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EXTENSIONAL QUOTIENTS OF STRUCTURES AND APPLICATIONS TO THE STUDY OF THE AXIOM OF EXTENSIONALITY

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Introduction :

Let M be a structure of the following kind : $M = \langle A,E \rangle$, where A is a set and E some relation on $A (E \subset A \times A)$. Our metatheory will be ZF (=Zermelo-Fraenkel set theory ; cf [1] Appendix A), though we will take later weaker systems.

Suppose $\sim$ is an equivalence relation on M : $\sim \subset A \times A$; $\forall a \in A$ $a \sim a$; $\forall a,b \in A$ $a \sim b \Rightarrow b \sim a$; $\forall a,b,c \in A$ $a \sim b \wedge b \sim c \Rightarrow a \sim c$.

Define the equivalence classes of $\sim$ by :

$[x]_\sim = \{y \in A \mid x \sim y\}$. Then the relation E on A induces in a natural way a relation E' on $A/\sim = \{[x]_\sim \mid x \in A\}$:

$$[x]_\sim \, E' \, [y]_\sim \text{ iff (by definition) } \exists x',y' \in A \;\; x' \sim x \wedge y' \sim y \wedge x' \, E \, y'$$

To simplify our notations, we will write "E" too for this relation E' on $A/\sim$.

By définition, $M/\sim$ will be the structure : $\langle A/\sim, E \rangle$.

We are interested now by untrivial equivalences $\sim$ on M, having the property : $M/\sim \models$ EXT, where EXT is the well-known axiom of extensionality :

$$\text{EXT} \equiv [\forall t(t \in x \Leftrightarrow t \in y)] \Rightarrow x = y.$$

In fact, such equivalences are known when M is a well-founded structure ($\forall B \subset A$: if B is nonempty then $\exists b \in B \; \forall b' \in B \, \neg \, b' E \, b$; "$\neg$" is the negation symbol). For such structures M, Mostowski defined a function f

inductively by :

$$f(a) = \{f(b) \mid b \mathrel{E} a\} \qquad (a \in A)$$

It is well-known that $B = \{f(a) \mid a \in A\}$ is a transitive set $(\forall x,y$ $(x \in y \in B \Rightarrow x \in B))$, and so $N = \langle B, \in \rangle$ is a model of EXT. Now, if we define $\sim_f$ by : $x \sim_f y \Leftrightarrow f(x) = f(y)$, it is easy to see that N is isomorphic to $M/\sim_f$.

So $M/\sim_f$ is a model of EXT. Now, the property "$f(a) = \{f(b) \mid bEa\}$" which defines completely f can be written : "$\{f(b) \mid f(b) \in f(a)\} = \{f(b) \mid bEa\}$" or, using the isomorphism $N \cong M/\sim_f$

$$\text{"}\{[x]_{\sim_f} \mid [x]_{\sim_f} \mathrel{E} [y]_{\sim_f}\} = \{[x]_{\sim_f} \mid xEy\}\text{"} \quad .$$

Thinking about M and $M/_{\sim_f}$ as models for set theory, this has the following sense : the "elements" of the equivalence class of a "set" y are exactly the equivalence classes of the "elements" of the "set"y. Equivalences having this property will be called "final". In a precise way :

Definition :

an equivalence $\sim$ on M is final

iff

$\forall y \in |M|$: $\{[x]_\sim \mid [x]_\sim E\, [y]_\sim\} = \{[x]_\sim \mid xEy\}$

(if $M = \langle A,E \rangle$, $|M|$ is the universe of M : $|M| = A$).

Moskowski's construction shows that for each well-founded structure M there exists an equivalence $\sim$ having the properties : 1) $\sim$ is final 2) $M/\sim \models$ EXT/

This leads us to the following definition :

Definition : an equivalence $\sim$ on M is a contraction

iff

$\sim$ is final and $M/\sim \ \models$ EXT.

The main result of chapter 1 is a generalization of Mostowski's theorem ("each well-founded structure has a contraction (in the sense just defined)") : we prove in fact that each structure has a contraction. This notion of contraction is useful to give simple proofs of results about the axiom of extensionality obtained by M. Boffa, D.Scott and R.O. Gandy (see chapters 5 and 6). Further, it gives new informations about EXT in set theory (see chapters 4 and 7). The important fact about contractions is that they change any structure M into a model $M/\sim$ of EXT, but in a way which preserves some important properties of M. This preservation is essentially due to the "finality" condition.

CHAPTER 1.

Before proving that each structure M admits a contraction, we need some elementary results and some new definitions :

Proposition 1.

Let $\sim$ be an equivalence on $M = \langle A,E \rangle$.

Then $\sim$ is final iff $\forall a,b,b' \in A\ (a\ E\ b \wedge b \sim b' \Rightarrow \exists a' E b' a' \sim a)$.

Proof :

1) Suppose $\sim$ final and $aEb \wedge b \sim b'$; aEb implies $[a]\ E\ [b]$, so $[a] \in \{[x] \mid [x]\ E\ [b]\}$; as $[b] = [b']$, $[a] \in \{[x] \mid [x]\ E\ [b']\} = \{[x] \mid xEb'\}$; so for some x we have : $x\ E\ b' \wedge [x] = [a]$; this

proofs : $\exists$ a' (a' E b' $\wedge$ a' $\sim$ a).

2) Suppose $\forall$ a,b,b' $\in$ A (a E b $\wedge$ b $\sim$ b' $\Rightarrow$ $\exists$ a' E b' a' $\sim$ a) ; if [x] E [y] , then $\exists$ x',y' (x'Ey' $\wedge$ x' $\sim$ x $\wedge$ y' $\sim$ y) ; so $\exists$ x''E y x'' $\sim$ x' ; this shows that [x] $\in$ {[z] | z E y} ; conversely, if [x] $\in$ {[z] | z E y}, then for some x' we have x' E y $\wedge$ x' $\sim$ x ; so [x] E [y] and [x] $\in$ {[z] | [z] E [y]}.

