibited, and let x be a variable not occurring in A. Then:

$$\vdash L^C(x) \mathbin{\&} y_1,\ldots,y_n \in L \Longrightarrow \qquad [A(x \mid y_1,\ldots,y_n) \Longleftrightarrow A(x \mid y_1{}^*,\ldots,y_n{}^*)]$$

<u>Proof</u>: The proof is by induction on the construction of A as a member of Π. Let x be fixed, and assume $L^C(x)$. If A contains no quantifiers (as a member of Π), then $A(x \mid \ldots)$ does not contain x. The statement for A in this case follows easily from 3.12, 3.8 and 1.7. If A is $\neg B$, $(B \vee C)$ or $(B \mathbin{\&} C)$, then the statement for A follows from the same for B (and C) by elementary logic. The case of the universal quantifier can be reduced to the above and the case for the existential quantifier, so it remains to consider this case. If A is $(\exists y_o)(y_o \in L \mathbin{\&} B)$, then $A(x \mid \ldots)$ is $(\exists y_o)(y_o \in^s x \mathbin{\&} B(x \mid \ldots))$. Suppose the statement holds for B, and let $y_1,\ldots,y_n$ be fixed elements of L such that $y_i \in L$ $(i = 1,\ldots,n)$.

9 (i) If $y_o \in^s x$, then by 3.14 $Od_s(y_o) < Od_s(x)$. Thus there is a θ with $Od_s(y_o) = \theta^*$, and hence $y_o = s(\{\theta\})^*$ by 3.12.

(ii) Assume $A(x \mid y_1,\ldots,y_n)$, so that for some $y_o \in^s x$, $B(x \mid y_o,y_1,\ldots,y_n)$. Evidently $y_o \in L$, so by the induction assumption we have $B(x \mid y_o^*, y_1^*, \ldots, y_n^*)$. But by 3.8, and the fact that $x = x^*$, this implies $y_o^* \in^s x$, and hence $A(x \mid y_1^*, \ldots, y_n^*)$.

(iii) Assume $A(x \mid y_1^*, \ldots, y_n^*)$, so that for some $y_o \in^s x$, $B(x \mid y_o, y_1^*, \ldots, y_n^*)$. Then by (i) there is a $z_o \in L$ with $z_o^* = y_o$, and evidently $z_o \in^s x$. Thus by the induction assumption we conclude $B(x \mid z_o, y_1, \ldots, y_n)$, and hence $A(x \mid y_1, \ldots, y_n)$, completing the proof.

3.28. Lemma: Let $A(y_1, \ldots, y_n)$ be a formula in Π with only the exhibited free variables, and let x be a variable not occurring in A. Then:

$$\vdash L^c(x) \;\&\; L^c(y_1) \;\&\ldots\&\; L^c(y_n) \Longrightarrow \\ [A^c(x \mid y_1, \ldots, y_n) \Longleftrightarrow A(x \mid y, \ldots, y_n)]$$

Proof: The proof is again by induction on the construction of A as a member of Π. If A contains no quantifiers as a member of Π, then $A^c(x \mid \ldots) = A(x \mid \ldots)$, and the statement holds trivially. If A is $\neg B$, $(B \vee C)$ or $(B \;\&\; C)$, the statement follows from that for B (and C) by elementary logic. It thus suffices to consider the case where A is $(\exists y_o)(y_o \in L \;\&\; B)$. In this case, $A^c(x \mid \ldots)$ is $(\exists y_o)(L^c(y_o) \;\&\; y_o \in^s x \;\&\; B^c(x \mid \ldots))$ and $A(x \mid \ldots)$ is $(\exists y_o)(y_o \in^s x \;\&\; B(x \mid \ldots))$. Suppose the statement holds for B, and suppose $x, y_1, \ldots, y_n$ are fixed sets with $L^c(x)$

and $L^C(y_i)$ holding (for each $i = 1,\ldots,n$).

If $A^C(x \mid y_1,\ldots,y_n)$, then $A(x \mid y_1,\ldots,y_n)$ follows immediately by the induction assumption.

If $A(x \mid y_1,\ldots,y_n)$ holds, let θ_o be the least θ with $s(\{\theta\}) \in^s x$ and $B(x \mid s(\{\theta\}),y_1,\ldots,y_n)$. We will show $\theta_o = \theta_o{*}$: if not, and $\theta_o{*} < \theta_o$, then

$$s(\{\theta_o{*}\}) = s(\{\theta_o\}){*} \in^s x$$

(by 3.12 and 3.8) and we have $B(x|s(\{\theta_o{*}\}),y_1,\ldots,y_n)$ by 3.27; but this contradicts the choice of θ_o. If, on the other hand $\theta_o < \theta_o{*}$, there is a $\delta < \theta_o$ with $\delta{*} = \theta_o$. Then one also has $s(\{\delta\}) \in^s x$, and by 3.27 and 3.12 we have $B(x \mid s(\{\delta\}),y_1,\ldots,y_n)$, again contradicting the choice of θ_o. Hence $s(\{\theta_o\}) = s(\{\theta_o\}){*}$; that is, $L^C(s(\{\theta_o\}))$. But this implies

$$L^C(s(\{\theta_o\})) \mathbin{\&} s(\{\theta_o\}) \in^s x \mathbin{\&} B^C(x \mid s(\{\theta_o\}),y_1,\ldots,y_n)$$

by the induction assumption. Hence $A^C(x \mid y_1,\ldots,y_n)$, and we are done.

From 3.27 and 3.28 it follows that the C-translate of every bounded axiom of subsets is a theorem of NF. [That is, the C-translate of each of the statements expressing "the subset of x defined by $\mathfrak{A}(y,y_1,\ldots,y_n)$ exists" (where the quantifiers of $\mathfrak{A}$ are relativized to x and $y_1,\ldots,y_n$ are elements of x) is a theorem of NF.] The C-translate of such a defining condition $\mathfrak{A}$ is of the form $A^C(x \mid y, y_1, \ldots, y_n)$, where A is a formula in Π. If we suppose that $L^C(x)$, $L^C(y_1), \ldots, L^C(y_n)$ hold, and let

$$z = \{\theta \mid \theta \in NO_s \; \& \; A(x \mid s(\{\theta\}), y_1, \ldots, y_n) \; \& \; s(\{\theta\}) \in^s x$$

Then evidently $z \subseteq x$ and $z \in L$ (by 3.18). Moreover, as in the arguments for 3.27 and 3.28, we easily show $z = z^*$, that is $L^C(z)$. Finally, if $L^C(y)$, by 3.28 we have

$$y \in^s z \iff y \in^s x \; \& \; A^C(x \mid y, y_o, \ldots, y_n).$$

Thus z is the element of L whose existence it was required to prove.

It does not seem possible to prove that the C-translate of every instance of the full axiom of subsets is a theorem of NF. The difficulty arises when we attempt to obtain analogues of 3.27 and 3.28 for arbitrary formulas in Π, and essentially stems from the fact that $L^C(x)$ is not a stratified formula.

Let B_o denote the system of set theory which consists of the axioms of extensionality, foundation, infinity, union set, power set, and all axioms of the form

$$(x)(y_1)\ldots(y_n)(\exists z)(y)[y \in z \iff y \in x \; \& \; \mathfrak{A}^x(y, y_1, \ldots, y_n)]$$

(where z does not occur in $\mathfrak{A}^x$, and every quantifier of $\mathfrak{A}^x$ is relativized to x).

3.29. <u>Theorem:</u> The operation of passing from a formula in the language of B_o to its C-translate provides a relative interpretation of B_o in NF.

The only axiom whose C-translate has not already been proved is the axiom of infinity. However $s(\{\omega_o\})$ is evidently an L^C-set, and the statement expressing the fact that $s(\{\omega_o\})$ is L^C-infinite is easily proved. [For if $Od_s(x) < \omega_o$, then $Od_s(\{x\}^s) < \omega_o$ as well. Moreover, if $L^C(x)$, then $L^C(\{x\}^s)$, by 3.17.].

<u>Remark:</u> B_o is roughly equivalent in strength to the cumulative theory of types with an axiom of infinity. One can in fact prove that the C-translate of the axiom of constructibility, $(V = L)$, is a theorem of NF, although we will not do so here. However, a relative interpretation of $B_o + (V = L)$ can be obtained in B_o by the standard Gödel techniques, and thus $B_o + (V = L)$ has a relative interpretation in NF.

B_o is apparently the strongest standard set theory for which a relative interpretation in NF can be obtained by the present techniques. If we extend NF by adding the sentence $(\omega_{\omega_o} \in NO)$ as an additional axiom, it is not hard to show that $\overset{s}{\in}$ restricted to $\in^s$-elements of $s(\{\omega_{\omega_o}\})$ provides a relative interpretation of Zermelo set theory (in this extension). [Any subset of a set $x \in^s s(\{\omega_{\omega_o}\})$ which is definable over $s(\{\omega_{\omega_o}\})$ (in terms of $\in^s$) is in L, by 3.18, and is an $\in^s$-member of $s(\{\omega_{\omega_o}\})$ by the analogue of T28 of [7] for s. Thus the full axiom of subsets is provided for in this interpretation. It is an easy matter to check that the other axioms of Z pass to theorems of the system NF + $(\omega_{\omega_o} \in NO)$ under this interpretation.]

The remainder of this section is devoted to proving that the C-translate of each instance of the axiom of replacement of Zermelo-Fraenkel set theory is a theorem in NF_1 (see Section 2). This will be accomplished by proving analogues of 3.27 and 3.28 for arbitrary formulas of Π, where of course each statement will be a theorem of NF_1 rather than NF.

3.30. <u>Theorem</u>: $\vdash_{NF_1} C(\theta) \iff stC(\theta)$

<u>Proof</u>: If $stC(\theta)$, then $\delta < \theta \implies \delta = \delta^*$. Thus

$$\{\delta \mid \delta < \theta\} = \{\delta \mid \delta < \theta\}^*$$

and hence $\theta = \theta^*$ by 1.15. Conversely, by 2.25, 2.26 and 1.23, if $C(\theta)$ then $stC(\theta)$.

Let $R(x_1,\ldots,x_n)$ be any formula whose free variables are $x_1,\ldots,x_n$ and which can be stratified in such a way that all of the free variables of R are assigned the same type. We let $K(R)$ denote the set

$$\{\theta \mid \theta \in Val(J_o) \;\&\; (\exists x_1)\ldots(\exists x_n)[x_1,\ldots,x_n \in^s s(\{\theta\}) \\ \;\&\; \neg(R(x_1,\ldots,x_n) \iff R(x_1{}^*,\ldots,x_n{}^*))] \;.$$

[Note that under the assumptions made above on R, $K(R)$ is a constant, stratified term.] If $K(R)$ is empty, then R is symmetric (with respect to the *-operation) for all elements of L. If not, then let $\theta_o = \min_{\leq} K(R)$. We show next (under this assumption) that

$$(\exists\theta)[\theta \in Val(J_o) \;\&\; \theta < \theta_o \\ \;\&\; (\delta)(\delta \in Val(J_o) \Longrightarrow (\delta \leq \theta \text{ v } \theta_o \leq \delta))] \qquad (1)$$

That is, θ_o has an immediate predecessor in $Val(J_o)$ relative to the ordering $\leq$.

<u>Proof</u>: Let $x_1,\ldots,x_n$ be such that $x_1,\ldots,x_n \in^s s(\{\theta_o\})$ and

$$\neg(R(x_1,\ldots,x_n) \iff R(x_1{}^*,\ldots,x_n{}^*)). \qquad (2)$$

For each $i = 1,\ldots,n$ let $\theta_i = Od_s(x_i)$, so that $\theta_i < \theta_o$ for each such i. Let δ be the maximum of $\theta_1,\ldots,\theta_n$, so that $\delta < \theta_o$.

There can be no element λ of $Val(J_o)$ with $\delta < \lambda < \theta_o$, since otherwise $\lambda \in K(R)$, contradicting the choice of θ_o.

Moreover there are ordinals δ_1, δ_2 and an ordinal $\lambda_1 \leq \theta_o$ such that

$$J(<\lambda_1, \delta_1, \delta_2>) = \delta. \tag{3}$$

Then if $\delta \in \mathrm{Val}(J_o)$ we are done, so we may suppose $\lambda_1 > 0$. But then $J(<0, \delta_1, \delta_2>) \in \mathrm{Val}(J_o)$ and $J(<0, \delta_1, \delta_2>) < \theta_o$. Moreover there can be no $\lambda \in \mathrm{Val}(J_o)$ with

$$J(<0, \delta_1, \delta_2>) < \lambda < \theta_o,$$

since this would imply $\delta < \lambda < \theta_o$. That is, $J(<0, \delta_1, \delta_2>)$ is the desired ordinal, and the proof is complete.

If t is any constant, stratified term, we let $b(t,R)$ denote the term t if $K(R) = \Lambda$, and we let it denote the term $\min_{\leq}(t, \theta_1)$ if $K(R) \neq \Lambda$ and θ_1 is the predecessor in $\mathrm{Val}(J_o)$ of $\min_{\leq} K(R)$. That is:

$$\begin{aligned} b(t,R) =_{df} \ & (\dot{\iota}\theta)[(K(R) = \Lambda \ \& \ \theta = t) \\ & \vee (\exists\delta)[\delta \in \mathrm{Val}(J_o) \ \& \ \delta < \min_{\leq} K(R) \ \& \ \theta = \min_{\leq}(\delta, t) \\ & \qquad \& \ (\lambda)(\lambda \in \mathrm{Val}(J_o) \Longrightarrow \neg(\delta < \lambda < \min_{\leq} K(R)]]. \end{aligned}$$

Then $b(t,R)$ is a constant, stratified term; we see immediately that $b(t,R) \leq t$ if $t \in NO$. Moreover, $b(t,R) \in \mathrm{Val}(J_o)$ if $t \in \mathrm{Val}(J_o)$, and in that case, for any $x_1, \ldots, x_n \in^s s(\{b(t,R)\})$ we have (by the definition of $K(R)$)

$$R(x_1, \ldots, x_n) \Longleftrightarrow R(x_1^*, \ldots, x_n^*).$$

Finally, if $(\theta)(C(\theta) \Longrightarrow \theta < t)$ then the same is true for $b(t,R)$; that is

$$(\theta)(C(\theta) \Longrightarrow \theta < b(t,R)).$$

Indeed, either $b(t,R)$ is t, or the successor of $b(t,R)$ in $Val(J_o)$ is $\min_{\leq}K(R)$. In the first case the statement is trivial. In the second case, if $C(b(t,R))$ then $stC(b(t,R))$ by 3.30. But this implies $stC(\min_{\leq}K(R))$ (since there are exactly 8 ordinals between two adjacent elements of $Val(J_o)$) which contradicts the definition of $K(R)$.

Thus we have proved:

3.31. <u>Lemma</u>: Let R and t be as above and suppose $t \in Val(J_o)$ and $(\theta)(C(\theta) \Longrightarrow \theta < t)$ are theorems of NF_1. Then:

(i) $\vdash_{NF_1} b(t,R) \leq t \;\&\; b(t,R) \in Val(J_o)$

(ii) $\vdash_{NF_1} (\theta)(C(\theta) \Longrightarrow \theta < b(T,R))$

(iii) $\vdash_{NF_1} (x_1)\ldots(x_n)[x_1,\ldots,x_n \in^s s(\{b(t,R)\}) \Longrightarrow (R(x_1,\ldots,x_n) \Longleftrightarrow R(x_1^*,\ldots,x_n^*))]$.

For each A in Π we define the <u>bounded form of A, A'</u>, and a term $t(A)$ as follows:

(i) if A is constructed in Π using only truth-functional connectives, then A' is A and $t(A)$ is Ω_o.

(ii) $(\neg A)' = \neg(A')$ and $t(\neg A)$ is $t(A)$

(iii) $(A \vee B)' = (A' \vee B')$ and $t(A \vee B)$ is $\min_{\leq}(t(A),t(B))$

(iv) $(A \& B)' = (A' \& B')$ and $t(A \& B)$ is $\min_{\leq}(t(A),t(B))$

(v) if A is $(\exists x)(x \in L \& B)$, then A' is $(\exists x)(x \in^{s} s(\{t(B)\}) \& B')$ and $t(A)$ is $b(t(B),A')$.

(vi) if A is $(x)(x \in L \Longrightarrow B)$ then A' is $(x)(x \in^{s} s(\{t(B)\}) \Longrightarrow B')$ and $t(A)$ is $b(t(B), A')$.