Definitions :

1) $F(M) = \{\sim | \sim$ is a final equivalence on $M\}$
2) $C(M) = \{\sim | \sim$ is a contraction on $M\}$
3) $x \sim_{EXT} y$ iff (by definition) $\forall t \in |M|$ $t\,Ex \Leftrightarrow tEy$
4) If $\sim_1$ is an equivalence on M and $\sim_2$ an equivalence on $M/\sim_1$, then $\sim_1/\sim_2$ is defined on M by : $x\ (\sim_1/\sim_2) y$ iff $[x]_{\sim_1} \sim_2 [y]_{\sim_1}$
5) "+" is defined by : $\sim^+$ is $\sim/\sim_{EXT}$ ($\sim$ on M ; $\sim_{EXT}$ on $M/\sim$)
6) "$\leq$" is defined on F(M) by :

$$\sim_1 \leq \sim_2 \text{ iff } \forall x,y \in |M| \ (x \sim_1 y \Rightarrow x \sim_2 y)$$

The following propositions are easy to prove :

Propositions :

2) $\leq$ is a partial ordering on F(M).
3) $x \sim^+ y \Leftrightarrow [(\forall tEx\ \exists t'Ey\ t\sim t') \wedge (\forall t'Ey\ \exists tEx\ t\sim t')]$
4) $\forall \sim_1,\sim_2 \in F(M)\ (\sim_1 \leq \sim_2 \Rightarrow \sim_1^+ \leq \sim_2^+)$
5) $\sim_1 \in F(M) \wedge \sim_2 \in F(M/\sim_1) \Rightarrow \sim_1/\sim_2 \in F(M)$
6) $\sim_1 \in F(M) \wedge \sim_2 \in C(M/\sim_1) \Rightarrow \sim_1/\sim_2 \in C(M)$
7) $\sim_1 \in F(M) \wedge \sim_2 \in F(M/\sim_1) \Rightarrow \sim_1 \leq \sim_1/\sim_2$

8) $\forall \sim, \sim' \in F(M) \; (\sim \leq \sim' \Leftrightarrow \exists \sim'' \in F(M/\sim) \; \sim' = \sim/\sim'')$

9) $\forall \sim \in F(M) \quad (\sim^+ \in F(M) \wedge \sim \leq \sim^+)$

10) $\forall \sim \in F(M) \quad (\sim \; \in C(M) \Leftrightarrow \sim = \sim^+)$

11) $(M/\sim_1)/_{\sim_2}$ is isomorphic to $M/_{(\sim_1/\sim_2)}$

We are now able to prove our main theorem :

Theorem 1 : Each structure has a contraction.

Proof : We give two proofs of this theorem.

Proof 1: this proof uses ordinals and cannot be reproduced in too weak systems ; here we give the proof in ZF (cf [6]). The kind of constructions done here are used too (independently) in [7] (p.17-18).

Take $M = \langle A,E \rangle$ and define $\sim_\alpha$ by :

$\sim_o$ is = (on M)

$\sim_{\alpha+1}$ is $(\sim_\alpha)^+$

$\sim_\gamma$ is $\bigcup\limits_{\alpha<\gamma} \sim_\alpha$ for γ limit ordinal

(in ZF we define a relation as being a set of ordered pairs).

Let us first prove by induction on α that all the $\sim_\alpha$ are final.

$\sim_o$ is trivially final ; if $\sim_\alpha \in F(M)$, then $(\sim_\alpha)^+ \in F(M)$; so $\sim_{\alpha+1} \in F(M)$; suppose $\forall \alpha < \gamma$ (γ limit ordinal) $\sim_\alpha \in F(M)$ and $xEy \wedge y \sim_\gamma y'$: then for some $\alpha < \gamma$ we have $xEy \wedge y \sim_\alpha y'$, and so $\exists \; x'Ey' \;\; x' \sim_\alpha x$; this proves : $\exists x' \; E \; y' x \sim_\gamma x'$.

In ZF, we can define the following set : $X = \{\sim_\alpha \in P\ (A\times A) \mid \alpha \text{ is an ordinal}\}$ ($PB = \{z \mid z \subset B\}$).

It is clear that $\alpha < \beta \Rightarrow \sim_\alpha \leq \sim_\beta$, so $\leq$ is a well-ordering on X. So for some ordinal δ we must have : $\sim_\delta = \sim_{\delta+1}$ (otherwise $<X,\leq>$ would have an order type bigger than any ordinal α) ; this $\sim_\delta$ is a contraction by proposition 10. So $M/\sim\delta \models$ EXT.

We will call "unextensionality degree" of a structure M the smallest ordinal δ such that $\sim_\delta = \sim_{\delta+1}$ (cf. [6]).

We can prove more now : suppose δ is the unextensionality degree of M. Then $\forall \sim \in C(M)\ \sim_\delta \leq \sim$. In other words, $\sim_\delta$ is the smallest contraction of M. Indeed it is easy to prove by induction on α that $\forall\alpha$ (ordinal) $\sim_\alpha \leq \sim$ (for any $\sim \in C(M)$).

Proof : $\sim_0 \leq \sim$ is trivial, if $\sim_\alpha \leq \sim$, then $(\sim_\alpha)^+ \leq \sim^+$ (by proposition 4), so $\sim_{\alpha+1} \leq \sim^+ = \sim$; suppose $\sim_\alpha \leq \sim$ for each $\alpha < \gamma$ (limit ordinal) : if $x \sim_\gamma y$, then $\exists \alpha < \gamma\ x \sim_\alpha y$, so $x \sim y$.

Notation : the least element of C(M) (whose existence was just proved) will be written $\sim_{Min(M)}$.

Proof 2 : this proof can be reproduced in very weak systems and will be used later (see Scott's result).

In fact we show here that there is a maximum element in $<F(M),\leq>$; as $\forall \sim \in F(M)\ \ \sim \leq \sim^+$, that maximum element is a contraction.

We need the following lemma :

<u>Lemma</u> : Suppose $B \subset F(M)$, B non empty. Then there is some element $\sim$ of $F(M)$ such that $\forall \sim' \in B \;\; \sim' \leq \sim$ (this property will be written simply : $B \leq \sim$).

<u>Proof of the lemma</u> :

If $B \subset F(M)$, and $z \subset A = |M|$, z will be said to be closed under B iff $\forall t \in z \; \forall t' \in A \; \forall \sim' \in B \quad (t \sim' t' \to t' \in z)$.

Define the equivalence relation $\sim$ by its equivalence classes :$[x]_\sim = \cap \{z | x \in z \wedge z$ is closed under $B\}$.