Note that $t(A)$ is a constant, stratified term, for each A in Π. Using 3.31, we prove next the analogue of 3.27 for arbitrary formulas in Π.

3.32. <u>Lemma</u>: Let A be any formula of Π with free variables $x_1,\ldots,x_n$. Then

(i) $\vdash_{NF_1} t(A) \leq \Omega_o \;\&\; t(A) \in Val(J_o)$

(ii) $\vdash_{NF_1} (\theta)(C(\theta) \Longrightarrow \theta < t(A))$

(iii) $\vdash_{NF_1} (x_1)\ldots(x_n)[x_1,\ldots,x_n \in^{s} s(\{t(A)\}) \Longrightarrow (A'(x_1,\ldots,x_n) \Longleftrightarrow A'(x_1{*},\ldots,x_n{*}))]$.

<u>Proof</u>: The proof proceeds by induction on the construction of A as a member of Π.

(i) If A is $x \in^s y$ or $x = y$, then $t(A) = \Omega_o$, so (i) and (ii) are trivial. Moreover, (iii) follows from 3.8 or 1.7.

(ii) If A is $\neg B$, $(B \vee C)$ or $(B \& C)$, then (i) and (ii) follow trivially from the induction assumption. Then $t(A) \leq t(B)$ (and $t(A) \leq t(B)$), and all are in $Val(J_o)$, so that $s(\{t(A)\}) \subseteq s(\{t(B)\})$ (and $s(\{t(A)\}) \subseteq s(\{t(C)\})$). Then (iii) for A follows from the induction assumption for B (and C) by elementary logic.

(iii) If A is $(\exists x)(x \in L \& B)$, then (i) and (ii) for $t(A)$ follow from the induction assumption for B and 3.31. Moreover, $t(A) = b(t(B),A')$, so that (iii) follows by 3.31.iii.

3.33. Lemma: Let A be any formula in Π with free variables $x_1,\ldots,x_n$. Then:

$$\vdash_{NF_1} (x_1)\ldots(x_n)[L^C(x_1) \& \ldots \& L^C(x_n) \Longrightarrow (A^C(x_1,\ldots,x_n) \Longleftrightarrow A'(x_1,\ldots,x_n))]$$

Proof: Again, we proceed by induction on the construction of A in Π. If A is $x \in^s y$ or $x = y$ then $A = A' = A^C$ and the statement is trivial. If A is $\neg B$, $(B \& C)$ or $(B \vee C)$, the statement follows from the induction assumption for B (and C) by elementary logic. It thus suffices to consider the case where A is $(\exists x)(x \in L \& B)$. Then A^C is $(\exists x)(L^C(x) \& B^C)$, and A' is $(\exists x)(x \in^{s} s(\{t(B)\}) \;\&\; B')$.

Let $x_1,\ldots,x_n$ be fixed elements of L such that $L^C(x_n)$ holds for each $i = 1,\ldots,n$.

(i) If $A^C(x_1,\ldots,x_n)$ holds, then for some x with $L^C(x)$ we have $B^C(x,x_1,\ldots,x_n)$. Thus by the induction assumption $B'(x,x_1,\ldots,x_n)$. Moreover, since $x = x^*$ and $t(B) \in \mathrm{Val}\,(J_o)$, we have $x \in^s s(\{t(B)\})$, so $A'(x_1,\ldots,x_n)$ holds.

(ii) Conversely, suppose $A'\,(x_1,\ldots,x_n)$ holds, and let

$$z = \{\theta \mid s(\{\theta\}) \in^s s(\{t(B)\}) \;\&\; B'(x,x_1,\ldots,x_n)\} \quad (1)$$

Evidently z is non-empty, and we set $\theta_o = \min_{\leq} z$. (Note that the condition is stratified, since each of the terms occurring in $B'(y_1,\ldots y_n)$ are constant and stratified.) We will show $\theta_o = \theta_o{}^*$: if $\theta_o < \theta_o{}^*$, then for some $\delta < \theta_o$ we have $\theta_o = \delta^*$. Moreover, $\theta_o < t(B)$ by 3.14, and hence $s(\{\delta\}) \in^s s(\{t(B)\})$, since $t(B) \in \mathrm{Val}(J_o)$. Hence by 3.32 we have

$$B'(s(\{\delta\}),\; x_1,\ldots,x_n)$$

which contradicts the choice of θ_o. If $\theta_o{}^* < \theta_o$, then $s(\{\theta_o{}^*\}) \in^s s(\{t(B)\})$ and $B'(s(\{\theta_o{}^*\}),x_1,\ldots,x_n)$ by similar reasoning, again contradicting the choice of θ_o.

But $\theta_o = \theta_o{}^*$ implies $L^C(s(\{\theta_o\}))$, so that by the induction assumption for B we have

$$(\exists x)(L^C(x) \;\&\; B^C(x,x_1,\ldots,x_n)).$$

That is, $A^C(x_1,\ldots,x_n)$, and the proof is complete.

3.34. <u>Theorem</u>: Let A be a formula of $\prod$ with the free variables $x,y,x_1,\ldots,x_n$. Let $B_1(z,x_1,\ldots,x_n)$ be the formula

$$(x)[x \in^s z \Longrightarrow (\exists_1 y)(A(x,y,x_1,\ldots,x_n))]$$

and let $B_2(z,x_1,\ldots,x_n)$ be the formula

$$(\exists u)(y)[y \in^s u \Longleftrightarrow (\exists x)(x \in^s z \ \& \ A(x,y,x_1,\ldots,x_n))]$$

Then

$$\vdash_{NF_1} [(z)(x_1)\ldots(x_n)[B_1(z,x_1,\ldots,x_n) \Longrightarrow B_2(z,x_1,\ldots,x_n)]^C$$

<u>Proof</u>: Let $z,x_1,\ldots,x_n$ be fixed elements of L such that $L^C(z),L^C(x_1),\ldots,L^C(x_n)$ all hold. Then by 3.32, for any $x,y \in^s s(\{t(A)\})$ we have

$$A'(x,y,x_1,\ldots,x_n) \Longleftrightarrow A'(x^*,y^*,x_1,\ldots,x_n) \qquad (1)$$

Suppose $B_1{}^C(z,x_1,\ldots,x_n)$, so that by 3.33 we have

$$B_1{}'(z,x_1,\ldots,x_n). \qquad (2)$$

(i) Then (2) implies that for each $x \in^s z$ there is a $y \in^s s(\{t(A)\})$ such that $A'(x,y,x_1,\ldots,x_n)$ holds. In fact, by (1) and the fact that $L^C(z)$ holds, there is such a y with $y = y^*$. (Take the y with minimal $Od_s(y)$.)

(ii) We can show that for each $x \in^s z$, the y given in (i) is in fact the only set satisfying

$$y \in^s s(\{t(A)\}) \ \& \ A'(x,y,x_1,\ldots,x_n). \qquad (3)$$

For if not, then there are two elements of L, say y_o and y_1, both satisfying (3) (for y) and such that $L^C(y_o)$ and $L^C(y_1)$. Indeed, we need only consider the first and second such y (when ordered according to the ordering on $Od_s(y)$) and apply (1), as well as 3.33.

Thus if we set

$$u = \{\theta \mid (\exists x)(x \in^s z \ \& \ A'(x, s(\{\theta\}), x_1, \ldots, x_n) \ \& \ \theta \in s(\{t(A)\})\} \qquad (4)$$

we have $u = u^*$ by (i) and (ii).

Moreover, with the help of 3.18 we can show $u \in L$. Although the quantifiers in A' are not relativized to a single variable, we can construct an equivalent formula which satisfies conditions of 3.18. That is, let t be a term which contains each of the terms appearing in A' (as relativizations of quantified variables) as $\in^s$-elements and subsets. Then an equivalent formula to A' in which each quantifier is relativized to $\in^s$-elements of t can easily be constructed.

But then $L^C(u)$; thus $B_2'(z, x_1, \ldots, x_n)$ holds (by 3.32), and thus $B_2^C(z, x_1, \ldots, x_n)$ by 3.33. This completes the proof of 3.34.

By proving 3.34 we have shown that the C-translate of each replacement axiom of Z-F is a theorem of NF_1. Since each instance of the axiom of subsets is derivable in B_o

from the replacement axioms we have proved:

Main Theorem I: The operation of passing from any formula of the language of Z-F to its C-translate in terms of $\in^s$ and L provides a relative interpretation of Z-F in NF_1.

Remark: As noted in the remark on page 72, a relative interpretation of Z-F + (V = L) in NF_1 can now be obtained be well-known techniques. It is an open question, and apparently a very difficult one, as to whether or not NF_1 (or even NF) is free from contradiction. Consistency proofs have been given for some fragments of NF (see [1] and [5]), but the proofs can be carried out in a system of elementary number theory. Thus, by the Gödel Incompleteness Theorem and the fact that NF is evidently adequate for number theory, these fragments fail by a wide margin to achieve the strength of NF.

In Section 6 we show that Rosser's Counting Axiom (Axiom 14 of [R]) is not a theorem of NF (unless NF is inconsistent). Since this statement, which is equivalent to $stC(\omega_o)$, is a theorem of NF_1 (by 3.30) we see that NF_1 is a non-trivial extension of NF (unless NF is inconsistent). This would seem to reinforce the view that NF is not itself a sufficiently strong set theory to possess a relative interpretation of Z-F. However the situation is somewhat more complicated. We consider in Section 4 a second extension NF_2 of NF which results from adding as axioms a class of stratified statements, and the method of proof of Section 6 is not sufficient to show that NF_2 is a non-trivial extension of NF. (Of course it may be.) Moreover, by a variation of the above argument for NF_1, we show that Z-F has a relative interpretation in NF_2.

It will be more convenient in carrying out the argument in Section 4 mentioned above to deal with a relation between ordinals rather than one between subsets of NO. The remainder of this section is devoted to the construction of the appropriate relation.

3.35. <u>Definition</u>: $\Delta_s =_{df} \Delta_{NO_s}$

Note that Δ_s is a constant, stratified term, and thus can be assigned any type. Moreover, since $\mathrm{Val}(J_o) \subseteq NO_s$ (and evidently $\mathrm{Val}(J_o)$ has the same cardinality as NO) we have $\mathrm{Nc}(NO) = \mathrm{Nc}(NO_s)$. But then

$$\leq \mathrm{smor}_{\Delta_s}(NO_s \upharpoonright \leq \upharpoonright NO_s)$$

(see 2.7). Moreover, since by 3.8 and 3.12 we have

$$\theta \in NO_s \iff \theta^* \in NO_s, \tag{1}$$

for any $\theta \in NO$, we can prove

$$\Delta_s(\theta^*) = \Delta_s(\theta)^* \qquad (2)$$

for every $\theta \in NO$. (See the proof of 2.12; the proof of (2) is essentially the same, and depends only on (1)). That is, we have proved:

3.36. <u>Lemma</u>: (i) $\vdash \ \underline{<}\mathrm{smor}_{\Delta_s} \ (NO_s \upharpoonright \leq \upharpoonright NO_s)$

(ii) $\vdash \ (\theta)(\Delta_s(\theta^*) = \Delta_s(\theta)^*)$.

We now define the relation on NO to be used in Section 4:

3.37. <u>Definition</u>: $R(\theta,\delta) =_{df} \Delta_s(\theta) \in s(\{\Delta_s(\delta)\})$.

$\hat{R} =_{df} \{<\theta,\delta> \mid R(\theta,\delta)\}$.

$R(\theta,\delta)$ is stratifiable in such a way that θ and δ are assigned the same type, and hence $\hat{R}$ is a constant, stratified term.

$\hat{R}$ differs from $\in^s$ (restricted to L) only in form, as the next theorem shows.

3.38. <u>Theorem</u>: $\vdash$ RUSC$(\hat{R})$ smor $(\in^s \upharpoonright L)$.

<u>Proof</u>: Let $f(\{\theta\}) = s(\{\Delta_s(\theta)\})$ for each $\theta \in NO$. Then evidently $f \in$ Funct and Arg(f) = USC(NO). More-

over, since $\mathrm{Val}(\Delta_s) = NO_s$, we see that $\mathrm{Val}(f) = L$. Also, $\Delta_s \in 1\text{-}1$ and $(USC(NO_s) 1 s) \in 1\text{-}1$, so $f \in 1\text{-}1$. That is,

$$AV(RUSC(R)) \; sm_f \; AV(\in^s \restriction L).$$

We also have

$$\begin{aligned} R(\theta,\delta) &\Longleftrightarrow \Delta_s(\theta) \in s(\{\Delta_s(\delta)\}) \\ &\Longleftrightarrow s(\{\Delta_s(\theta)\}) \in^s s(\{\Delta_s(\delta)\}) \end{aligned}$$

which completes the proof.

<u>Remark</u>: For any $\theta \in NO$, $f(\{\theta^*\}) = f(\{\theta\})^*$, by 3.12 and 3.36. That is,

$$f(\{\theta^*\}) = s(\{\Delta_s(\theta^*)\}) = s(\{\Delta_s(\theta)^*\}) = s(\{\Delta_s(\theta)\})^*.$$

As we did for $\in^s$ and L, we define the class Σ of formulas which are constructed from $R(\theta,\delta)$ and $\theta = \delta$ by means of truth-functional connectives and quantifiers relativized to NO (ie. quantifiers with variables denoted by lower case Greek letters.) That is, Σ is the smallest class of formulas satisfying:

(i) $R(\theta,\delta)$ and $\theta = \delta$ are in Σ, for any variables θ,δ (ranging over NO);

(ii) if A,B are in Σ, so are $\neg A$, $(A \vee B)$ and $(A \,\&\, B)$;

(iii) if A is in Σ, then so are $(\exists\theta)A$ and $(\theta)A$.

Every formula of Σ can be stratified in such a way that all its free variables are assigned the same type.

For each A in Σ we define the relativization of A to Cantorian ordinals, A^C, in the obvious way. Then using the ismorphism f given in the proof of 3.38 we obtain:

3.39. Theorem: Let A be any formula of Σ, with free variables $\theta_1,\ldots,\theta_n$ and bound variables $\theta_{n+1},\ldots,\theta_k$ (as a member of Σ). Let $\hat{A}$ be the formula of Π obtained from A by replacing each variable θ_i by x_i, by relativizing each resulting quantified variable to L and by replacing $R(\theta_i,\theta_j)$ by $x_i \in^S x_j$. Let f be the isomorphism defined in the proof of 3.38. Then:

(i) $\vdash (\theta_1)\ldots(\theta_n)[A(\theta_1,\ldots,\theta_n) \Longleftrightarrow \hat{A}(f(\{\theta_1\}),\ldots,f(\{\theta_n\}))]$.

(ii) $\vdash (\theta_1)\ldots(\theta_n)[C(\theta_1) \& \ldots \& C(\theta_n) \Longrightarrow$
$(A^C(\theta_1,\ldots,\theta_n) \Longleftrightarrow \hat{A}^C(f(\{\theta_1\}),\ldots,f(\{\theta_n\})))]$.

Proof: The proofs proceed by induction on the construction of A as a member of Σ. (i) follows immediately from 3.38, and (ii) follows immediately from 3.38 and the remark following it (for, $C(\theta) \Longleftrightarrow L^C(f(\{\theta\}))$ by that remark).

Thus if we define the C-translate (relative to Σ) of a formula of set theory in terms of $R(\theta,\delta)$ rather than in terms of $\in^s$, the results obtained in this section carry over. Also, if $A(\theta,\theta_1,\ldots,\theta_n)$ is a formula of Σ, with free variables $\theta,\theta_1,\ldots,\theta_n$, in which every quantifier introduced in its construction as a member of Σ occurs in the form $(\delta)(R(\delta,\theta_i) \Longrightarrow \ldots)$ or $(\exists\delta)(R(\delta,\theta_i) \mathbin{\&} ---)$ (for some $i = 1,\ldots,n$), then

$$\vdash (\theta_o)\ldots(\theta_n)(\exists\delta)(\theta)[R(\theta,\delta) \Longleftrightarrow (R(\theta,\theta_o) \mathbin{\&} A(\theta,\theta_1,\ldots,\theta_n))].$$

(where δ does not occur in A). That is, if every quantifier of A (as a member of Σ) is relativized to the $\hat{R}$-elements of one of the free parameters of A, then there is a δ whose $\hat{R}$-elements are exactly the ordinals θ satisfying $A(\theta,\theta_1,\ldots,\theta_n)$ and which are $\hat{R}$-elements of θ_o. This follows, by 3.39, from 3.18 and the adjustments made in the course of the proof of 3.34.