Then $\sim$ is defined precisely by : $x \sim y \leftrightarrow [x]_\sim = [y]_\sim$.

It is easy to see that $\sim' \leq \sim$ for each $\sim' \in B$. (this results form the fact that for each $\sim' \in B$: $[x]_{\sim'} \subset [x]_\sim$).

We have still to prove that $\sim$ is final. Suppose $\sim$ is not final : then for some $x,y,y' \in A$ we have $x \; E \; y \wedge y \sim y' \wedge \forall x' \; \neg(x \sim x' \wedge x' \; E \; y')$. Then the following subset D of $[y]_\sim$ is not empty :
$D = \{y'' | y'' \sim y \wedge \forall x'' \; \neg(x \sim x'' \wedge x'' \; E \; y'')\}$

Then there must be $y'' \in D$ and $y''' \notin D$ and $\sim' \in B$ such that $y'' \sim' y'''$, for otherwise $[y]_\sim \setminus D$ would be closed under B, contradicting the fact that $[y]_\sim$ is the smallest set containing y and closed under B.

So take $y'' \in D$, $y''' \notin D$, $\sim' \in B$ such that $y'' \sim' y'''$: from $y''' \notin D$, we deduce : $\exists\, x'''$ such that $x''' \sim x \wedge x''' \, E \, y'''$. As $y'' \sim' y'''$, we have $\exists\, x'' \, E \, y'' \; x'' \sim' x'''$. From $x'' \sim' x''' \wedge x''' \sim x \wedge \sim' \leq \sim$ we deduce : $x'' \sim x$.

So we proved : $\exists\, x'' \, E \, y'' \quad x'' \sim x$.
This contradicts the fact that $y'' \in D$.

Now the proof 2 of theorem 1 goes as follows : take $B = F(M)$ and define $\sim$ by : $[x]_\sim = \cap \{z \mid x \in z \wedge z$ is closed under $B\}$.

By the lemma, we have : $\forall \sim' \in F(M) \; \sim' \leq \sim$. This implies $\sim^+ \leq \sim$; so, as $\sim \leq \sim^+$ is always true, we have $\sim = \sim^+$ and $\sim$ is a contration. This contraction will be written $\sim_{Max(M)}$.

CHAPTER 2 : The structures $\langle F(M), \leq \rangle$ and $\langle C(M), \leq \rangle$

Theorem 2 : $\langle F(M), \leq \rangle$ and $\langle C(M), \leq \rangle$ are complete lattices.

Before giving the proof of this theorem, it will be useful to give some definitions :

Definitions : Let $\leq$ be a partial ordering on a set K.

1) if $B \subset K$, and $x \in K$, then $B \leq x$ means that $\forall y \in B \; y \leq x$

2) $\sup_K B = z$ iff z is the smallest element x of K having the property $B \leq x$.

3) $\inf_K B = z$ iff z is the greatest element x of K having the property $x \leq B$.

4) $\langle K, \leq \rangle$ is a complete lattice iff $\forall B \subset K$ (B non empty) $\exists z_1, z_2 \; (z_1 = \sup_K B \wedge z_2 = \inf_K B)$

Proof of the theorem 2 :

1) $\langle F(M), \leq \rangle$:

Take $B \subset F(M)$ (B not empty) and constrict $\sim_1$ by : $[x]_{\sim_1} = \cap \{z \mid x \in z \wedge z$ is closed under $B\}$.

We have proved (proof of the lemma of chapter 1) that $B \leq \sim_1$.

To prove that $\sim_1 = \sup_{F(M)} B$ it suffices to prove that if $\sim^* \in F(M) \wedge B \leq \sim^*$, $\sim_1 \leq \sim^*$. Indeed, suppose $B \leq \sim^*$. Then $[x]_{\sim^*}$ is closed under B. By définition of $[x]_{\sim_1}$, we have $[x]_{\sim^*} \supset [x]_{\sim_1}$. This implies $\sim_1 \leq \sim^*$.

Take $B \subset F(M)$ (B non empty).

Define $\underline{B} = \{\sim' \in F(M) \mid \sim' \leq B\}$ and $\sim_2 = \sup_{F(M)} \underline{B}$. Then $\sim_2 = \inf_{F(M)} B$.

Let us show first that $\sim_2 \leq B$: if $\sim \in B$, then $[x]_{\sim_2} = \cap \{z \mid x \in z \wedge z$ is closed under $\underline{B}\}$ and the fact that (z is closed under $\sim \rightarrow z$ is closed under $\underline{B}$) implies $[x]_{\sim_2} \subset [x]_{\sim}$. So we have $\sim_2 \leq \sim$.

Now as $\sim_2 = \sup_{F(M)} \underline{B} \wedge \sim_2 \in \underline{B}$, we have trivially : $\forall \sim (\sim \leq B \rightarrow \sim < \sim_2)$.

So $\sim_2 = \inf B$.

2) $\langle C(M), \leq \rangle$

By theorem 1 (chapter 1), C(M) is non empty. Let us prove first that C(M) has a least element (without using proof 1 of theorem 1 ; we want to be able to prove this result in very weak systems). By point 1 of this proof, we know the existence of $\sim^* = \inf_{F(M)} C(M)$. Let us show that $\sim^* \in C(M)$. We know that $\sim^* \leq (\sim^*)^+$. Suppose $\sim \in C(M)$, then, as $\sim^* \leq \sim$, we have $(\sim^*)^+ \leq \sim^+ = \sim$. This shows that $(\sim^*)^+ \leq C(M)$. As $\sim^*$ is $\inf_{F(M)} C(M)$, we have : $(\sim^*)^+ \leq \sim^*$, and so $\sim^* = (\sim^*)^+$.

Conclusion : $\sim^* \in C(M)$. This proofs (without using ordinals) that each structure has a minimum contraction.

Let us show now that $<C(M), \leq >$ is a complete lattice.

Take $B \subset C(M)$ (B non empty). Define $\sim_1 = \sup_{F(M)} B$; $\sim_1$ is an element of $F(M)$ but not necessarily of $C(M)$

Define $\sim_2$ = the minimum contraction of $M/\sim_1$, and $\sim^* = \sim_1/\sim_2$. Then $\sim^*$ is $\sup_{C(M)} B$.

Indeed : if $\sim \in B$, then $\sim \leq \sup_{F(M)} B = \sim_1 \leq \sim_1/\sim_2 = \sim^*$

Suppose $\sim'$ has the property $B \leq \sim'$. We have to show that $\sim^* \leq \sim'$.

As $\sim_1 = \sup_{F(M)} B$, we have $\sim_1 \leq \sim'$. So by proposition 8 (chapter 1), $\exists \sim'' \in F(M/\sim_1)$ $\sim' = \sim_1/\sim''$. This implies trivially $\sim_1/\sim_2 \leq \sim_1/\sim''$ and so : $\sim^* \leq \sim'$. So we showed that $\sim^*$ is $\sup_{C(M)} B$.

Take $\sim = \inf_{F(M)} B$. Then $\sim$ is $\inf_{C(M)} B$.

Indeed : the only thing to prove is that $\sim \in C(M)$.

We have trivially : $\sim \leq \sim^+$.

If $\sim' \in B$, then $\sim \leq \sim'$. So $\sim^+ \leq (\sim')^+ = \sim'$. So $\sim^+ \leq B$. As $\sim$ is $\inf_{F(M)}$, we deduce : $\sim^+ \leq \sim$.

Conclusion : $\sim = \sim^+$

<u>Remarks</u> : 1) The proof of theorem 2 shows for $B \subset C(M)$ (B non empty) :

$$\begin{cases} \sup_{F(M)} B \leq \sup_{C(M)} B \\ \inf_{F(M)} B = \inf_{C(M)} B \end{cases}$$

2) simple examples (with finite M) show that generally $<C(M), \leq >$ is not a boolean algebra.

3) if M is a well-founded structure, then $C(M)$ has only one element. The proof is "by induction" on E.

CHAPTER 3 : Properties of contractions

Definitions : If $M = \langle A,E\rangle$ $(E \subset A \times A)$ and $B \subset A$, then $\langle B,E\rangle$ will be the substructure of M obtained by restricting E to B.
$N = \langle B,E\rangle$ will be called an initial substructure of M iff $\forall x,y \in A\ (x\,Ey \wedge y \in B \Rightarrow x \in B)$. In that case M will be said to be an end extension of N (notation : $N \ll M$).

Property 1 : If $\sim_{Max(M)} = \sup_{F(M)} F(M) = \sup_{C(M)} C(M)$
Then $x \sim_{Max(M)} y \leftrightarrow \exists \sim \in F(M)\quad x \sim y$ (Proof : trivial)

Property 2. If $\sim_{Min(M)} = \inf_{F(M)} C(M) = \inf_{C(M)} C(M)$
Then $x \sim_{Min(M)} y \leftrightarrow \forall \sim \in C(M)\quad x \sim y$ (Proof trivial)

Property 3. Suppose $N \ll M$ and $\sim\restriction_N$ is the restriction of $\sim$ to N.
Then 1) $\forall \sim \in F(M)\quad \sim\restriction_N \in F(N)$
2) $\forall \sim \in F(N)\quad \exists \sim' \in F(M)\quad \sim'\restriction_N = \sim$

Proof : 1) Suppose $xEy \wedge y \sim y' \wedge x,y,y' \in |N|$.
Then $\exists x' \in |M|\ (x'Ey' \wedge x' \sim x)$
By $N \ll M$ we have $\exists x' \in |N|\ (x'Ey' \wedge x' \sim\restriction_N x)$.

2) Take $\sim \in F(N)$ and define $\sim'$ by :
$x \sim' y \leftrightarrow (x,y \in |N| \wedge x \sim y) \vee (x,y \notin |N| \wedge x \sim_{EXT} y)$
(Remember that $x \sim_{EXT} y \leftrightarrow \forall t \in |M|\ (t\,E\,x \leftrightarrow t\,E\,y)$).
Then $\sim' \in F(M) \wedge \sim = \sim'\restriction_N$

Property 4. Suppose $N \ll M$ and $\sim_N$ is $\sup_{C(N)} C(N)$ and $\sim_M$ is $\sup_{C(M)} C(M)$.

Then $(\sim_M)\restriction_N = \sim_N$.

Proof : 1) if $x,y \in |N|$ and $x \sim_N y$, then, by property 3, $\exists \sim' \in F(M)$ $\sim'\restriction_N = \sim_N$ and so : $x \sim' y$. By property 1, this implies $x \sim_M y$.

2) if $x,y \in |N|$ and $x \sim_M y$, then by property 3 we have $x \sim' y$ for $\sim' = (\sim_M)\restriction_N$. By property 1 : $x \sim_N y$

Property 5. $(\sim_M = \sup_{C(M)} C(M)$; $\sim_N = \sup_{C(N)} C(N))$

$$x \sim_M y \leftrightarrow \forall N \ll M \ (x,y \in |N| \rightarrow x \sim_N y)$$
$$\leftrightarrow \exists N \ll M \ (x,y \in |N| \wedge x \sim_N y)$$

Proof : 1) if $x \sim_M y$ and $x,y \in |N|$ with $N \ll M$, then by property 4 : $x \sim_N y$.

2) the implication $\forall N \ll M \ldots \rightarrow \exists N \ll M \ldots$ is trivial

3) if $\exists N \ll M (x,y \in |N| \wedge x \sim_N y)$, by property 4, we have $\sim_N = (\sim_M)\restriction_N$. So $x \sim_M y$.

Property 5 is very important for the following reason : to know whether $x \sim_M y$ is true or false, we do not have to look to the whole structure M but only to some $N \ll M$ containing x and y. This fact will be useful in chapter 5.

Property 6 : Suppose $B \subset A$ and $M = \langle A,E \rangle$. If $\langle B,E \rangle \models EXT$ and $\sim$ is the minimum contraction on M, then the map $f : \langle B,E \rangle \rightarrow M/\sim$ such that $f(b) = [b]_\sim$ is an embedding (= injective morphism).

Proof : Define $\sim_\alpha$ as in proof 1 of theorem 1 : $\sim_0$ is = ; $\sim_{\alpha+1} = (\sim_\alpha)^+$;
$\sim_\gamma = \bigcup_{\alpha<\gamma} \sim_\alpha$ for γ limit ordinal.