The last properties of R needed in Section 4 are (i) that the $\hat{R}$-elements of each Ω_k are just the predecessors of Ω_k under $\leq$, and (ii) that $\hat{R}$ has a symmetry relative to the *-operation. We also show $R(\theta,\delta)$ implies $\theta < \delta$.

3.40. <u>Lemma</u>: (i) $\vdash (\theta)(\delta)(R(\theta,\delta) \Longleftrightarrow R(\theta^*,\delta^*))$

(ii) $\vdash (\theta)(R(\theta,\Omega_k) \Longleftrightarrow \theta < \Omega_k)$

for each integer $k \geq 0$.

Proof: (i) follows immediately from 3.38 by the remark following it. That is,

$$\begin{aligned} R(\theta,\delta) &\iff f(\{\theta\}) \in^{s} f(\{\delta\}) \\ &\iff f(\{\theta\})^{*} \in^{s} f(\{\delta\})^{*} \quad \text{(by 3.8)} \\ &\iff f(\{\theta^{*}\}) \in^{s} f(\{\delta^{*}\}) \\ &\iff R(\theta^{*},\delta^{*}) \end{aligned}$$

(ii) Since $\Omega_k \in \mathrm{Val}(J_o)$, we have

$$\Delta_s(\theta) \in s(\{\Omega_k\}) \iff \Delta_s(\theta) < \Omega_k.$$

Hence if $\Delta_s(\theta_k) = \Omega_k$ (for each $k \geq 0$) then by 3.36,

$$\Delta_s(\theta) \in s(\{\Omega_k\}) \iff \theta < \theta_k. \tag{1}$$

It thus suffices to show $\Delta_s(\Omega_k) = \Omega_k$ for each $k \geq 0$, for in that case, (1) reduces to

$$\Delta_s(\theta) \in s(\{\Delta_s(\Omega_k)\}) \iff \theta < \Omega_k \tag{2}$$

or, $$R(\theta,\Omega_k) \iff \theta < \Omega_k.$$

It is easy to show by induction over $\leq$ that $\theta \leq \Delta_s(\theta)$ for all $\theta \in \mathrm{NO}$, (since $\mathrm{NO}_s \subseteq \mathrm{NO}$). Thus $\theta_o \leq \Delta_s(\theta_o) = \Omega_o$. If $\theta_o < \Omega_o$, then for any $\theta \in \mathrm{NO}$, $\Delta_s(\theta^{*}) = \Delta_s(\theta)^{*} < \Omega_o$, and hence $\theta^{*} < \theta_o$. But this contradicts 2.10, and thus

$$\Omega_o = \Delta_s(\Omega_o).$$

Then using 3.36.ii it follows that $\Omega_k = \Delta_s(\Omega_k)$ for all $k \geq 0$: that is,

$$\begin{aligned} \Delta_s(\Omega_{k+1}) &= \Delta_s(\Omega_k{}^*) \\ &= \Delta_s(\Omega_k)^* \\ &= \Omega_k{}^* = \Omega_{k+1}; \end{aligned}$$

and the result follows by induction over the non-negative integers.

3.41. <u>Lemma</u>: $\vdash (\theta)(\delta)(R(\theta,\delta) \implies \theta < \delta)$.

<u>Proof</u>: If $R(\theta,\delta)$ then $\Delta_s(\theta) \in s(\{\Delta_s(\delta)\})$. Thus by 3.14

$$\Delta_s(\theta) \lessdot \Delta_s(\delta)$$

so that $\theta < \delta$ by 3.36.

Section 4:

In this section we show that in a second extension of NF (NF_2, defined below) the difficulties in proving the C-translate of each replacement axiom of Z-F can also be overcome. The technique used is, as for NF_1, to extend each unstratified formula of the form $A^C(\theta_1,\ldots,\theta_n)$ to a stratified one $A'(\theta_1,\ldots,\theta_n)$ (by changing the relativizations on quantifiers) which has symmetry relative to the *-operation up to some ordinal greater than every Cantorian ordinal. To do this we need first to prove a basic theorem concerning functions which are defined on a segment of $\leq$, and which possess a certain symmetry property. Namely;

$$(\theta)(\theta \in \mathrm{Arg}(f) \Longrightarrow [(\theta^* \in \mathrm{Arg}(f)) \;\&\; (f(\theta^*) = f(\theta)^*)]) \quad (1)$$

The central difficulty in dealing with such functions is that the condition (1) on f is not stratified. However we deal with this in much the same fashion as in the proof that the ω-function is symmetric in this sense. (See Theorem 2.12 and the remark which follows it.)

4.1. Definition: Let $A(f)$ be the conjunction of the following: (i) $f \in \mathrm{Funct}$

(ii) $f \subseteq \mathrm{NO}\times\mathrm{NO}$

(iii) $(\theta)[\theta \in \mathrm{Arg}(f) \Longrightarrow \theta < f(\theta)]$.

Then we define:

$$F =_{df} \{f \mid A(f)\}.$$

Evidently F is a stratified term, and can be assigned any type. Moreover, $\vdash f \in F \iff A(f)$.

4.2. <u>Lemma</u>: $\vdash f \subseteq NO \times NO \implies (f \in F \iff f^* \in F)$.

<u>Proof</u>: Let $f \subseteq NO \times NO$. Then if $f \in F$, by 1.10 $f^* \in$ Funct. Also, if $\theta \in Arg(f^*)$, then by 1.9, $\theta = \delta^*$ for some $\delta \in Arg(f)$. Then $\delta < f(\delta)$, so by 1.3 $\delta^* < f(\delta)^*$, or $\delta^* < f^*(\delta^*)$ (by 1.11). Hence $f \in F \implies f^* \in F$.

The converse is proved in a similar way.

4.3. <u>Definition</u>: $\Delta(f,\theta) =_{df} Clos(\{\theta\},f)$

<u>Remark</u>: In [R, p.245-248] Rosser defines Clos(A,P) for arbitrary terms A and P, and presents certain conditions under which it is a useful term. For our purposes, however, there is no difficulty, since

$$Clos(\{\theta\},f) = \bigcap\{x \mid \theta \in x \;\&\; (y)(z)[(y \in x \;\&\; \langle y,z\rangle \in f) \implies z \in x)]\}.$$

This is clearly stratified if $\{\theta\}$ and f are assigned the same type, and in such a case $Clos(\{\theta\},f)$ has that type.

In particular, the set

$$\{x \mid \theta \in x \ \& \ (y)(z)[(y \in x \ \& \ \langle y,z\rangle \in f) \Longrightarrow z \in x)]\}$$

exists, so that the antecedent in each of the Theorems IX.5.12-15 of [R] is provable for our A and P. Moreover, the antecedent of IX.5.16 is also provable; in addition to stating that the above set exists, it requires the existence of the set

$$\{x \mid x \in \theta \ v \ (\exists y)(y \in \mathrm{Clos}(\{\theta\},f) \ \& \ \langle y,x\rangle \in f)\}.$$

But since the condition is stratified the existence of this set is provable in NF.

Immediate from these considerations are the following four Lemmas, each a specialization of one of the theorems of [R] mentioned above (together with the equivalence, under the assumption of $f \in \mathrm{Funct}$, of the statement $\langle y,z\rangle \in f$ and the statement $y \in \mathrm{Arg}(f) \ \& \ f(y) = z$):

4.4. <u>Lemma</u>: $\vdash \ \theta \in \mathrm{Clos}(\{\theta\},f)$

<u>Proof</u>: Theorem XI.5.13 of [R].

4.5. <u>Lemma</u>:

$$\vdash \ f \in \mathrm{Funct} \Longrightarrow [(x \in \mathrm{Clos}(\{\theta\},f) \ \& \ x \in \mathrm{Arg}(f)) \\ \Longrightarrow (f(x) \in \mathrm{Clos}(\{\theta\},f))].$$

<u>Proof</u>: Theorem IX.5.14 of [R].

4.6. <u>Lemma</u>:

$\vdash$ f $\in$ Funct $\Longrightarrow$

$$[[(\theta \in x) \;\&\; (z)(z \in x \;\&\; z \in \mathrm{Arg}(f) \Longrightarrow f(z) \in x)] \Longrightarrow (\mathrm{Clos}(\{\theta\},f) \subseteq x)].$$

<u>Proof</u>: Theorem IX.5.15 of [R].

4.7. <u>Lemma</u>: $\vdash f \in \mathrm{Funct} \Longrightarrow [z \in \mathrm{Clos}(\{\theta\},f) \Longleftrightarrow$ $(z = \theta \;.\mathrm{v}.\; (\exists x)(x \in \mathrm{Clos}(\{\theta\},f) \;\&\; x \in \mathrm{Arg}(f) \;\&\; f(x) = z))]$.

<u>Proof</u>: Theorem IX.5.16 of [R].

4.8. <u>Lemma</u>: $\vdash f \in F \Longrightarrow [\theta \in \Delta(\delta,f) \Longrightarrow \delta \leq \theta]$.

<u>Proof</u>: Let $A = \{\theta \mid \delta \leq \theta\}$. Then $\delta \in A$. Also, if $\theta \in A$ and $\theta \in \mathrm{Arg}(f)$, then $\theta < f(\theta)$, so $f(\theta) \in A$ as well. Then by 4.6 $\Delta(\delta,f) \subseteq A$, which is the desired conclusion.

4.9. <u>Lemma</u>: $\vdash f \in F \Longrightarrow$

$$[\theta \in \mathrm{Arg}(f) \Longrightarrow (\Delta(\theta,f) = \{\theta\} \cup \Delta(f(\theta),f)]$$

<u>Proof</u>: (i) Let $A = \{\theta\} \cup \Delta(f(\theta),f)$. Then $\theta \in A$; also if $\delta \in A$ and $\delta \in \mathrm{Arg}(f)$, then $f(\delta) \in \Delta(f(\theta),f)$

(since either $\delta = \theta$ or $\delta \in \Delta(f(\theta),f)$: then use 4.5). Thus $\Delta(\theta,f) \subseteq A$ by 4.6.

(ii) By 4.8, and since $\theta < f(\theta)$, we see that $\{\theta\} \cap \Delta(f(\theta),f) = \Lambda$. Thus, since $\theta \in \Delta(\theta,f)$ (by 4.4).

$$\{\theta\} \cup \Delta(f(\theta),f) \subseteq \Delta(\theta,f) \iff \Delta(f(\theta),f) \subseteq \Delta(\theta,f)-\{\theta\}.$$

Then, since $\theta \in \mathrm{Arg}(f)$, we have $f(\theta) \in \Delta(\theta,f)$ and $\theta \neq f(\theta)$, so that $f(\theta) \in \Delta(\theta,f)-\{\theta\}$. Furthermore, if $\delta \in \Delta(\theta,f)-\{\theta\}$ and $\delta \in \mathrm{Arg}(f)$, then $f(\delta) \in \Delta(\theta,f)$ (and $\theta < \delta < f(\delta)$, so $f(\delta) \in \Delta(\theta,f)-\{\theta\}$. Thus by 4.6. $\Delta(f(\theta),f) \subseteq \Delta(\theta,f)-\{\theta\}$ which completes the proof.

4.10. Lemma:

(i) $\vdash [f \in F \;\&\; \delta \in \Delta(\theta,f)] \Longrightarrow [\Delta(\delta,f) \subseteq \Delta(\theta,f)]$

(ii) $\vdash [h,f \in F \;\&\; h \subseteq f] \Longrightarrow [\Delta(\theta,h) \subseteq \Delta(\theta,f)]$.

Proof: Both proofs use 4.6, and are obvious.

4.11. Lemma:

$\vdash [f \in F \;\&\; \delta \in \Delta(\theta,f)] \Longrightarrow \Delta(\theta,f)-\Delta(\delta,f) \in \mathrm{Fin}$.

Proof: Suppose $f \in F$. If $\theta \notin \mathrm{Arg}(f)$, then $\Delta(\theta,f) = \{\theta\}$, so the statement is trivially true. Thus we may assume $\theta \in \mathrm{Arg}(f)$. If for some $\delta \in \Delta(\theta,f)$ we have $\Delta(\theta,f)-\Delta(\delta,f) \notin \mathrm{Fin}$, define

$$\delta_o = \min_{\leq}\{\delta \mid \delta \in \Delta(\theta,f) \ \& \ \Delta(\theta,f)-\Delta(\delta,f) \notin \mathrm{Fin}\}. \quad (1)$$

Then $\delta_o \in \Delta(\theta,f)$, so that by 4.7, there is a $\delta_1 \in \Delta(\theta,f)$ such that $\delta_o = f(\delta_1)$. (Note that $\delta_o \neq \theta$, by (1)). Then $\Delta(\delta_1,f) = \{\delta_1\} \cup \Delta(\delta_o,f)$ (by 4.9), so that

$$\Delta(\theta,f)-\Delta(\delta_o,f) = (\Delta(\theta,f)-\Delta(\delta_1,f)) \cup \{\delta_1\}. \quad (2)$$

(since $\delta_1 < f(\delta_1) = \delta_o$). But this is a contradiction, since the right side of (2) is in Fin, and the left side is not (by (1)). Thus the Lemma is proved.

4.12. Lemma:

(i) $\vdash (f \in F \ \& \ \max_{\leq}\Delta(\theta,f) \in NO) \Longrightarrow \Delta(\theta,f) \in \mathrm{Fin}$

(ii) $\vdash (f \in F \ \& \ \delta = \max_{\leq}\Delta(\theta,f)) \Longrightarrow$

$$(\Delta(\theta,f)-\mathrm{Arg}(f) = \{\delta\}).$$

Proof: (i) Suppose $f \in F$ and $\delta = \max_{\leq}\Delta(\theta,f)$. Then $\delta \notin \mathrm{Arg}(f)$, (since otherwise $f(\delta) \in \Delta(\theta,f)$ and $\delta < f(\delta)$) so that $\Delta(\delta,f) = \{\delta\}$. Then,

$$\Delta(\theta,f) = (\Delta(\theta,f)-\Delta(\delta,f)) \cup \{\delta\}$$

so that by 4.11, $\Delta(\theta,f) \in \mathrm{Fin}$.

(ii) By (i), we may induct on the cardinality of $\Delta(\theta,f)$. That is, let $A(n)$ be the formula

$$(f)(\theta)(\delta)[(f \in F \;\&\; \delta = \max_{\leq}\Delta(\theta,f) \;\&\; \Delta(\theta,f) \in n) \Longrightarrow (\Delta(\theta,f)\text{-}\mathrm{Arg}(f) = \{\delta\})].$$

Then by (i), $(n)(n \in Nn \Longrightarrow A(n))$ is equivalent to the statement to be proved. Also, $A(n)$ is evidently stratified, so by [R, p.277] we may use the schema of induction over Nn.

(a) If $n = 1$: if $\Delta(\theta,f) \in 1$, then $\Delta(\theta,f) = \{\theta\}$. Thus $\theta = \max_{\leq}\Delta(\theta,f)$, and $\theta \notin \mathrm{Arg}(f)$. Hence $\Delta(\theta,f)\text{-}\mathrm{Arg}(f) = \{\theta\}$, so that $A(1)$ is proved.

(b) Assume $A(n)$: Suppose $f \in F$, $\Delta(\theta,f) \in n+1$ and $\delta = \max_{\leq}\Delta(\theta,f)$. We have then $\theta \in \mathrm{Arg}(f)$. Also $\Delta(\theta,f) = \{\theta\} \cup \Delta(f(\theta),f)$ (by 4.9) and $\delta = \max_{\leq}\Delta(f(\theta),f)$ (since $\theta < f(\theta)$). But $\Delta(f(\delta),f) \in n$ (by 4.8) so that

$$\Delta(f(\theta),f)\text{-}\mathrm{Arg}(f) = \{\delta\}$$

by the induction assumption. Then since $\theta \in \mathrm{Arg}(f)$,

$$\Delta(\theta,f)\text{-}\mathrm{Arg}(f) = \{\delta\}$$

also, establishing $A(n+1)$. Thus, by induction

$$(n)(n \in Nn \Longrightarrow A(n))$$

completing the proof of (ii).