Then for some γ , $\sim_\gamma$ is the minimum contraction $\sim$ of M. Let us prove by induction on α that $\forall\alpha\ (x \sim_\alpha y \leftrightarrow x = y)$ (if $x,y \in B$)
For $\alpha = 0$ it is trivial ; suppose $\sim_\alpha$ is = : if $x \sim_{\alpha+1} y$, then $(\forall t\ Ex\ \exists\ t'\ Ey\ t \sim_\alpha t') \wedge (\forall t'\ E\ y\ \exists\ t\ E\ x\ \ t \sim_\alpha t')$, by proposition 3 of chapter 1. So we have $\forall t\ \ t\ E\ x \leftrightarrow t\ E\ y$(in M). This implies $\forall\ t \in B\ (tEx \leftrightarrow tEy)$, and by $<B,E> \models$ EXT, we have $x = y$. Suppose $\sim_\alpha$ is = for all $\alpha < \gamma$ (limit ordinal) ; if $x \sim_\gamma y$, then $\exists\ \alpha < \gamma\ x \sim_\alpha y$, so $x = y$

The function $f : <B,E> \to M/\sim$ such that $f(b) = [b]_\sim$ is injective :
$f(b) = f(b') \to [b]_\sim = [b']_\sim \to b \sim_\delta b' \to b = b'$.

It is an embedding : if x E y, then $[x]_\sim E[y]_\sim$;
if $[x]_\sim$ E $[y]_\sim$, then for some x' we have $x' \sim x \wedge x'\ E\ y$. But as $x,y,x' \in B$, this implies $x' = x$ and x E y.

The following property gives information about how to construct untrivial final equivalences :

Property 7 : Suppose σ is an automorphism of $N << M$ ($\leftrightarrow \sigma : N \to N$ is 1-1 and $xEy \leftrightarrow \sigma(x)\ E\ \sigma(y)$). Define $\sim_\sigma$ on M by : $x \sim_\sigma y \leftrightarrow$ $(x,y \in |N| \wedge \exists k$ (a natural number) such that $x = \sigma^k(y) \vee y = \sigma^k(x))$ $\vee\ (x,y \notin |N| \wedge x = y)$. Then $\sim_\sigma \in F(M)$.

Proof : elementary

<u>Property 8</u>. Call a structure M "uncontractable" iff $C(M) = \{=_M\}$ ($=_M$ is the equality restricted to M). Then if M is uncontractable and $N << M$, N has no (untrivial) automorphism.

<u>Proof</u> : Suppose $N << M$ and σ is an automorphism of M. By property 7, $\sim_\sigma$ is a final equivalence on M. Let $\sim_{Max(M)}$ be the maximum contraction on M. Then $\sim_\sigma \leq \sim_{Max}$. As M is uncontractable, we have : $(=_M) \leq (\sim_\sigma) \leq (\sim_{Max}) = (=_M)$. So $\sim_\sigma$ is $=_M$. If $y=\sigma(x)$, by definition of $\sim_\sigma$, we have $x \sim_\sigma y$, so $x = y$. This shows that σ is trivial on N.

CHAPTER 4 : Preservated formulas. More about EXT

Add to $\mathcal{L}_{ZF}$ (the language of ZF) the following symbol : $\sim^*$ and define the following formulas in this enriched language $(\mathcal{L}_{\sim^*})$:

Eq($\sim^*$) $\equiv \sim^*$ is an equivalence

$\equiv (\forall x\ x \sim^* x) \wedge (\forall x,y\ \ x \sim^* y \to y \sim^* x) \wedge \forall x,y,z\ (x \sim^* y \wedge y \sim^* z \to x \sim^* z)$

Fin($\sim^*$) $\equiv \sim^*$ is a final equivalence

$\equiv \mathrm{Eq}(\sim) \wedge \forall x,y,y'\ (x \in y \wedge y \sim y' \to \exists x' \in y'\ x' \sim^* x)$

Contr ($\sim^*$) $\equiv \sim^*$ is a contraction (of the universe)

$\equiv \mathrm{Fin}(\sim^*) \wedge [(\forall t \in x\ \exists t' \in y\ \ t \sim^* t') \wedge (\forall t' \in y\ \exists t \in x\ t \sim^* t')$

$\Rightarrow x \sim^* y]$

If $\varphi(X_1,X_2,\ldots,X_n)$ (sometimes written $\varphi(\vec{X})$) is a formula of $\mathcal{L}_{ZF}$, define $\varphi^*(\vec{X})$ (in the language $\mathcal{L}_{\sim^*}$) as being the result of replacing = by $\sim^*$ and $\in$ by $\in^*$ in $\varphi(\vec{X})$, where $\in^*$ is defined by : $x \in^* y$ iff $\exists x' \sim^* x\ \exists y' \sim^* y$ $x' \in y'$

<u>Definitions</u> : Let T be a theory in $\mathcal{L}_{ZF}$ (T is a set of closed formulas). Then :

1) $\varphi(X)$ is <u>T-preserved</u> (under contractions)

iff

$T + \mathrm{Contr}(\sim^*) \vdash \forall \vec{x}\ [\varphi(\vec{x}) \Rightarrow \varphi^*(\vec{x})]$

2) $\varphi(X)$ is <u>T-copreserved</u> (under contractions)

iff

$T + \mathrm{Contr}(\sim^*) \vdash \forall \vec{x}\ [\varphi^*(x) \to \exists \vec{y} \sim^* \vec{x}\ \varphi(\vec{y})]$

(where $\vec{y} \sim^* \vec{x}$ means $y_1 \sim^* x_1 \wedge y_2 \sim^* x_2 \wedge \ldots \wedge y_n \sim^* x_n$)

3) $\varphi(\vec{X})$ is <u>preserved</u> iff $\varphi(\vec{x})$ is $\emptyset$-preserved ($\emptyset$ being the empty set)

4) $\varphi(\vec{X})$ is <u>copreserved</u> iff $\varphi(\vec{x})$ is $\emptyset$ -copreserved.

Proposition 1 : Suppose T is a theory in $\mathcal{L}_{ZF}$ and σ is a sentence in $\mathcal{L}_{ZF}$

Then $T + Contr(\sim^*) \vdash \sigma \leftrightarrow T \vdash \sigma$

<u>Proof</u> : $T + Contr(\sim^*)$ is a theory in $\mathcal{L}_{\sim^*}$. The models for that language $\mathcal{L}_{\sim^*}$ are of the form $N = \langle A,E,\sim\rangle$ where E and $\sim$ are relations on A. Suppose $T + Contr(\sim^*) \vdash \sigma$ and $M = \langle A,E\rangle$ is a model of T (if T is inconsistent, the proof is trivial). Let $\sim$ be a contraction of M. Then $N = \langle A,E,\sim\rangle$ is a model of $T+Contr(\sim)$ if we interpret $\sim^*$ by $\sim$ So $N \models \sigma$. As σ does not contain the symbol $\sim^*$, this implies that $M \models \sigma$. So we prove that $\forall M\ (M \models T \Rightarrow M \models \sigma)$. This shows $T \vdash \sigma$.

Proposition 2. Let $\varphi(\vec{X})$ be a formula in $\mathcal{L}_{ZF}$.

Then $Eq(\sim^*) \vdash \forall \vec{X},\vec{Y}(\vec{X} \sim^* \vec{Y} \Rightarrow (\varphi^*(\vec{X}) \leftrightarrow \varphi^*(\vec{Y})))$.

<u>Proof :</u> By induction on the lenght of φ.

Proposition 3. Let $\varphi(X_1,X_2,\ldots,X_n)$ be a formula in $\mathcal{L}_{ZF}$. Let T be a consistant theory in $\mathcal{L}_{ZF}$.

Then φ is T-preserved iff $\forall M \models T\ \forall \sim \in C(M)\ \forall \vec{a} \in |M|$ $(M \models \varphi(a_1\ldots a_n) \Rightarrow M/\sim \models \varphi([a_1],\ldots,[a_n]))$.

<u>Proof</u> : 1) Suppose φ is T-preserved and $M \models \varphi(\vec{a})$. Let $\sim^*$ be a contraction on M. Then $T + Contr(\sim^*) \vdash \forall\vec{X}\ [\varphi(\vec{X}) \Rightarrow \varphi^*(\vec{X})]$. So $N = \langle A,E,\sim^*\rangle$ is a model of $\varphi(\vec{a})$ and $N \models \varphi^*(\vec{a})$. It is easy to see that $N \models \varphi^*(\vec{a})$ is equivalent to $M/\sim^* \models \varphi([a_1],\ldots[a_n])$. (by induction on φ)

2) Suppose $T + Contr(\sim^*) \nvdash \forall \vec{X}[\varphi(\vec{X}) \Rightarrow \varphi^*(\vec{X})]$.
Then for some model $N = \langle A,E,\sim^* \rangle \models T + Contr(\sim^*) + \varphi(\vec{a})$ we will have $N \models \neg\varphi^*(\vec{a})$ (for some $\vec{a} \in |M|$). But then we have $M = \langle A,E \rangle \models \varphi(\vec{a})$ and $M/\sim \models \neg\varphi([a_1],[a_2]\ldots[a_n])$.

The set of all T-preserved formulas (Pres(T)) and the set of all T-copreserved formulas (Copres (T)) are not known exactly. But we give here some simple properties of these sets, which will be useful in chapter 6.

Preservation properties : Let $\varphi,\Psi,\ldots$ be formulas in $\mathcal{L}_{ZF}$.

1) "φ" $\in$ Pres(T) $\Rightarrow$ "$\neg\varphi$" $\in$ Copres(T)

2) if "σ" is a sentence (=closed formula), then "σ" $\in$ Copres(T) $\Leftrightarrow$ "$\neg\sigma$" $\in$ Pres(T)

3) the atomic formulas "$X \in y$ ","$X \in X$", " $X = Y$","$X = X$" are $\emptyset$-preserved.

4) the atomic formulas "$X \in y$", $X = Y$" are $\emptyset$-copreserved

5) if "φ" , "Ψ" $\in$ Pres (T) then the following formulas are T-preserved :
"$\varphi \wedge \Psi$, "$\varphi \vee \Psi$","$\forall x\ \varphi$", " $\exists x\ \varphi$" "$\forall x \in y\ \varphi$", "$\exists x \in y\ \varphi$", $\forall \vec{x}\ [\theta(\vec{x}) \Rightarrow \varphi]$
where θ does contain no other free variables then $\vec{x}$ and is T-copreserved.

6) if "φ","Ψ" $\in$ Copres(T) then the following formulas are T-copreserved :
"$\varphi \vee \Psi$","$\varphi \wedge \Psi$" where "φ" and "Ψ" have no common free variable, "$\exists x\ \varphi$"

Using these properties, it is easy to prove results as : "X is empty" is preserved, "Y = a ∪ b" is preserved, "X is not empty" is preserved, "Y = {a,b}" is preserved,...

<u>Theorem 1 :</u> Suppose T is a theory in $\mathcal{L}_{ZF}$ and $\forall\sigma \in T$ σ is T-preserved. Take a sentence θ which is T-copreserved. Then $T + EXT \vdash \theta \Leftrightarrow T \vdash \theta$

<u>Proof</u> : Suppose $T + EXT \vdash \theta$; if M is a model of T such that $M \models \neg\theta$, then for any contraction $\sim$ on M : $M/\sim \models \neg\theta$ (as θ is T-copreserved, $\neg\theta$ is T-preserved). As all the axioms σ of T are T-preserved, we have : $M/\sim \models T + EXT$, implying : $M/\sim \models \theta$.

This contradicts $M/\sim \models \neg\theta$.

<u>Theorem 2 :</u> Suppose $\sim^*$ is a definable relation in T such that $T \vdash Contr(\sim^*)$; if $\forall\sigma \in T$, $T \vdash \sigma^*$ then T and T+ EXT are equiconsistent because each of them can be interpreted in the other.

<u>Proof</u> : We suppose that there is a formula $\theta(x,y)$ such that, if we write $x \sim^* y$ instead of $\theta(x,y)$ we have : $T \vdash Contr(\sim^*)$. Our interpretation of T + EXT into T is obtained by interpreting = by $\sim^*$ and $\in$ by $\in^*$. We have indeed : $T \vdash (T + EXT)^*$.

That there is an interpretation of T in T + EXT is trivial.

This theorem will be useful in chapter 6 to prove that Z (Zermelo's set theroy) and Z' = Z without EXT are equivalent (from the point of view of relative interpretability).

CHAPTER 5 : Amalgamation property for extensional structures

In [2], M. Boffa proved the following result :

Theorem : Suppose $M \ll M_1$; $M \ll M_2$; $M, M_1, M_2 \models EXT$.