4.13. <u>Lemma</u>:

(i) $\vdash [f,h \in F \;\&\; h \subseteq f \;\&\; \theta \in \mathrm{Arg}(h) \;\&\; \max_{\leq}\Delta(\theta,h) \notin \mathrm{Arg}(f)] \Longrightarrow [\Delta(\theta,f) = \Delta(\theta,h)]$

(ii) $\vdash [f,h \in F \;\&\; h \subseteq f \;\&\; \theta \in Arg(h) \;\&\; \max_{\leq}\Delta(\theta,h) \in Arg(f)]$
$\qquad\qquad \Longrightarrow [\Delta(\theta,f) = \Delta(\theta,h) \cup \Delta(f(\max_{\leq}\Delta(\theta,h)),f)]$

<u>Proof</u>: (i) Assume the antecedent holds for f,h,θ. If $\Delta(\theta,h)$ has no maximum, then evidently $\Delta(\theta,h) \subseteq Arg(h)$. Thus, for any $\delta \in \Delta(\theta,h)$, one has $\delta \in Arg(f)$, and $f(\delta) \in \Delta(\theta,h)$ (since $h \subseteq f$). Thus, by 4.6. $\Delta(\theta,f) \subseteq \Delta(\theta,h)$. But this implies $\Delta(\theta,f) = \Delta(\theta,h)$, by 4.10.

Thus we may syppose that $\Delta(\theta,h)$ has a maximum element, say δ_o. Then, by 4.12

$$\Delta(\theta,h)\text{-}Arg(h) = \{\delta_o\} \qquad (1)$$

Assume $\Delta(\theta,h) \neq \Delta(\theta,f)$ and let $\delta_1 = \min_{\leq}(\Delta(\theta,f)-\Delta(\theta,h))$. ($\delta_1$ exists by 4.10) Then for some $\lambda \in \Delta(\theta,f)$ we have $f(\lambda) = \delta_1$ (by 4.7, since clearly $\theta < \delta_1$). Moreover, since $f \in F$, we have $\lambda < \delta_1$, so that $\lambda \in \Delta(\theta,h)$. However, $\lambda \notin Arg(h)$ (since otherwise $f(\lambda) = h(\lambda) \in \Delta(\theta,h)$) so that by (1), $\lambda = \delta_o$. But this implies $\delta_o \in Arg(f)$, contradicting the initial assumptions, so that $\Delta(\theta,h) = \Delta(\theta,f)$, completing the proof of (i).

(ii) Let f,h,θ be as in the antecedent, and let $\delta_o = \max_{\leq}\Delta(\theta,h)$, so that $\delta_o \in Arg(f)$. Evidently $f(\delta_o) \in \Delta(\theta,f)$, since $\Delta(\theta,h) \subseteq \Delta(\theta,f)$ by 4.10. Therefore

$$\Delta(\theta,h) \cup \Delta(f(\delta_o),f) \subseteq \Delta(\theta,f) \qquad (2)$$

(again by 4.10). To prove the opposite inclusion, we use

4.7: Clearly $\theta \in \Delta(\theta,h)$. If $\delta \in \Delta(\theta,h)$, then by (1) either $\theta \in \mathrm{Arg}(h)$, (whence $f(\delta) = h(\delta) \in \Delta(\theta,h)$) or $\delta = \delta_o$ (whence $f(\delta) = f(\delta_o) \in \Delta(f(\delta_o),f)$). If $\delta \in \Delta(f(\delta_o),f)$ and $\delta \in \mathrm{Arg}(f)$, then $f(\delta) \in \Delta(f(\delta_o),f)$ by 4.5. Thus, by 4.7

$$\Delta(\theta,f) \subseteq \Delta(\theta,h) \cup \Delta(f(\delta_o),f)$$

so that with (2) we have

$$\Delta(\theta,f) = \Delta(\theta,h) \cup \Delta(f(\delta_o),f)$$

completing the proof of (ii).

4.14. Lemma:

$\vdash [f \in \mathrm{Funct}\ \&\ f \in \mathrm{NO}\times\mathrm{NO}] \Longrightarrow [\Delta(\theta,f)^* = \Delta(\theta^*,f^*)]$

Proof: (i) Evidently $\theta^* \in \Delta(\theta,f)^*$. Also, if $\delta \in \Delta(\theta,f)^*$ and $\delta \in \mathrm{Arg}(f^*)$, then by 1.6 and 1.9, there is a λ with $\lambda \in \mathrm{Arg}(f)$, $\lambda \in \Delta(\theta,f)$, and $\delta = \lambda^*$. Then $f(\lambda) \in \Delta(\theta,f)$, so $f^*(\lambda^*) = f(\lambda)^* \in \Delta(\theta,f)^*$. Thus, by 4.7

$$\Delta(\theta^*,f^*) \subseteq \Delta(\theta,f)^*.$$

(ii) To prove the opposite inclusion, we note that $\Delta(\theta^*,f^*) \subseteq \{\theta^*\} \cup \mathrm{Arg}(f^*) \cup \mathrm{Val}(f^*) \subseteq \mathrm{NO}^*$ (since $f \subseteq \mathrm{NO}\times\mathrm{NO}$). Thus there is a set $A \subseteq \mathrm{NO}$ such that $A^* = \Delta(\theta^*,f^*)$; it suffices to show $A \supseteq \Delta(\theta,f)$, by 1.7. But $\theta \in A$ follows from $\theta^* \in \Delta(\theta^*,f^*)$. Moreover, if $\delta \in A$ and $\delta \in \mathrm{Arg}(f)$, then $\delta^* \in \Delta(\theta^*, f^*)$ and $\delta^* \in \mathrm{Arg}(f^*)$, so that $f^*(\delta^*) \in \Delta(\theta^*, f^*)$. However, by 1.11, this implies $f(\delta) \in A$; thus by 4.7 $\Delta(\theta, f) \subseteq A$, and the proof is complete.

We restrict our attention, for the next few pages, to those members of F which are defined on some subset of NO of the form $\{\theta \mid \delta_1 \leq \theta < \delta_2\}$ and which are increasing relative to $\leq$. We thus make the following definition.

4.15. <u>Definition</u>:

(i) $F_1 =_{df} \{f \mid f \in F \ \& \ (\exists\delta_1)(\exists\delta_2)(\mathrm{Arg}(f) = \{\theta \mid \delta_1 \leq \theta < \delta_2\})$
$\& \ (\theta_1)(\theta_2)[(\theta_1, \theta_2 \in \mathrm{Arg}(f) \ \& \ \theta_1 \leq \theta_2) \Longrightarrow f(\theta_1) \leq f(\theta_2)]$

(ii) $m(f) =_{df} \min_{\leq} \mathrm{Arg}(f)$

(iii) $l(f) =_{df} \mathrm{lub}[\mathrm{Arg}(f)]$

Note that F_1, $m(f)$ and $l(f)$ are all stratified terms, and that F_1 may be assigned any type, while $m(f)$ and $l(f)$ are assigned the next lower type than that of f.

4.16. <u>Lemma</u>:

(i) $\vdash f \subseteq NO \times NO \Longrightarrow (f \in F_1 \Longleftrightarrow f^* \in F_1)$

(ii) $\vdash f \in F_1 \Longrightarrow [\mathrm{Arg}(f) = \{\theta \mid m(f) \leq \theta < l(f)\}]$

(iii) $\vdash f \in F_1 \Longrightarrow [m(f^*) = m(f)^* \ \& \ l(f^*) = l(f)^*]$.

<u>Proof</u>: (i) Let $f \subseteq NO \times NO$. Then if $f \in F_1$ (so that $f^* \in F$ by 4.2) one has $Arg(f) = \{\theta \mid \delta_1 \leq \theta < \delta_2\}$. Hence $Arg(f^*) = \{\theta \mid \delta_1{}^* \leq \theta < \delta_2{}^*\}$ (by 1.3 and 1.4). Since f is increasing, f^* must be also, by 1.3 and 1.11. Thus $f^* \in F_1$.

The converse is proved similarly.

(ii) This follows from the fact that if $A = \{\theta \mid \delta_1 \leq \theta < \delta_2\}$ then evidently $\delta_1 = \min_{\leq} A$ and $\delta_2 = \mathrm{lub}A$.

(iii) Obvious, after (ii).

4.17. <u>Lemma</u>: $\vdash [f \in F_1 \ \& \ \theta_1, \theta_2 \in Arg(f) \ \& \ \theta_1 \leq \theta_2] \Longrightarrow$
$[\Delta(\theta_1, f) \in Fin \Longrightarrow \Delta(\theta_2, f) \in Fin]$.

<u>Proof</u>: The proof is by induction on the cardinal of $\Delta(\theta_1, f)$ (hence over Nn). The appropriate formula $A(n)$ is stratified so that by [R, p.277] the induction scheme applies.

(i) $A(0)$ holds trivially, since one has

$$1 \leq_c Nc(\Delta(\theta_1, f)).$$

(ii) Assume $A(n)$ (for $n \in Nn$) and suppose that $f \in F_1$, $\theta_1, \theta_2 \in Arg(f)$, $\theta_1 \leq \theta_2$ and $\Delta(\theta_1, f) \in n+1$. Then by 4.9,

$$\Delta(\theta_1, f) = \{\theta_1\} \cup \Delta(f(\theta_1), f)$$

and

$$\Delta(\theta_2, f) = \{\theta_2\} \cup \Delta(f(\theta_2), f) \qquad (1)$$

Hence (by 4.8, since $f \in F_1$) $\Delta(f(\theta_1),f)$ is an element of n. Moreover, $\theta_1 \leq \theta_2$ implies $f(\theta_1) \leq f(\theta_2)$. Thus either $f(\theta_2) \notin \text{Arg}(f)$ (whence $\Delta(f(\theta_2),f) \in \text{Fin}$) or $f(\theta_2) \in \text{Arg}(f)$ (whence $f(\theta_1) \in \text{Arg}(f)$ as well, since $m(f) \leq \theta_1 < f(\theta_1) \leq f(\theta_2) < l(f)$ in that case). In the latter case, by the induction assumption $\Delta(f(\theta_2),f) \in \text{Fin}$, so that by (1) $\Delta(\theta_2,f) \in \text{Fin}$ also. Thus in any case we have established $A(n) \Longrightarrow A(n+1)$ for any $n \in \text{Nn}$, and hence the proof is complete.

The next theorem applies 4.14 to members of F_1 which possess a certain symmetry: namely $m(f) = m(f)^*$, $l(f)^* < l(f)$, and for $\theta \in \text{Arg}(f)$, $f(\theta^*) = f(\theta)^*$. In particular, we show that if such a function satisfies $f(l(f)^*) \geq l(f)$, then there is a Cantorian ordinal between $l(f)$ and $f(l(f)^*)$. In the sequel this result (or its corollary 4.20) is of great use, especially in showing that for certain ordinals θ, there is a Cantorian ordinal δ with $\theta < \delta$. That is, for such ordinals θ we construct a function f with the desired symmetry and which satisfies $l(f) = \theta$ and $f(l(f)^*) \geq \theta$. Thus by 4.19 there is a Cantorian ordinal $\delta > \theta$.

In fact, given such an f, the desired ordinal can be defined by a stratified formula with free variable f.

4.18. Definition:

$$t(f) =_{df} \max_{\leq}\Delta(\min_{\leq}\{\theta \mid \theta \in Arg(f) \ \&\ \Delta(\theta,f) \in Fin\},f)$$

Note that $t(f)$ is a stratified term, and is assigned a type one lower than that of f. Moreover $t(f) \in NO$ holds exactly when there is a $\theta \in Arg(f)$ such that $\Delta(\theta,f) \in Fin$.

4.19. Theorem:

$$\vdash [f \in F_1 \ \&\ f^* \subseteq f \ \&\ m(f) = m(f^*) \ \&\ l(f) \leq f(l(f)^*)] \Longrightarrow$$
$$[t(f) = t(f)^* \ \&\ l(f) < t(f) \leq f(l(f)^*)]$$

Proof: (i) Since $f(l(f)^*) \geq l(f)$ (which implies $f(l(f)^*) \in NO$, and hence $m(f) \leq l(f)^* < l(f)$) we have

$$\Delta(l(f)^*,f) = \{l(f)^*,f(l(f)^*)\} \in Fin.$$

(ii) If $\theta \in Arg(f)$ and $\Delta(\theta^*,f) \in Fin$, then $\Delta(\theta,f) \in Fin$: Indeed, $\Delta(\theta,f)^* = \Delta(\theta^*,f^*)$ by 4.14, and since $f^* \in F$, $\Delta(\theta^*,f^*) \subseteq \Delta(\theta^*,f)$ (by 4.10). Thus if $\Delta(\theta^*,f) \in Fin$ we have $\Delta(\theta,f)^* \in Fin$, and hence $\Delta(\theta,f) \in Fin$ by 1.18 and [R, p.421].

(iii) If $\theta \in Arg(f)$ and $\Delta(\theta,f) \in Fin$ then $\Delta(\theta^*,f) \in Fin$: If $\Delta(\theta,f) \in Fin$, then $\Delta(\theta,f)^* \in Fin$ (1.18 and [R, p.241]) so that $\Delta(\theta^*,f^*) \in Fin$ by 4.14. Let $\delta_o = \max_{\leq}\Delta(\theta^*,f^*)$, so that (since $f^* \in F$, and $\theta < \delta_o$)

$$l(f)^* < \delta_o.$$

Then, by 4.17, if $\delta_o \in \mathrm{Arg}(f)$

$$\Delta(\theta^*,f) = \Delta(\theta^*,f^*) \cup \Delta(f(\delta_o),f)$$

or, since $f \in F_1$, (and hence $l(f) \leq f(l(f)^*) \leq f(\delta_o)$)

$$\Delta(\theta^*,f) = \Delta(\theta^*,f^*) \cup \{f(\delta_o)\}. \qquad (1)$$

Otherwise, $\delta_o \notin \mathrm{Arg}(f)$, and by 4.17

$$\Delta(\theta^*,f) = \Delta(\theta^*,f^*) \qquad (2)$$

In either case, $\Delta(\theta^*,f) \in \mathrm{Fin}$.

(iv) Let $\theta_o = \min_{\leq} \{\theta \mid \theta \in \mathrm{Arg}(f) \;\&\; \Delta(\theta,f) \in \mathrm{Fin}\}$. Then $\theta_o = \theta_o^*$ and $\Delta(\theta_o,f) = \Delta(\theta_o,f)^*$:

$\theta_o \in \mathrm{NO}$, since $l(f)^* \in \mathrm{Arg}(f)$ and $\Delta(l(f)^*,f) \in \mathrm{Fin}$. If $\theta_o \neq \theta_o^*$, then either $\theta_o^* < \theta_o$ (whence $\theta_o^* \in \mathrm{Arg}(f)$ and $\Delta(\theta_o^*,f) \in \mathrm{Fin}$ by (iii)) or $\theta_o < \theta_o^*$. In the latter case, by 1.4 there is a $\theta_1 < \theta_o$ such that $\theta_o = \theta_1^*$, and evidently $\theta_1 \in \mathrm{Arg}(f)$. Moreover, by (ii), $\Delta(\theta_1,f) \in \mathrm{Fin}$. In either case, the definition of θ_o is contradicted, so we have $\theta_o = \theta_o^*$.

As in the argument for (iii) above, (since $\Delta(\theta_o,f) \in \mathrm{Fin}$) if $\delta_o = \max_{\leq}\Delta(\theta_o,f^*)$, then either

$$\Delta(\theta_o^*,f) = \Delta(\theta_o^*,f^*) \cup \{f(\delta_o)\}$$

or
$$\Delta(\theta_o^*,f) = \Delta(\theta_o^*,f^*).$$

That is, using 4.14, either

$$\Delta(\theta_o,f) = \Delta(\theta_o,f)^* \cup \{f(\delta_o)\} \tag{3}$$

or

$$\Delta(\theta_o,f) = \Delta(\theta_o,f)^* \tag{4}$$

But if (3) holds, and we set $n_o = Nc(\Delta(\theta_o,f))$,(so $n_o \in Nn$), then by 1.18 $n_o = T(n_o) + 1$, contradicting 5.4 of [13]. Thus we have proved (4), and hence (iv).

Evidently $t(f) = \max_{\leq}\Delta(\theta_o,f)$, so by (iv) and 1.14, $t(f) = t(f)^*$. Moreover, since $\theta_o \in Arg(f)$, so that $\theta_o < t(f) \notin Arg(f)$, we have $l(f) \leq t(f)$. In fact $l(f) < t(f)$, since $l(f)^* < l(f)$ (see (i)). Finally, since we also have $t(f) = \max_{\leq}\Delta(\theta_o,f^*)$, there is a $\delta < l(f)^*$ such that $t(f) = f^*(\delta)$ (= $f(\delta)$ since $f^* \subseteq f$), and hence $t(f) = f(\delta) \leq f(l(f)^*)$, completing the proof of 4.19.