Then there exists a structure N and embeddings $h_1 : M_1 \to N$, $h_2 : M_2 \to N$ such that :

1) $\forall x \in |M| \quad h_1(x) = h_2(x)$

2) $h_1(M_1) \ll N$

3) $h_2(M_2) \ll N$

4) $N \models EXT$

Proof by contractions :

Let us take the following notations :

$M_1 = \langle A_1, E_1\rangle$, $M_2 = \langle A_2, E_2\rangle$, $M = \langle A, E_1\rangle = \langle A, E_2\rangle$, $A = A_1 \cap A_2$, $E_1\restriction_A = E_2\restriction_A$

Contruct first $N' = \langle A', E'\rangle$, where :

$A' = A \cup [(A_1 \setminus A) \times \{1\}] \cup [(A_2 \setminus A) \times \{2\}]$ (*)

and E' is defined by :

if $X, Y \in A$; $XE'Y \Leftrightarrow XE_1Y \Leftrightarrow XE_2Y$

if $X, Y \in A_i \setminus A$: $\langle X, i\rangle E' \langle Y, i\rangle \Leftrightarrow XE_iY$ $(i = 1,2)$

if $X \in A, Y \in A_i \setminus A$: $XE'\langle Y, i\rangle \Leftrightarrow XE_iY$ $(i = 1,2)$

Clearly M_1 is isomorphic to $\bar{M}_1 = \langle A \cup [(A_1 \setminus A) \times \{1\}], E'\rangle$ by the isomorphism $g_1 : M_1 \to \bar{M}_1$ defined by :

$$g_1(x) = x \qquad \text{if } x \in A$$

$$g_1(x) = \langle x, 1 \rangle \qquad \text{if } x \in A_1 \setminus A.$$

in the same way, define $\bar{M}_2$ and g_2;

Remark that $\bar{M}_1 \ll N'$, $\bar{M}_2 \ll N'$ and $g_i(x) = x$ if $x \in A$ ($i = 1,2$).

Let $\sim$ be the minimum contraction on N'. The structure N we search is $N'/\sim$.

Indeed : $N \models EXT$ is trivial ; take $M'_1 = \langle \{[x]_\sim \mid x \in |\bar{M}_1|\}, E' \rangle$ and $M'_2 = \langle \{[x]_\sim \mid x \in |\bar{M}_2|\}, E' \rangle$. Define $h_i : M_i \to N$ by : $h_i(x) = [g_i(x)]_\sim$ ($i = 1,2$). Then $h_i(M_i) = M'_i \ll N$ and h_i is an embedding ($i = 1,2$) : this last fact results from property 6 (chapter 3) and the fact that g_i is an isomorphism ($i = 1,2$); that $h_1(x) = h_2(x)$ if $x \in |M| = A$ is trivial.

(**) We may suppose that A does not contain elements of the kind : $\langle x, 1 \rangle''$, $\langle x, 1 \rangle$; in fact A' is wanted to be the disjoint union of A, $A_1 \setminus A$ and $A_2 \setminus A$.

CHAPTER 6. Proofs by contractions of results of Scott and Gandy

1) Scott's result

In [3] Scott proved that the two versions of Zermelo's set theory Z are (cf. [1] appendix A) equivalent for relative interpretability. In fact, he proved somwhat more then this : the system $Z^{\neq}$ in which he gives an interpretation of Z is in fact weaker than simply Z without EXT ; it should be noted that our interpretation works too for Z and the system $Z^{\neq}$ defined by Scott : it is only to give a clear idea of our construction that we prefer here to work with Z and Z', as those systems are probably more familiar to the reader.

Before giving the proof, it is necessary to remark that there are some difficulties when one works in a theory which drops the axiom of extensionality. In such theories, a term as $\{X|\varphi(x)\}$ (where φ is a formula) does not represent a unique object, so its use is ambiguous. Therefore, we will take the following convention : we will only use such terms in formulas, and never alone as representing objects. For example, the formula $y = \{X|\varphi(x)\}$ has to be understood as meaning : $\forall t\ (t \in y \leftrightarrow \varphi(t))$; in the same way : $y = Px$ means $\forall t(t \in y \leftrightarrow t \subset x)$; $t \subset z$ means $\forall\, u \in t\ u \in z$; $y = Ux$ means $\forall t\ (t \in y \leftrightarrow \exists z \in x\ t \in z)$; and so on. Formulas as $Px \in y$ will be understood in the evident way : $\exists z\ (z = Px \wedge z \in y)$. With this convention, we can go on using terms to clarify the sense of our formulas.

<u>Theorem (D.Scott)</u> : Z and Z' = Z without EXT are equivalent for relative interpretability.

Proof by contractions :

First, define CL ("closure axiom") by :

CL $\equiv \forall x \ \exists t \ (x \in t \wedge t$ is a transitive set) where "t is a transitive set" means : $\forall a,b \ (a \in b \in t \rightarrow a \in t)$. It is easy to see that Z' is equivalent to Z" = Z' + CL . Indeed, if $H = \{x \mid \exists t \ (t \text{ is transitive} \wedge x \in t)\}$ then $<H,\in,=>$ is an interpretation of Z" in Z'.

We want to prove now that Z" is equivalent to Z. It suffices to apply theorem 2 (chapter 4) in the case T = Z". We have in fact to show two things :

1) it is possible to define in Z" a relation $\sim^{\star}$ such that Z" $\vdash$ Contr $(\sim^{\star})$.

2) for each axiom σ of Z" $\vdash$ Contr $\sigma^{\star}$.