[Note that the conclusion of 4.19 implies that $f(l(f)^*) > l(f)$.]

We can prove by an entirely analogous argument, a counterpart of 4.19 for functions with $f \subseteq f^*$. That is,

4.20. <u>Theorem</u>:

$$\vdash [f \in F_1 \ \&\ f \subseteq f^* \ \&\ m(f) = m(f)^* \ \&\ l(f)^* \leq f(l(f))] \Longrightarrow [t(f) = t(f)^* \ \&\ l(f)^* < t(f) \leq f^*(l(f))].$$

<u>Proof</u>: The proof is similar to that of 4.19, and we omit it. The only new result needed is that if $n \in Nn$,

then $T(n) \neq n + 1$ (needed in the counterpart of (iv)). This is proved exactly the same way as 5.4 of [13].

In section 5 we obtain some consequences of 4.19 and 4.20 which generalize and sharpen some of Specker's results concerning the ordering $\leq_c$. For now, we illustrate their use in:

4.21. <u>Theorem</u>: (i) $\vdash (\theta^* < \theta \ \& \ \theta^* + \delta = \theta) \Longrightarrow \delta \neq \delta^*$

(ii) $\vdash (\theta < \theta^* \ \& \ \theta + \delta = \theta^*) \Longrightarrow \delta \neq \delta^*$

<u>Proof</u>: (i) Assume the antecedent for θ and δ. Let f be the function defined on $\{\lambda \mid \lambda < \theta\}$ by $f(\lambda) = \lambda + \delta$. Then, since $\delta \neq 0$, we see $f(\lambda) > \lambda$ for every $\lambda < \theta$. Moreover, $\lambda \leq \lambda' \Longrightarrow f(\lambda) \leq f(\lambda')$(see [R, p.475-6]). Thus $f \in F_1$.

Since $\theta = l(f)$ (and $f(\theta^*) = \theta$), in order to apply 4.19 it remains only to show $f^* \subseteq f$. That is, we need to show: if $\lambda < \theta$, then $f(\lambda)^* = f(\lambda^*)$. But it is an easy consequence of Rosser's definition of ordinal addition that $(\lambda_1 + \lambda_2)^* = \lambda_1{}^* + \lambda_2{}^*$. [cf [R, p.475]. We leave this proof to the reader]. Thus if $\lambda < \theta$ then $f(\lambda)^* = (\lambda + \delta)^* = \lambda^* + \delta^*$, and $f(\lambda^*) = \lambda^* + \delta$. Hence, if $\delta = \delta^*$ we have $f^* \subseteq f$.

But applying 4.19 (and the remark immediately following the proof of 4.19) we see that this is a contradiction, proving $\delta \neq \delta^*$.

(ii) The proof proceeds similarly, using 4.20 instead of 4.19.

<u>Remark</u>: 1. One can show that the order isomorphism between $seg_{\omega_0} \leq$ and $Nn \upharpoonleft \leq_c \upharpoonright Nn$ is also sum-preserving. (See the remark following 2.23.) Since this isomorphism also commutes with the type raising operations ($\delta \longrightarrow \delta^*$ and $n \longrightarrow T(n)$) we conclude from 4.21: if $n \in Nn$ and $n \neq T(n)$, and if $m = T(m)$, then $n \neq T(n) + m$ and $T(n) \neq n + m$. This extends the result of Specker [13] used above. (Specker proved the first inequality for $m = 1,2$.)

2. While there are no results in the literature indicating that the statement $(\exists\theta)(\theta \in NO \;\&\; \theta < \theta^*)$ is provable in NF, this sentence is consistent with NF (even if attention is restricted to ordinals $\theta < \omega_0$) as we prove in Section 6. Thus, although 4.20 and the second part of 4.21 will play no part in the results of this section, we include them for completeness.

It will be useful, in applications, to have a version of 4.19 which applies to a wider class of functions than F_1.

4.22. <u>Theorem</u>: There is a stratified term s(f) with f as its only free variable, (and which is assigned a type one lower than that of f) such that:

$$\vdash [f \in \text{Funct} \;\&\; \text{Arg}(f) = \{\theta \mid \delta_1 \leq \theta < \delta_2\} \;\&\; \delta_1 = \delta_1{}^*$$
$$\&\; f^* \subseteq f \;\&\; (\exists\theta)((\theta \leq \delta_2) \;\&\; f(\theta^*) \geq \delta_2)] \Longrightarrow$$
$$[s(f) = s(f)^* \;\&\; \delta_2 < s(f)]$$

<u>Proof</u>: Assume the antecedent holds for f, δ_1 and δ_2. Then for any $\theta \in \text{Arg}(f^*)$, we define

$$g(\theta) = \text{lub}[\{f^*(\lambda) \mid \delta_1 \leq \lambda \leq \theta\} \cup \{\theta\}].$$

(i) Since $f^* \subseteq f$, for any $\theta \in \text{Arg}(f^*)$ one has $f^*(\theta) < \Omega_o$, so that $g(\theta)$ is defined for any $\theta \in \text{Arg}(f^*)$. Moreover, g is defined by a stratified term (with free variable f, since δ_1, $\text{Arg}(f^*)$ are definable thus in terms of f) and has type one higher than that of f. Evidently $g \in \text{Funct}$, $g \subseteq NO\times NO$ and $\theta \in \text{Arg}(g) \Longrightarrow \theta < g(\theta)$. Furthermore, if $\theta_1 \leq \theta_2$, and both are in $\text{Arg}(g)$, then:

$$\{f^*(\lambda) \mid \delta_1 \leq \lambda < \theta_1\} \subseteq \{f^*(\lambda) \mid \delta_1 \leq \lambda < \theta_2\}$$

so $g(\theta_1) \leq g(\theta_2)$, and $g \in F_1$.

(ii) We show next that $g^* \subseteq g$: If $\theta \in \text{Arg}(g)$, then $\theta^* \in \text{Arg}(g)$, and:

$$\{f^*(\lambda) \mid \delta_1 \leq \lambda \leq \theta^*\} = \{f^*(\lambda) \mid \delta_1 \leq \lambda \leq \theta\}^*$$

(since $f^{**} \subseteq f^*$ and $\delta_1 = \delta_1{}^*$) and $\{\theta^*\} = \{\theta\}^*$. Thus, by 1.14, $g(\theta^*) = g(\theta)^*$; that is, $g^* \subseteq g$.

(iii) Finally, we show $g(l(g)^*) \geq l(g)$:

By construction, $l(g) = l(f^*) = \delta_2{}^*$. Moreover, by assumption, there is a $\theta \leq \delta_2$ such that $f(\theta^*) \geq \delta_2$. That is, $f^*(\theta^{**}) \geq \delta_2{}^*$, and $\theta^{**} \leq \delta_2{}^{**}$. But, by the construction of g, this implies

$$l(g) = \delta_2{}^* \leq g(\delta_2{}^{**}) = g(l(g)^*)$$

Thus the antecedent of 4.19 holds for g, so that we have:

$$t(g) = t(g)^* \ \&\ l(g) < t(g) \qquad (1)$$

Thus if we define $s(f) =_{df} (\dot{\iota}x)(x^* = t(g))$, then $s(f) = s(f)^* = t(g)$, and (by (1)) $\delta_2 < s(f)$. [s(f) is stratified since t(g) is, as a term with free variable f; the type of s(f) is as claimed since that of g is one higher than f]. The proof of 4.22 is complete.

We now introduce a second extension of NF which bears some similarity to NF_1, except that all additional axioms are stratified. Specifically, the axioms of NF_2 are just the axioms of NF plus all instances of the following two axiom schemes:

(i) $[t_1 \leq t_2 \ \&\ t_2 \in NO \ \&\ t_2 = t_2{}^*] \Longrightarrow t_1 = t_1{}^*$

where t_1, t_2 are any two constant, stratified terms.

(ii) $t \in WS \Longrightarrow Nc(USC(t) \leq_c Nc(t)$

where t is any constant, stratified term.

<u>Remark</u>: One apparent advantage of NF_2 is that all allowed instances of (i) and (ii) are stratified sentences (since constant, stratified terms can be assigned any type). As we shall see in Section 6, NF_1 is a non-trivial extension of NF (unless NF is inconsistent). However the method of proof used there does not apply to NF_2, so that the question of whether or not NF_2 extends NF in a non-trivial way is open. Moreover, it is easy to prove (using for example an analysis of cut-free proofs in a Herbrand-type system of quantification theory) that any stratified theorem of quantification theory (in the language of NF) has a proof every line of which is a stratified formula. It thus may be somewhat easier to prove the consistency of NF_2 relative to NF than that of NF_1, if an approach via the analysis of proofs is used.

More importantly, axiom scheme (i) is apparently somewhat weaker than the axiom added to NF in obtaining NF_1; as will be shown in Section 6, if NF_2 is consistent, then there are models of NF_2 which do not even satisfy Rosser's Counting Axiom (and thus are not models of NF_1). Although axiom scheme (ii) is not restricted to sets each of which is a subset of some well-ordered, Cantorian set (as the additional axiom of NF_1 and schema (i) are), it applies only to sets which have stratified definitions. After some investigation, we have been able to find no argument within NF which would indicate a way of deriving a contradiction in NF_2. However, the remainder of this section is devoted to demonstrating that the full Zermelo-Fraenkel set theory has a relative interpretation in NF_2, a fact which indicates that this system is capable of somewhat more complex argument than seems at first to be the case.

4.23. Lemma: Let t be a constant, stratified term. Then:

$$\vdash_{NF_2} t \in NO \Longrightarrow t^* \leq t$$

Proof: Suppose $t \in NO$, and by way of contradiction, let $t < t^*$. Let $A = \{\theta \mid \theta < t\}$. Then $A \subseteq NO$, and A is a constant, stratified term. Thus by axiom scheme (ii), and 1.18,

$$Nc(A^*) \leq_c Nc(A).$$

That is, by 1.15,

$$Card(t^*) \leq_c Card(t)$$

However, by [R, p.478] $Card(t) \leq_c Card(t^*)$, so we have

$$Card(t^*) = Card(t) \qquad (1)$$

If $t < \omega_o$, then (1) implies $t = t^*$. Thus we may suppose $\omega_o \leq t$. Now let $\theta_o = \min_{\leq}\{\theta \mid \theta \in W \ \& \ Card(\theta) = Card(t)\}$. Then θ_o is a stratified term, and by (1) and 1.21,

$$Card(\theta_o{}^*) = Card(\theta_o)$$

so that $\theta_o = \theta_o^*$ by the remark following Definition 2.3. Then if we define $\theta_1 = \min_{\leq} \{\theta \mid \theta \in W \ \& \ \theta_o < \theta\}$ (so $\theta_1 \in NO$, since $\theta_o < \Omega_o \in W$), we have $\theta_1 = \theta_1^*$ and $t < \theta_1$. But θ_1 is also a constant, stratified term, so by axiom scheme (i) $t = t^*$, contradicting the initial assumptions. That is, $t^* \leq t$, and the proof is complete.

The next proposition will be used to show that in NF_2 the extension of formulas A^C can be accomplished in an especially uniform way.

4.24. <u>Lemma</u>: Let $R(x,y)$ be a formula with x,y as its free variables and which can be stratified when x and y are assigned the same type. Let k be any integer; then:

$$\vdash_{NF_2} (\theta)(\delta)[\theta,\delta < \Omega_k) \Longrightarrow (R(\theta,\delta) \Longleftrightarrow R(\theta^*,\delta^*))] \Longrightarrow$$

$$(\theta)[\theta < \Omega_{k+1} \Longrightarrow$$

$$[(\exists\delta)(\delta < \Omega_k \ \& \ R(\theta,\delta)) \Longleftrightarrow (\exists\delta)(\delta < \Omega_k \ \& \ R(\theta^*,\delta))]]$$

<u>Proof</u>: Assume the antecedent, and let $S(x)$ denote the formula $(\exists)(\delta < \Omega_k \ \& \ R(x,\delta))$. Then S is stratifiably definable, and contains only x free. We prove first:

$$(\theta)[(\theta < \Omega_k \ \& \ S(\theta)) \Longrightarrow S(\theta^*)] \qquad (1)$$

For, if $\theta < \Omega_k$ and $S(\theta)$, then $R(\theta,\delta)$ for some $\delta < \Omega_k$.

But then by assumption, we have $R(\theta^*, \delta^*)$. Moreover, $\delta^* < \Omega_k^* < \Omega_k$, so $S(\theta^*)$ holds, and (1) is proved.

Evidently (in view of (1)) the conclusion of 4.24 is equivalent to

$$(\theta)[(\theta < \Omega_{k+1} \ \& \ S(\theta^*)) \Longrightarrow S(\theta)] \qquad (2)$$

If, for every $\theta < \Omega_k$, $S(\theta^*) \Longrightarrow S(\theta)$, then we are done. Otherwise, we define

$$\theta_o = \min_{\leq} \{\theta \mid \theta < \Omega_k \ \& \ S(\theta^*) \ \& \neg S(\theta)\}. \qquad (3)$$

and have $\theta_o \in NO$. Since θ_o is clearly a constant, stratified term, we have $\theta_o^* \leq \theta_o$. (By 4.23) Moreover, since $S(\theta_o^*)$ and $\neg S(\theta_o)$ holds we have

$$\theta_o^* < \theta_o. \qquad (4)$$

Since (by (1)), for all $\theta < \theta_o$ one has $S(\theta) \Longleftrightarrow S(\theta^*)$, it suffices to show $\Omega_{k+1} \leq \theta_o$.

For each $\theta < \theta_o$ we define:

$$g(\theta) = \begin{cases} \min_{\leq} \{\delta \mid R(\theta,\delta)\} & \text{if } S(\theta) \\ 0 \ \text{if} \neg S(\theta) \end{cases} \qquad (5)$$

That is, we set g equal to

$$\{<\theta,\lambda> \mid \theta < \theta_o \ \& \ [(S(\theta) \ \& \ \lambda = \min_{\leq} \{\delta \mid R(\theta,\delta)\}) \vee (\neg S(\theta) \ \& \ \lambda = 0)] \ .$$

Then evidently g is a stratified, constant term; moreover

$g \in \mathrm{Funct}$ and $\mathrm{Arg}(g) = \{\theta \mid \theta < \theta_o\}$. We show next that $g^* \subseteq g$:

(i) $\theta < \theta_o \Longrightarrow (g(\theta) = 0 \Longleftrightarrow g(\theta^*) = 0)$:

But this follows immediately, since $\theta < \theta_o$ implies $S(\theta) \Longleftrightarrow S(\theta^*)$.

(ii) $\theta < \theta_o \Longrightarrow (g(\theta^*) \leq g(\theta)^*)$:

By (i), (and the fact that $0 \leq 0^*$) we may assume that $S(\theta)$ and $S(\theta^*)$ hold. Thus $R(\theta,g(\theta))$ holds, and evidently $g(\theta) < \Omega_k$ ($S(\theta)$ implies $R(\theta,\delta)$ for <u>some</u> $\delta < \Omega_k$, and $g(\theta)$ is the least such δ). Then $R(\theta^*,g(\theta)^*)$ holds, so $g(\theta^*) \leq g(\theta)^*$.

(iii) $\theta < \theta_o \Longrightarrow (g(\theta)^* \leq g(\theta^*))$:

Again, by (i) we may assume $S(\theta)$ and $S(\theta^*)$. If, by way of contradiction, we assume $g(\theta^*) < g(\theta)^*$, then for some $\delta < g(\theta)$, $g(\theta^*) = \delta^*$. Moreover, we have $R(\theta^*,g(\theta^*))$, that is $R(\theta^*,\delta^*)$. But since (as in (ii)) $\delta < \Omega_k$, we have $R(\theta,\delta)$, contradicting the definition of $g(\theta)$. Thus (i), (ii) and (iii) establish

$$g^* \subseteq g. \tag{6}$$

Now suppose, contrary to what we wish to prove, $\theta_o < \Omega_{k+1}$. Then, since $S(\theta_o^*)$ holds (by (3)) we have $R(\theta_o^*,g(\theta_o^*))$ and $g(\theta_o^*) < \Omega_k$. If $g(\theta_o^*) < \Omega_{k+1}$, then for some $\delta < \Omega_k$, we have $g(\theta_o^*) = \delta^*$. However this implies $R(\theta_o,\delta)$, and hence $S(\theta_o)$, a contradiction. Thus we have

$$\theta_o < \Omega_{k+1} \Longrightarrow (g(\theta_o^*) \geq \Omega_{k+1} > \theta_o). \qquad (7)$$

Thus the conditions of Theorem 4.22 are satisfied for g, if $\theta_o < \Omega_{k+1}$. But then the constant, stratified term s(g) satisfies

$$s(g) = s(g)^* \ \& \ s(g) \in NO \ \& \ \theta_o < s(g). \qquad (8)$$

Thus, by an instance of axiom scheme (i) of NF_2 we have $\theta_o = \theta_o^*$ (since θ_o is also a constant, stratified term) which is a contradiction. Hence $\Omega_{k+1} \leq \theta_o$ is established, and thus the proof of 4.24 is complete.

Using Theorem 2.28, we obtain an extension of 4.24 to formulas with any number of free variables.

4.25. <u>Corollary</u>: Let $R(x_1,\ldots,x_n,y)$ be any formula with only $x_1,\ldots,x_n,y$ as free variables, and which can be stratified when all the free variables are assigned the same type. Let k be any integer; then:

$$\vdash_{NF_2} (\theta_1)\ldots(\theta_n)(\delta)[\theta_1,\ldots,\theta_n,\delta < \Omega_k \Longrightarrow (R(\theta_1,\ldots,\theta_n,\delta) \Longleftrightarrow R(\theta_1^*,\ldots,\theta_n^*,\delta^*))] \Longrightarrow$$

$$(\theta_1)\ldots(\theta_n)[\theta_1,\ldots,\theta_n < \Omega_{k+1} \Longrightarrow [(\exists\delta)(\delta < \Omega_k \ \& \ R(\theta_1,\ldots,\theta_n,\delta) \Longleftrightarrow (\exists\delta)(\delta < \Omega_k \ \& \ R(\theta_1^*,\ldots,\theta_n^*,\delta))]]$$

<u>Proof</u>: Let $A = \{\theta \mid \theta < \Omega_k\}$. Then if f is the similarity function given by 2.28, let $g = A^n \upharpoonleft f$. Since $f(<\theta_1,\ldots,\theta_n>)^* = f(<\theta_1{}^*,\ldots,\theta_n{}^*>)$ (and $Val(f) = NO$) we see immediately that $Val(g) = A$, and

$$g(<\theta_1,\ldots,\theta_n>)^* = g(<\theta_1{}^*,\ldots,\theta_n{}^*>$$

for any $<\theta_1,\ldots,\theta_n> \in A^n$. Evidently g is also a constant, stratified term. If we then define

$$S(x,y) =_{df} (\exists x_1)\ldots(\exists x_n)[g(<x_1,\ldots,x_n>) = x \;\&\; R(x_1,\ldots,x_n,y)]$$

it is clear that S is stratifiable when x,y are assigned the same type, and S has only x,y free. Moreover, the statement to be proved for R is equivalent (in NF) to the statement

$$(\theta)(\delta)[\theta,\delta < \Omega_k \Longrightarrow (S(\theta,\delta) \Longleftrightarrow S(\theta^*,\delta^*))] \Longrightarrow$$

$$(\theta)[\theta < \Omega_{k+1} \Longrightarrow [(\exists\delta)(\delta < \Omega_k \;\&\; S(\theta,\delta)) \Longleftrightarrow (\exists\delta)(\delta < \Omega_k \;\&\; S(\theta^*,\delta))]].$$

(This follows easily from the properties of g stated above.) Since this is a theorem in NF_2, by 4.24, the proof of 4.25 is complete.

We now return to the formula R(x,y) defined in Section 3 (see Definition 3.37). For the remainder of this section 'R(x,y)' will refer to this formula, and 'Σ' to the class of formulas constructed from $R(\theta,\delta)$ and $\theta = \delta$ using truth functional connectives and quantifiers relativized to NO (ie. quantifiers whose variables are denoted by lower case Greek letters.) Recall that every formula A of Σ is stratifiable by an assignment which gives the same type to all the free variables of A. We assume that the free variables of any A in Σ are also relativized to NO.

For each formula A in Σ we define the degree of A ($\underline{d(A)}$) as the maximum number of nested quantifiers in A (based on the construction of A as a member of Σ). That is, we define $d(A)$ inductively by

(i) $d(A) = 0$ if A is the formula $R(\theta,\delta)$ or $\theta = \delta$;

(ii) $d(A \& B) = d(A \vee B) = \max(d(A), d(B))$;

(iii) $d(\neg A) = d(A)$;

(iv) $d((\theta)A) = d((\exists\theta)A) = 1 + d(A)$.

Then for each A in Σ we define the bounded relativization of A, ($\underline{A'}$) inductively, as follows:

(i) $A' = A$ if A is $R(\theta,\delta)$ or $\theta = \delta$;

(ii) $(A \& B)' = A' \& B'$, $(A \vee B)' = A' \vee B'$;

(iii) $(\neg A)' = \neg(A')$;

(iv) $((\theta)A)' = (\theta)(\theta < \Omega_k \Longrightarrow A')$

$((\exists\theta)A)' = (\exists\theta)(\theta < \Omega_k \& A')$.

Where $d(A) = k$.

Thus for any A in Σ, A' is a formula which can be stratified in such a way that all free variables of A' are assigned the same type. We apply Corollary 4.25 to show that for each A in Σ, A' is equivalent to A^C when the free variables are assumed to range over Cantorian ordinals, and A' has *-symmetry up to a certain, specified Ω_k. Namely:

4.26. <u>Lemma</u>: Let A be in Σ, with free variables $\theta_1,\ldots,\theta_n$, and let $d(A) = k$. Then

(i) $\vdash_{NF_2} (\theta_1)\ldots(\theta_n)[\theta_1,\ldots,\theta_n < \Omega_k \Longrightarrow$
$$(A'(\theta_1,\ldots,\theta_n) \Longleftrightarrow A'(\theta_1{}^*,\ldots,\theta_n{}^*))]$$

(ii) $\vdash_{NF_2} (\theta_1)\ldots(\theta_n)[C(\theta_1) \,\&\ldots\&\, C(\theta_n) \Longrightarrow$
$$(A'(\theta_1,\ldots,\theta_n) \Longleftrightarrow A^C(\theta_1,\ldots,\theta_n))]$$

<u>Proof</u>: (i) This follows by induction on the construction of A as an element of Σ. If A is $\theta_1 = \theta_2$, (i) follows by 1.3. If A is $R(\theta_1,\theta_2)$, then (i) follows by 3.40. If A is $\neg B$, $(B \vee C)$ or $(B \,\&\, C)$, and (i) holds for B (and C), then (i) holds for A by elementary logic. It thus suffices to consider the case where A is $(\exists\theta_o)\, B$ (since universal quantifiers can be dealt with as negations of existential ones.)

Thus we suppose (i) holds for B, with free variables $\theta_o,\theta_1,\ldots,\theta_n$. Now if $d(A) = k$ (so $k > 0$), then evidently $d(B) = k-1$. Thus we have $A' = (\exists\theta_o)(\theta_o < \Omega_{k-1} \,\&\, B')$. By the induction assumption (for B') we have

$$(\theta_o)\ldots(\theta_n)[\theta_o,\ldots,\theta_n < \Omega_{k-1} \Longrightarrow (B'(\theta_o,\ldots,\theta_n) \Longleftrightarrow B'(\theta_o{}^*,\ldots,\theta_n{}^*))].$$

Then, since B' satisfies the conditions of 4.25, we have

$$(\theta_1)\ldots(\theta_n)[\theta_1,\ldots,\theta_n < \Omega_k \Longrightarrow (A'(\theta_1,\ldots,\theta_n) \Longleftrightarrow A'(\theta_1{}^*,\ldots,\theta_n{}^*))]$$

which is (i) for A. Thus the proof of (i) is complete.

(ii) Again we induct over the method of construction of A (as an element of Σ). If A is $R(\theta_1,\theta_2)$, then $A = A' = A^C$, so the result is trivial, and the same is true if A is $\theta_1 = \theta_2$. If A is $\neg B$, $(B \vee C)$, or $(B \& C)$, and (ii) holds for B (and C), then we obtain (ii) for A by elementary logic. Thus, as in the case of (i), it suffices to consider the case where A is $(\exists\theta_o)B$, and where (ii) holds for B. Suppose $d(A) = k$ so that $k > 0$ and $d(B) = k-1$. Thus A' is $(\exists\theta_o)(\theta_o < \Omega_{k-1} \& B')$, and A^C is $(\exists\theta_o)(C(\theta_o) \& B^C)$.

Assume that $\theta_1,\ldots,\theta_n$ are fixed ordinals such that $C(\theta_o) \& \ldots \& C(\theta_n)$ holds. Then, by the induction assumption

$$(\theta_o)(C(\theta_o) \Longrightarrow (B'(\theta_o,\theta_1,\ldots,\theta_n) \Longleftrightarrow B^C(\theta_o,\ldots,\theta_n)) \quad (1)$$

Furthermore, by part (i),

$$(\theta_o)[\theta_o < \Omega_{k-1} \Longrightarrow (B'(\theta_o,\theta_1,\ldots,\theta_n) \Longleftrightarrow B'(\theta_o{}^*,\theta_1,\ldots,\theta_n))] \quad (2)$$

(since $C(\theta)$ implies $\theta < \Omega_n$ for every integer n).

Now, if $A^C(\theta_1,\ldots,\theta_n)$ holds, there is a θ_o (with $C(\theta_o)$) such that $B^C(\theta_o,\theta_1,\ldots,\theta_n)$. Hence by (1) we have $B'(\theta_o,\theta_1,\ldots,\theta_n)$, and since $\theta_o < \Omega_{k-1}$, this implies $A'(\theta_1,\ldots,\theta_n)$.

To go in the other direction, suppose $A'(\theta_1,\ldots,\theta_n)$ holds. Let

$$\theta_o = \min_{\leq} \{\theta \mid B'(\theta,\theta_1,\ldots,\theta_n)\}. \quad (3)$$

Then evidently $\theta_o < \Omega_{k-1}$; we will show $C(\theta_o)$ holds. Otherwise $\theta_o < \theta_o{}^*$ or $\theta_o{}^* < \theta_o$. In the first case, there is a $\delta < \theta_o$ such that $\theta_o = \delta^*$. Then $\delta < \Omega_{k-1}$, so by (2) we have $B'(\delta,\theta_1,\ldots,\theta_n)$, contradicting (3). In the second case, we have $B'(\theta_o{}^*,\theta_1,\ldots,\theta_n)$, again by (2), which also contradicts (3). Therefore, we have $C(\theta_o)$ and $B'(\theta_o,\theta_1,\ldots,\theta_n)$. However, by (1) this implies $B^C(\theta_o,\theta_1,\ldots,\theta_n)$ & $C(\theta_o)$, and hence $A^C(\theta_1,\ldots,\theta_n)$.

That is, we have proved

$$A'(\theta_1,\ldots,\theta_n) \iff A^C(\theta_1,\ldots,\theta_n)$$

whenever $C(\theta_1)$ &...& $C(\theta_n)$ holds. Thus the proof of (ii) is complete.

We are now in a position to show that the C-translate of every replacement axiom of Z-F is a theorem of NF_2.

4.27. <u>Theorem</u>: Let A be a formula in Σ with free variables $\delta_1,\delta_2,\theta_1,\ldots,\theta_n$. Let $B_1(\theta_o,\theta_1,\ldots,\theta_n)$ be the formula

$$(\delta_1)(\delta_2)(\delta_3)[(R(\delta_1,\theta_o) \mathbin{\&} A(\delta_1,\delta_2,\theta_1,\ldots,\theta_n) \mathbin{\&} A(\delta_1,\delta_3,\theta_1,\ldots,\theta_n)) \Longrightarrow (\delta_2 = \delta_3)] \mathbin{\&} (\delta_1)[R(\delta_1,\theta_o) \Longrightarrow (\exists\delta_2)A(\delta_1,\delta_2,\theta_1,\ldots,\theta_n)]$$

and let $B_2(\theta_o,\theta_1,\ldots,\theta_n)$ be the formula

$$(\exists\lambda)(\delta_2)[R(\delta_2,\lambda) \Longleftrightarrow (\exists\delta_1)(R(\delta_1,\theta_o) \mathbin{\&} A(\delta_1,\delta_2,\theta_1,\ldots,\theta_n)].$$

Then: $$\vdash_{NF_2}[(\theta_o)(\theta_1)\ldots(\theta_n)[B_1(\theta_o,\theta_1,\ldots,\theta_m) \Longrightarrow B_2(\theta_o,\theta_1,\ldots,\theta_n)]]^C.$$

<u>Proof</u>: Suppose $\theta_o,\theta_1,\ldots,\theta_n$ are such that $C(\theta_o) \mathbin{\&} C(\theta_1) \mathbin{\&}\ldots\mathbin{\&} C(\theta_n)$ holds. Then evidently it is necessary to show that $B_1{}^C(\theta_o,\theta_1,\ldots,\theta_n)$ implies $B_2{}^C(\theta_o,\theta_1,\ldots,\theta_n)$. Equivalently, by 4.25, we suppose $B_1'(\theta_o,\theta_1,\ldots,\theta_n)$ holds, and must prove $B_2'(\theta_o,\theta_1,\ldots,\theta_n)$. By 3.41 and the fact that $\theta_o = \theta_o{}^*$, we see that for every integer k,

$$R(\delta_1,\theta_o) \Longrightarrow \delta_1 < \Omega_k. \qquad (1)$$

Thus, if $d(A) = k$, and if $\delta_2 < \Omega_{k+1}$ and $\delta_3 < \Omega_k$, we have

$$[R(\delta_1,\theta_o) \mathbin{\&} A'(\delta_1,\delta_2,\theta_1,\ldots,\theta_n) \mathbin{\&} A'(\delta_1,\delta_3,\theta_1,\ldots,\theta_n)] \Longrightarrow (\delta_2 = \delta_3) \qquad (2)$$

(from $B'(\theta_o,\ldots,\theta_n)$ and 4.25). But then if δ_2,δ_3 are any ordinals $< \Omega_k$ and the antecedent of (2) holds for $\delta_1,\delta_2,\delta_3$, then we obtain

$$R(\delta_1{*},\theta_o) \;\&\; A'(\delta_1{*},\delta_2{*},\theta_1,\ldots,\theta_n) \;\&\; A'(\delta_1{*},\delta_3{*},\theta_1,\ldots,\theta_n)$$

by 4.25. Thus by (2) we have $\delta_2{*} = \delta_3{*}$, and hence $\delta_2 = \delta_3$. That is,

$$[R(\delta_1,\theta_o) \;\&\; A'(\delta_1,\delta_2,\theta_1,\ldots,\theta_n) \;\&\; A'(\delta_1,\delta_3,\theta_1,\ldots,\theta_n)] \implies (\delta_2 = \delta_3) \tag{3}$$

for any $\delta_2,\delta_3 < \Omega_k$ and any $\delta_1 \in NO$. Also, from the second clause of $B_1'(\theta_o,\ldots,\theta_n)$ we have

$$R(\delta_1,\theta_o) \implies (\exists\delta_2)(\delta_2 < \Omega_k \;\&\; A'(\delta_1,\delta_2,\theta_1,\ldots,\theta_n)) \tag{4}$$

for any $\delta_1 \in NO$.

Now let

$$D = \left\{\delta_2 \mid \delta_2 < \Omega_k \;\&\; (\exists\delta_1)(R(\delta_1,\theta_o) \;\&\; A'(\delta_1,\delta_2,\theta_1,\ldots,\theta_n))\right\} \tag{5}$$

so $D \subseteq \{\theta \mid \theta < \Omega_k\}$. In fact, we show

$$D = D^* \tag{6}$$

(i) If $\delta_2 \in D$, so that for some δ_1, $R(\delta_1,\theta_o)$ and $A'(\delta_1,\delta_2,\theta_1,\ldots,\theta_n)$, hold then by 4.25 and (1) we have $R(\delta_1{*},\theta_o)$ and $A'(\delta_1{*},\delta_2{*},\theta_1,\ldots,\theta_n)$, and hence $\delta_2{*} \in D$.

(ii) Suppose $\delta_2{*} \in D$, so that for some δ_1 we have $R(\delta_1,\theta_o) \;\&\; A'(\delta_1,\delta_2{*},\theta_1,\ldots,\theta_n)$. But then $\delta_1 < \theta_o$ so that

$\delta_1 = \lambda_1{}^*$ for some λ_1 with $\lambda_1 < \theta_o$; moreover we have $R(\lambda_1, \theta_o)$. Thus by (4) there is a $\lambda_2 < \Omega_k$ with $A'(\lambda_1, \lambda_2, \theta_1, \ldots, \theta_n)$, and hence with $\lambda_2 \in D$. But then by 4.25, $A'(\lambda_1{}^*, \lambda_2{}^*, \theta_1, \ldots, \theta_n)$ holds, and hence by (3), $\lambda_2{}^* = \delta_2{}^*$. That is, $\delta_2 = \lambda_2 \in D$.

(iii) By (i) and (ii), $\delta_2 \in D \iff \delta_2{}^* \in D$, so $D^* \subseteq D$. If $D \neq D^*$, then there must be a $\delta_2 \in D$ with $\delta_2 \geq \Omega_o$. However, $D \subseteq \{\theta \mid \theta < \Omega_k\} \subseteq \{\theta \mid \theta < \Omega_o\}$, so this is impossible, and (6) is established.

Now by 3.40 (that is, since $\theta < \Omega_k \iff R(\theta, (\Omega_k)))$, the formula defining D (in (4)) is equivalent to a formula in which all quantifiers are bounded (relative to R). Thus, by the remarks preceding 3.40 there is a $\lambda \in NO$ such that

$$(\delta)(\delta \in D \iff R(\delta, \lambda)). \tag{7}$$

Moreover, by 3.40 and (6), we have $\lambda = \lambda^*$. Thus we have

$$(\delta_2)[\delta_2 < \Omega_{k+1} \implies [R(\delta_2, \lambda) \iff (\exists\delta_1)(\delta_1 < \Omega_k \;\&\; R(\delta_1, \theta_o) \;\&\; A'(\delta_1, \delta_2, \theta_1, \ldots, \theta_n)]]$$

That is, since $\lambda < \Omega_{k+2}$, we have

$$B_2{}'(\theta_o, \theta_1, \ldots, \theta_n)$$

which completes the proof of 4.27.

Thus if A^C is the C-translate of any axiom of Z-F, $\vdash_{NF2} A^C$, completing the proof of:

Main Theorem II: The operation of passing from any formula of the language of Z-F to its C-translate (in terms of $R(\theta,\delta)$) provides a relative interpretation of Z-F in NF_2.

Section 5:

In this section we make some straightforward applications of Theorems 4.19, 4.20 and 4.22. In particular, we show that in NF no well ordered set is similar to its subset class, generalizing a result of Specker [13]. Moreover for "large" well-ordered sets x, Nc(x) is not smaller than Nc(SC(x)). Specifically, this holds for sets $x \in WS$ which have $Nc(USC(x)) <_c Nc(x)$ and are not subsets of any well-ordered Cantorian set. We also prove that the statement $(\omega_{\Omega_o} \in NO)$ is a theorem in both NF_1 and NF_2. Each of the proofs proceeds by constructing a function f which satisfies the conditions of 4.19, 4.20 or 4.22 under the assumption that the statement to be proved is false. Then the existence of the particular Cantorian ordinal t(f) (or s(f)) is shown to be contradictory.

Note that $2^{Card(\theta)} \in NC$ for every $\theta \in NO$, since $Card(\theta) <_c Nc(NO) <_c Nc(USC(V))$. Also, if $2^n \in NC$, then

$$(i) \qquad 2^{T(n)} = T(2^n)$$

(see 5.9 of [13]).

5.1. Theorem: $\vdash (x)[x \in WS \Longrightarrow \neg(x \text{ sm } SC(x))]$

Proof: Suppose the statement is false, and let x_o be a set such that $x_o \in WS$ and x_o sm $SC(x_o)$. Then the same holds true for $USC^2(x_o)$. That is, $USC^2(x_o) \in WS$ and $USC^2(x_o)$ sm $SC(USC^2(x_o))$ (see [R, p.368]). Then let $\theta_o = Cd(x_o)$ (see Definition 2.21), so that $\theta_o \in W$ or $\theta_o < \omega_o$ and $Card(\theta_o) = Nc(USC^2(x_o))$. Thus we have

$$Card(\theta_o) = 2^{T(Card(\theta_o))}$$

(by 4.2 of [13]) or

$$Card(\theta_o) = 2^{Card(\theta_o*)} \tag{1}$$

by 1.21.

Moreover, if $\theta \leq \theta_o$, then $\theta* \leq \theta_o*$ so that $Card(\theta*) \leq_c Card(\theta_o*)$. Hence

$$2^{Card(\theta*)} \leq_c Card(\theta_o)$$

and thus $2^{Card(\theta*)}$ is the cardinal of a well-ordered set. In fact, it is the cardinal of a well-ordered set of the form $USC^2(x)$, so that for some δ,

$$2^{Card(\theta*)} = Card(\delta) \tag{2}$$

Then for each $\theta < \theta_o*$ we define

$$f(\theta) = \min_{\leq} \left\{\delta \mid 2^{Card(\theta)} = Card(\delta)\right\}. \tag{3}$$

By (2), $Arg(f) = \left\{\theta \mid \theta < \theta_o*\right\}$, and evidently $f \in Funct$. [Note that the defining condition is stratified when θ and δ are assigned the same type.]

Since $\mathrm{Card}(\theta) <_c 2^{\mathrm{Card}(\theta)}$ for all $\theta \in \mathrm{NO}$ (by [R, p.390]), we have

$$\theta < \theta_o^* \Longrightarrow (\theta < f(\theta)) \tag{4}$$

so $f \in F$. Moreover, if $\theta \leq \delta < \theta_o^*$, then $\mathrm{Card}(\theta) \leq_c \mathrm{Card}(\delta)$, and hence $2^{\mathrm{Card}(\theta)} \leq_c 2^{\mathrm{Card}(\delta)}$. That is,

$$\theta \leq \delta < \theta_o^* \Longrightarrow (f(\theta) \leq f(\delta) \tag{5}$$

so that $f \in F_1$. By similar argument, (1) implies $\theta_o^* < \theta_o$, and thus $\theta_o^{**} < \theta_o^*$. Moreover, by 1.21 and the symmetry (i) of 2^n, we have

$$\mathrm{Card}(\theta_o^*) = 2^{\mathrm{Card}(\theta_o^{**})} \tag{6}$$

from (1), and hence $f(\theta_o^{**}) = \theta_o^*$.

Finally, if $\theta < \theta_o^*$, so that

$$\mathrm{Card}(f(\theta)) = 2^{\mathrm{Card}(\theta)}$$

we have

$$\mathrm{Card}(f(\theta)^*) = 2^{\mathrm{Card}(\theta^*)}$$

(as in the proof of (6)), and thus

$$f(\theta)^* = f(\theta^*) \tag{7}$$

for all $\theta < \theta_o^*$. [This follows since $f(\theta) \in W$ or $f(\theta) < \omega_o$, by the definition (3), so $f(\theta)^* \in W$ or $f(\theta)^* < \omega_o$ as well, by 2.4 and 2.2. Then one also has $f(\theta^*) = 2^{\mathrm{Card}(\theta^*)}$, with $f(\theta^*) \in W$ or $f(\theta^*) < \omega_o$, so that $f(\theta^*)$ and $f(\theta)^*$ must be equal.]

Thus f satisfies the conditions of 4.19, and we conclude $\theta_o^* < t(f) = t(f)^* \leq f(\theta_o^{**})$, that is, $\theta_o^* < \theta_o^*$, a contradiction. Hence the proof is complete.

5.2. <u>Definition</u>: $SC^1(x) = SC(x)$

$$SC^{n+1}(x) =_{df} SC(SC^n(x)).$$

Using basically the same approach as in the proof of 5.1, we obtain the following for each $k \geq 1$:

5.3. <u>Theorem</u>: $\vdash (x)[x \in WS \Longrightarrow \neg(x \text{ sm } SC^k(x))]$.

<u>Proof</u>: The proof proceeds as in 5.1, except that instead of considering the function which takes n to 2^n, we consider the function taking n to the k^{th} iterate of the expotential, applied initially to n. We consider the case $k = 2$ as typical.

Suppose $x_o \in WS$ and x_o sm $SC^2(x_o)$, and let $\theta_o = Cd(x_o)$ and $n_o = Nc(USC^2(x_o))$. Then $n_o = Card(\theta_o)$, and moreover,

$$2^{2^{n_o}} = 2^{Nc(SC(USC(x_o)))} = Nc(SC^2(x_o)) = Nc(x_o)$$

That is,

$$T^2(2^{2^{n_o}}) = n_o,$$

or using the symmetry (i) of 2^n and 1.21,

$$2^{2^{\mathrm{Card}(\theta_o**)}} = \mathrm{Card}(\theta_o). \qquad (1)$$

Moreover, since $n_o <_c 2^{n_o} <_c 2^{2^{n_o}}$, we have

$$\mathrm{Card}(\theta_o**) < \mathrm{Card}(\theta_o)$$

or $\theta_o** < \theta_o$. Thus by 1.3, $\theta_o* < \theta_o$.

We then define a function f with $\mathrm{Arg}(f) = \{\theta \mid \theta < \theta_o**\}$ by

$$f(\theta) = \min_{\leq}\left\{\delta \;\middle|\; 2^{2^{\mathrm{Card}(\theta)}} = \mathrm{Card}(\delta)\right\}.$$

(By (1), if $\theta < \theta_o**$ then $2^{2^{\mathrm{Card}(\theta)}} \leq_c \mathrm{Card}(\theta_o)$, so f is defined for every such ordinal.)

We prove $f \in F_1$ by the same method as in the proof of 5.1, and note that (1) implies

$$f(\theta_o***) = \theta_o*$$

using the symmetry of 2^n and 1.21. Moreover, $\theta_o** < \theta_o*$ by 1.3. Thus the conditions of 4.19 are satisfied by f, and we conclude

$$\theta_o** < t(f) = t(f)* \leq f(\theta_o***).$$

That is, $\theta_o** < t(f) = t(f)* < \theta_o*$, which contradicts 1.3 and completes the proof for $k = 2$.

The proof for $k > 2$ proceeds similarly.

It is possible to sharpen 5.1 when we restrict attention to sets $x \in WS$ with $Nc(USC(x)) \leq_c Nc(x)$, as we show in the next theorem:

5.4. <u>Theorem</u>: $\vdash (x)[(x \in WS \ \&\ Nc(USC(x)) \leq_c Nc(x)) \Longrightarrow$

$[(Nc(x) \leq_c Nc(SC(x)) \Longrightarrow$

$(\exists y)(Can(y) \ \&\ y \in WS \ \&\ Nc(x) \leq_c Nc(y))]]$.

<u>Proof</u>: Suppose the statement is false, and let $x_o \in WS$ be such that $Nc(USC(x_o)) \leq_c Nc(x_o)$ and $Nc(x_o) \leq_c Nc(SC(x_o))$. We suppose further that $Nc(x_o) \leq_c Nc(USC(V))$ without loss of generality. Also we see that in fact $Nc(USC(x_o)) <_c Nc(x_o)$, since otherwise $Can(x_o)$, and the statement holds.

Then we have $Nc(USC^3(x)) <_c Nc(USC^2(x_o))$ and $Nc(USC^2(x_o)) \leq_c 2^{Nc(USC^3(x_o))}$ by the symmetries of $\leq_c$ and 2^n relative to T. Let $\theta_o = Cd(x_o)$ so that $Card(\theta_o) = Nc(USC^2(x_o))$. Hence by 1.21,

$$\theta_o^* < \theta_o \quad \text{and} \quad Card(\theta_o) \leq_c 2^{Card(\theta_o^*)}. \qquad (1)$$

For each $\theta < \theta_o^*$ we define

$$f(\theta) = \min_{\leq} \left\{\delta \mid \neg Card(\delta) \leq_c 2^{Card(\theta)}\right\} \qquad (2)$$

Now since it has been assumed that $Nc(x_o) \leq_c Nc(USC(V))$ we have $Card(\theta_o) \leq_c Nc(USC^3(V))$. Thus (since $SC(USC(V))$ has the same cardinality as $USC(V)$)

$$2^{Card(\theta_o^*)} \leq_c Nc(USC^3(V)).$$

That is

$$\neg\, Card(\Omega_o) \leq_c 2^{Card(\theta_o^*)}$$

since $\mathrm{Card}(\Omega_o) = \mathrm{Nc}(\mathrm{USC}(\mathrm{NO}))$. Therefore $f(\theta) \in \mathrm{NO}$ for every $\theta < \theta_o{}^*$, and in fact $f(\theta) \leq \Omega_o$ for every such θ.

If $\theta < \theta_o{}^*$, then $\mathrm{Card}(\theta) <_c 2^{\mathrm{Card}(\theta)}$, so that $\theta < f(\theta)$. Moreover, f is an increasing function on $\{\theta \mid \theta < \theta_o{}^*\}$ since

$$\begin{aligned}\theta_1 \leq \theta_2 &\Longrightarrow \mathrm{Card}(\theta_1) \leq_c \mathrm{Card}(\theta_2)\\ &\Longrightarrow 2^{\mathrm{Card}(\theta_1)} \leq_c 2^{\mathrm{Card}(\theta_2)}\end{aligned}$$

and thus

$$\neg\, \mathrm{Card}(\delta) \leq_c 2^{\mathrm{Card}(\theta_2)} \Longrightarrow \neg\, \mathrm{Card}(\delta) \leq_c 2^{\mathrm{Card}(\theta_1)}$$

by the transitivity of $\leq_c$.

Thus $f \in F_1$. We show next that $\theta < \theta_o{}^*$ implies $f(\theta^*) = f(\theta)^*$: By (2) we have

$$\neg\, \mathrm{Card}(f(\theta)) \leq_c 2^{\mathrm{Card}(\theta)}$$

so that by 1.21 and the symmetry of $\leq_c$ and 2^n

$$\neg\, \mathrm{Card}(f(\theta)^*) \leq_c 2^{\mathrm{Card}(\theta^*)}$$

That is, $f(\theta^*) \leq f(\theta)^*$. To prove the opposite inequality for f, suppose $f(\theta^*) < f(\theta)^*$, by way of contradiction. Then for some $\delta < f(\theta)$ we have $f(\theta^*) = \delta^*$ (by 1.3). Moreover, by (2) we have

$$\neg\, \mathrm{Card}(\delta^*) \leq_c 2^{\mathrm{Card}(\theta^*)}$$

so that by the symmetry of $\leq_c$ and 2^n, and by 1.21

$$\neg\, \mathrm{Card}(\delta) \leq_c 2^{\mathrm{Card}(\theta)}$$

But this implies $f(\theta) \leq \delta$, which is a contradiction.

Finally, from the second half of (1) we obtain

$$\mathrm{Card}(\theta_o{}^*) \leq_c 2^{\mathrm{Card}(\theta_o{}^{**})}$$

and hence $\theta_o{}^* < f(\theta_o{}^{**})$. Thus the conditions of 4.19 are satisfied by f, and hence

$$\theta_o{}^* < t(f) = t(f)^* \qquad (3)$$

But then $\theta_o < t(f) = t(f)^*$ as well, so that if $m = \mathrm{Card}(t(f))$, we have $m \in NC$ and

$$Nc(USC^2(x_o)) \leq_c m = T(m).$$

Thus by the symmetry of $\leq_c$ relative to T

$$Nc(x_o) \leq_c m = T(m). \qquad (4)$$

That is, if y is any element of m we have $y \in WS$ (since $t(f) \in NO$), $Can(y)$ and $Nc(x_o) \leq_c Nc(y)$ completing the proof of 5.4.

As an immediate consequence of Theorem 5.4 it follows that if $x \in WS$ satisfies $Nc(USC(x)) <_c Nc(x)$ and x is a subset of no Cantorian, well-ordered set, then $\neg Nc(x) \leq_c Nc(SC(x))$. In particular this is true of NO. (Indeed, one even has $\neg Nc(NO) \leq_c Nc(USC^2(V))$, as in [R, p.476].)

5.5. Corollary: $\vdash \neg \mathrm{Nc}(\mathrm{NO}) \leq_c \mathrm{Nc}(\mathrm{SC}(\mathrm{NO}))$.

We end this section by showing that the statement $(\omega_{\Omega_o} \in \mathrm{NO})$ is provable in NF_1 and NF_2. That is, in each of these extensions of NF the set W of ω-numbers has cardinality at least $\mathrm{Nc}(\mathrm{USC}(\mathrm{NO}))$.

5.6. Theorem: (i) $\vdash_{\mathrm{NF}_1} \omega_{\Omega_o} \in \mathrm{NO}$

(ii) $\vdash_{\mathrm{NF}_2} \omega_{\Omega_o} \in \mathrm{NO}$.

Proof: (i) Suppose the contrary, and let θ_o be such that $\omega_{\theta_o} = \Omega_o$. Then for any $\theta \in \mathrm{NO}$, if $\omega_\theta \in \mathrm{NO}$ we have $\omega_{\theta*} < \Omega_o$ by 2.12, and hence $\theta* < \theta_o$. Thus in particular $\theta_o* < \theta_o$.

Now let $f = \{\theta \mid \theta < \theta_o\} \upharpoonright \Delta_W$, so that $\mathrm{Arg}(f) = \{\theta \mid \theta < \theta_o\}$ and $f(\theta) = \omega_\theta$ whenever $\theta < \theta_o$. Moreover, by 2.12 we see that $f* \subseteq f$. Thus to show that f satisfies the conditions of Theorem 4.22 it remains only to show $f(\theta*) \geq \theta_o$ for some $\theta \leq \theta_o$. In fact we prove

$$f(\theta_o*) \geq \theta_o. \tag{1}$$

If otherwise, then $f(\theta_o*) < \theta_o$, or what is the same thing, $(\omega_{\theta_o})* < \theta_o$. That is $\Omega_1 < \theta_o$, which implies $\omega_{\Omega_1} \in \mathrm{NO}$. Indeed, $\omega_{\Omega_1} < \Omega_o$ so that there is a θ_1 with $\omega_{\Omega_1} = \theta_1*$.

Thus $\theta_1 \in W$, and there is a δ such that $\omega_\delta = \theta_1$. However by 2.12 this implies $\delta^* = \Omega_1$, and hence $\delta = \Omega_o$. Since $\omega_\delta \in NO$, this contradicts the initial assumption and establishes (1).

Thus by Theorem 4.22 we have

$$\theta_o < s(f) = s(f)^* \tag{2}$$

and hence (in NF_1, by 3.30) $\theta_o = \theta_o{}^*$. But this is a contradiction, and hence $\omega_{\Omega_o} \in NO$.

(ii) The proof proceeds exactly as for (i), except that we note that f and θ_o are defined by constant, stratified terms. Hence upon proving (2) we have $\theta_o = \theta_o{}^*$ by an insistence of axiom scheme (i) of NF_2. This again is a contradiction, which proves $\omega_{\Omega_o} \in NO$ in NF_2 as well.

Section 6:

In this section we use a model-theoretic argument to show that if NF is consistent then the extension of NF obtained by adding the statement

$$F: \quad (\exists m)(\exists n)[m,n \in Nn \ \&\ m <_c n \ \&\ m \neq T(m) \ \&\ n = T(n)]$$

is also consistent. [In fact the same is true for any consistent extension of NF all of whose axioms are stratified sentences.] Thus since F implies the negation of the Counting Axiom (which is equivalent to the statement $(n)(n \in Nn \Longrightarrow n = T(n))$) in NF, we also see that the Counting Axiom is not a theorem of any consistent extension of NF by stratified sentences.

Let $L(\Theta)$ denote the first order language in which $\in$ and Θ are the only non-logical constants, (the first a binary predicate and the second a constant term). We define formulas and terms in the usual way (allowing class abstractions as terms); a formula or term is stratified if it is stratified in the usual sense, where in addition any specific occurence of Θ may be assigned any type. Thus $\Theta \in \Theta$ is stratified, while $x \in x$ is not.

By a structure M, we mean a relational structure $M = \langle A,R,e \rangle$ appropriate to the language $L(\Theta)$ (so that A is a set, R is a binary relation on elements of A and e is an element of A.) A 1-1 function g from A onto itself is an *automorphism* of M if

(i) for any $a,b \in A$: $\langle a,b\rangle \in R$ if and only if $\langle g(a),g(b)\rangle \in R$, and

(ii) $g(e) = e$.

Given any automorphism g of M we define the k^{th} iterate of g, $\underline{g^k}$, by the recursion formula:

$$g^{o}(a) = a \quad \text{for all} \quad a \in A$$
$$g^{k+1}(a) = g(g^{k}(a)) \quad \text{for all} \quad a \in A.$$

Let $M = \langle A,R,e\rangle$ be a structure with automorphism g. We define a second structure M^g by setting M^g equal to $\langle A,R^g,e\rangle$ where R^g is defined by the equivalence

$$\langle a,b\rangle \in R^g \iff [a,b \in A \;\&\; \langle g(a),b\rangle \in R].$$

Note that for any $a,b \in A$ we have

$$M^g \models (x_o \in x_1)[a,b] \iff M \models (x_o \in x_1)[g(a),b]$$

(This construction derives from a technique of Specker's used in [15].

6.1. <u>Lemma</u>: Let g be an automorphism of a structure M. Then g is also an automorphism of M^g.

<u>Proof</u>: Let $M = \langle A,R,e \rangle$ so that $M^g = \langle A,R^g,e \rangle$. Thus it is only necessary to show that g satisfies condition (i) above for R^g. But for any $a,b \in A$ we have

$$\begin{aligned} \langle a,b \rangle \in R^g &\iff \langle g(a),b \rangle \in R \\ &\iff \langle g(g(a)),g(b) \rangle \in R \\ &\iff \langle g(a),g(b) \rangle \in R^g. \end{aligned}$$

which completes the proof.

For structures $M = \langle A,R,e \rangle$, formulas $\emptyset$ of $L(\circledast)$ and infinite sequences $[a_o,a_1,\dots]$ of elements of A we define the satisfaction relation

$$M \models \emptyset[a_o,a_1,\dots]$$

in the usual way.

The next result relates satisfaction by M to satisfaction by M^g for formulas which are stratified. It is a consequence of this result that whenever a stratified formula $\emptyset$ is a sentence (ie. when $\emptyset$ has no free variables)

$$M \models \emptyset \iff M^g \models \emptyset$$

6.2. Lemma: Let M be a structure with automorphism g. Let $\emptyset(x_1,\ldots,x_n)$ be a stratified formula of $L(\circledast)$ with only $x_1,\ldots,x_n$ free, and suppose that we are given a stratification assignment under which $x_1,\ldots,x_n$ are assigned $i_1,\ldots,i_n$ respectively. Then for any $a_1,\ldots,a_n$ in A,

$$M \models \emptyset[a_1,\ldots,a_n] \Longleftrightarrow M^g \models \emptyset[g^{i_1}(a_1),\ldots,g^{i_n}(a_n)]$$

Proof: The proof proceeds by induction on the syntactic structure of $\emptyset$.

(i) If $\emptyset$ is the formula $(x_1 \in x_2)$, then $i_2 = i_1 + 1$. Then we have

$$\begin{aligned} M \models \emptyset[a_1,a_2] &\Longleftrightarrow M \models \emptyset[g^{i_2}(a_1),g^{i_2}(a_1)] \\ &\Longleftrightarrow M^g \models \emptyset[g^{i_1}(a_1),g^{i_2}(a_1)], \end{aligned}$$

which is the desired equivalence.

(ii) If $\emptyset$ is the formula $(x_1 = \circledast)$ then we have

$$\begin{aligned} M \models \emptyset[a_1] &\Longleftrightarrow a_1 = e \\ &\Longleftrightarrow g^{i_1}(a_1) = e \quad \text{(since } g(e) = e) \\ &\Longleftrightarrow M^g \models \emptyset[g^{i_1}(a_1)]. \end{aligned}$$

(The formulas $(x_1 = x_2)$ and $(\circledast = x_1)$ are handled similarly.)

(iii) If $\emptyset$ is $\neg\emptyset_1$, (or $(\emptyset_1 \vee \emptyset_2)$ or $(\emptyset_1 \& \emptyset_2)$) and the Lemma is true for $\emptyset_1$ (and $\emptyset_2$), then the statement for $\emptyset$ follows immediately using properties of the satisfaction relation (and the fact that any stratification assignment for $\neg\emptyset_1$ (or $(\emptyset_1 \vee \emptyset_2)$ or $(\emptyset_1 \& \emptyset_2)$) also stratifies $\emptyset_1$ (and $\emptyset_2$)).

(iv) Suppose $\emptyset$ is $(\exists x_o)\emptyset_1$, where the statement is true for $\emptyset_1$: Let the given stratification assignment assign i_o to x_o. Then:

$$M \models \emptyset[a_1,\ldots,a_n] \iff M \models \emptyset_1[a_o,a_1,\ldots,a_n] \quad \text{for some } a_o \text{ in } A$$

$$\iff M^g \models \emptyset_1[g^{i_o}(a_o),g^{i_1}(a_1),\ldots,g^{i_n}(a_n)] \quad \text{for some } a_o \text{ in } A$$

$$\iff M^g \models \emptyset_1[b_o,g^{i_1}(a_1),\ldots,g^{i_n}(a_n)] \quad \text{for some } b_o \text{ in } A$$

(since g^{i_o} is a 1-1 function from A onto itself)

$$\iff M^g \models \emptyset[g^{i_1}(a_1),\ldots,g^{i_n}(a_n)].$$

Since universal quantifiers can be treated as negations of existential quantifiers, this completes the proof of 6.2.

Remark: Lemma 6.2 is analogous to a result in [15] which relates models of NF to certain models of simple type theory. Recently Jensen [5] has applied a similar method in proving the consistency of a weakened version of NF.

We apply Lemma 6.2 by showing that for any consistent set S of stratified sentences which contains the axioms of NF, if $S \cup \{\neg F\}$ is consistent, then $S \cup \{\neg F\}$ has a model M with an automorphism g such that $M^g \models F$. We do this by making use of a theorem of Ehrenfeucht and Mostowski [2] (or more precisely, a varient of their Theorem 6.1 noted by Vaught in [19]) on the existence of models with automorphisms.

In the following, S will denote a set of stratified sentences of $L(\Theta)$ containing the axioms of NF, but no element of which contains any occurrence of the constant Θ. S_1 will denote the result of adding to S the sentences $\Theta = T(\Theta)$, $\Theta \in Nn$, and $\neg\Theta = \underline{n}$ (for each $\underline{n} = 0,1,2,...$ a numeral of NF). Evidently we have:

6.3. __Lemma__: Let S and S_1 be as above. Then S_1 is consistent, if S is.

Let $(x_o <_c x_1)$ be the formula which defines the cardinal ordering on Nn, and let $U(x_o)$ be the formula $(x_o <_c \Theta)$. Then for any structure $M = \langle A,R,e\rangle$ let U^M be the ordered structure $\langle A_1,R_1\rangle$ where:

$$A_1 = \{a \mid a \in A \;\&\; M \models U(x_o)[a]\}$$

$$R_1 = \{\langle a,b\rangle \mid a,b \in A_1 \;\&\; M \models (x_o <_c x_1)[a,b]\}.$$

6.4. __Lemma__: Let S_1 be as above. Then if M is a model for S_1, then U^M is an infinite (total, linear) ordering.

Proof: Since S_1 contains the axioms of NF, M satisfies the statement expressing the fact that $<_c$ is a total, linear ordering on Nn. Moreover, since S_1 contains the sentences $\Theta \in Nn$ and $\neg\Theta = \underline{n}$ for all $\underline{n} = 0,1,2\ldots, U^M$ is infinite, and is an initial segment of the relation corresponding to $<_c$ over M. Thus U^M is also a total, linear ordering.

Let $\mathbb{Z}$ denote the ordered structure whose domain is the set of (positive and negative) integers and whose ordering relation is the usual one. By 6.4 and the Theorem of [19] noted above, if S_1 is consistent then there is a model M of S_1 such that

(i) $\mathbb{Z}$ is a sub-ordering of U^M

(ii) any automorphism of $\mathbb{Z}$ extends to an automorphism of M.

Using this fact we can now prove:

6.5. Theorem: Let S be a set of stratified sentences of $L(\Theta)$ (none of which contains any occurrence of the constant Θ) which contains the axioms of NF. Then if S is consistent, so is $S \cup \{F\}$. Thus if S is consistent it does not have $\neg F$ as a theorem. In particular, $\neg F$ is not a theorem of NF (if NF is consistent).

Proof: Suppose the contrary, so that $\neg F$ is a theorem of S and S is consistent. Then $\neg F$ is also a theorem of S_1, which is consistent by 6.3. Thus there is a model M for $S_1 \cup \{\neg F\}$ such that M and $\mathbb{Z}$ satisfy conditions (i) and (ii) above. Let g be an automorphism of M which is an extension of a non-trivial automorphism of $\mathbb{Z}$. Then by 6.2, M^g is a model of S_1 since every sentence in S_1 is stratified. We will show that $M^g \models F$, contradicting the initial assumption.

First, since the sentences $(\Theta = T(\Theta))$ and $\Theta \in Nn$ are in S_1 we have

$$M^g \models (x_o \in Nn \;\&\; x_o = T(x_o))[e].$$

Furthermore, since $\mathbb{Z}$ is a subordering of U^M, and the formula $(x_o <_c \Theta)$ is stratified,

$$M^g \models (x_o <_c x_1)[k,e] \tag{2}$$

for any k in the domain of $\mathbb{Z}$. Thus by (1) and (2), and since $M \models F$ we have

$$M \models (x_o = T(x_1))[k,k]$$

for any k in the domain of $\mathbb{Z}$. Hence by 6.2 (and the fact that $(x_o = T(x_1))$ is stratified when x_o is assigned one higher type than x_1), if k is in the domain of $\mathbb{Z}$ we have

$$M^g \models (x_o = T(x_1))[g(k),k]. \tag{3}$$

But if k is in the domain of $\mathbb{Z}$ then $g(k) \neq k$, so that

(3) implies

$$M^{g} \models \neg(x_o = T(x_o))[k] \tag{4}$$

Thus, by (1), (2) and (4), we see $M^{g} \models F$. However this contradicts the initial assumption and completes the proof of 6.5.

Since $\neg F$ is a weak version of Rosser's Counting Axiom (which is equivalent to the sentence $(n)[n \in Nn \Longrightarrow n = T(n)]$ we have immediately:

6.6. <u>Theorem</u>: If NF is consistent, then Rosser's Counting Axiom is not a theorem of NF, nor of any consistent extension of NF by stratified axioms.

<u>Remark</u>: Theorem 6.6 shows that, as has been suspected, if NF is consistent then it does not provide for every schema of induction over Nn. Indeed, if $K(n)$ is the formula $(n = T(n))$ then the sentences $K(0)$ and $(n)[n \in Nn \Longrightarrow (K(n) \Longrightarrow K(n+1))]$ are provable in NF. However the sentence $(n)[n \in Nn \Longrightarrow K(n)]$ is equivalent to the Counting Axiom, and thus is not a theorem of NF. Of course this does not imply that the Counting Axiom is inconsistent with NF, and this question remains open.

6.7. <u>Theorem</u>: If NF is consistent, then the sentence $(n)[n \in Nn \Longrightarrow T(n) \leq_c n]$ is not a theorem of NF, nor of any consistent extension of NF by stratified axioms.

<u>Proof</u>: We show the sentence

$$F \Longrightarrow (\exists n)[n \in Nn \ \& \ n <_c T(n)]$$

is a theorem of NF. Suppose F holds, and let $m,n \in Nn$ satisfy $m <_c n \ \& \ n = T(n) \ \& \ m \neq T(m)$. Then there is a $k \in Nn$ with $m + k = n$. If $m <_c T(m)$ we are done, and if not, $T(m) <_c m$. In this case

$$(T(m) + k) <_c (m + k) = n$$

But $n = T(n) = T(m + k) = (T(m) + T(k))$, and hence

$$(T(m) + k) <_c (T(m) + T(k))$$

so that $k <_c T(k)$, completing the proof.

<u>Remark</u>: In [17, p.294] Quine states that the formula $Nc(x) \leq_c Nc(USC(x))$ is equivalent in NF to the formula $Nc(x) = Nc(USC(x))$. This is false, however, even for finite sets, as Theorem 6.7 shows.

<u>Remark</u>: Since the sentence $\neg F$ is evidently a theorem of NF_1, Theorem 6.5 shows that if NF is consistent, then NF_1 is a nontrivial extension of it. In fact since the

axioms of NF_2 are all stratified sentences we see that $\neg F$ is not a theorem of NF_2 either (unless this system is inconsistent).

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