Let us first give here the list of axioms of Z :

A × 1 : $\exists x \forall t \ t \notin x$ (empty set axiom)

A × 2 : $\exists x \ x = \{a,b\}$ (pairing axiom)

A × 3 : $\exists x \ x = \cup z$ (union axiom)

A × 4 : $\exists x \ x = P z$ (power set axiom)

A × 5 : axiom of infinity : there are many (non equivalent) versions of this axiom. In [3] Scott takes the following:

$$\exists x \ [\forall t \ (\forall t(\forall z \ z \notin t \rightarrow t \in x) \wedge \forall a \in x \quad a \in x]$$

Scott's proof (and ours too) works still if we take a more classical form, as for example :

$$2 \ x \ [\forall t(\forall z \ z \notin t \rightarrow t \in x) \wedge \forall a \in x \ \ \{a\} \in x] .$$

A × 6 : for any formula not containing x free, we have the axiom :

$$\exists x \quad x = \{t \in a \mid \varphi \}$$

A × 7 : EXT $\equiv \forall t(t \in X \leftrightarrow t \in Y) \rightarrow X = Y$

Point 1 : define (in Z" = Z without EXT + CL) : $\sim^*$ by : $x \sim^* y$ iff

$$\exists t\ (x \in t \wedge y \in t \wedge \ t \text{ is transitive} \wedge x \sim_t y).$$

$\sim_t$ here is the maximum contraction on the structure $<t,\in>$; to avoid problems it is necessary to work here with <u>partitions</u> of t instead of <u>equivalence relations</u> on t to define what we mean by a contraction. In this way, it is possible to rewrite in Z" the proofs of most of the results obtained in chapters 1,2 and 3, applied to structures of the kind : $<t,\in>$ with t a transitive set . Using this fact, it will be easy to prove that $\sim^*$ is a"contraction" of the "structure" $<V,\in>$ (where V is the universe); in a precise way : Z" $\vdash$ Contr ($\sim^*$). Indeed

1) $x \sim^* x$: by axiom CL , we have $\exists t\ (x \in t \wedge t$ is transitive). So clearly : $x \sim_t x$.

2) $x \sim^* y \to y \sim^* x$: trivial

3) $x \sim^* y \wedge y \sim^* z \to x \sim^* z$.
Suppose $x \sim^* y \wedge y \sim^* z$. Take t (transitive) such that $x \sim_t y$ and t' such that $y \sim_{t'} z$. Take some t" such that t" = t $\cup$ t'.
By property 5 (chapter 3) $x \sim_t y \leftrightarrow x \sim_{t''} y$ and $y \sim_{t'} z \leftrightarrow y \sim_{t''} z$. So from $x \sim_{t''} y \wedge y \sim_{t''} z$ we deduce : $x \sim_{t''} z$, and so $x \sim^* z$.

4) $\sim^*$ is final : suppose $x \in y \wedge y \sim^* y'$. Then for some transitive t : $x \in y \wedge y \sim_t y'$
So $\exists x' \in y'\ x' \sim_t x$; this shows $\exists x' \in y'\ x' \sim^* x$.

5) $\sim^*$ is a contraction :
suppose $(\forall z \in x\ \exists z' \in y\ z \sim^* z') \wedge (\forall z' \in y\ \exists z \in x\ z \sim^* z')$. Take a set a such that a = x $\cup$ y and a set b such that b = P a. By axiom CL , there is a transitive t such that b $\in$ t ; so $x \in t$ and $y \in t$. By property 5, we have then : $(\forall z \in x\ \exists z' \in y\ z \sim_t z') \wedge (\forall z' \in y\ \exists z \in x\ z \sim_t z)$.
As $\sim_t$ is a contraction, this implies $x \sim_t y$ and so $x \sim^* y$.

Point 2 : we have to show that each axiom σ of Z" : Z" $\vdash \sigma^*$. This is very easy to show for axioms 1,2,3, simply using the preservation properties (chapter 4).

Let us look now to the other axioms :

A × 4 : let us show that the formula "$Y = \mathcal{P} X$" is Z" - preserved, for our definable contraction $\sim^*$. By this, we mean :

$$Z'' \vdash [(y = \mathcal{P} x) \Rightarrow (y = \mathcal{P} x)^*].$$

Suppose $y = \mathcal{P} x$.

Then $(y = \mathcal{P} x)^*$ is $\forall t (t \in^* y \leftrightarrow \forall z \in^* t\ z \in^* x)$. If $t \in^* y$, then $\exists\ t' \in y\ t' \sim^* t$. As $y = \mathcal{P} x$, we have : $z \in^* t \rightarrow \exists\ z' \in t'\ z' \sim^* z$; as $t' \in y = \mathcal{P} x$, $z' \in x$; so $z \in^* x$.

Conversely, suppose $\forall z \in^* t\ z \in^* x$. If $z \in t$, then $z \in^* t$, and so $z \in^* x$. This implies $\exists\ z' \in x\ z' \sim^* z$. Take a t' such that $t' = \{z' | z' \in x \wedge \exists\ z \in t\ \ z \sim^* z'\}$. Such a t' exists by A × 6. As we clearly have $(\forall z' \in t'\ \exists\ z \in t\ z' \sim^* z)$ and $(\forall z \in t\ \exists\ z \in t\ z' \sim^* z)$ and $(\forall z \in t\ \exists\ z' \in t'\ z' \sim^* z)$, and as $\sim^*$ is a contraction, we may conclude : $t \sim^* t'$. Then, as $t' \subset x$, we have $t' \in y$

From $t \sim^* t' \wedge t' \in y$, we conclude : $t \in^* y$.

A × 5 : It is now easy to prove that

$\exists x\ [\forall t\ (t \text{ is not empty} \rightarrow t \in x) \wedge \forall\ a \in x\ \mathcal{P} a \in x]$ is Z"-preserved

(for $\sim^*$) Simply use the preserving properties and the following facts :

1) "t is not empty" is $\emptyset$-copreserved : indeed : if $\exists z\ z \in^* t$, then for some $z' \sim^* z\ z' \in t$ and so $\exists\ z'\ z' \in t$.

2) "$\mathcal{P} a \in x$" is preserved : it is in fact the formula $\exists z\ (z = \mathcal{P} x \wedge z \in x)$; $z = \mathcal{P} x$ is preserved, as we proved for A × 4.

In the same way it is easy to prove that other forms of A × 5 are Z''-preserved (for $\sim^*$).

A × 6 : take a formula $\varphi(t,\ldots)$

We have to prove in Z'' :

$$[\exists X \ \forall t(t \in X \leftrightarrow t \in a \wedge \varphi(t,\ldots))]^*$$

Take some X such that $X = \{t \in a | \varphi^*(t,\ldots)\}$

As $\sim^*$ is definable in Z'', $\varphi^*(t,\ldots)$ is a formula in $\mathcal{L}_{ZF}$ and by A × 6 such a set X has to exist in Z''.

If $t \in^* X$, then $\exists t' \in x\ t' \sim^* t$. So $t' \in a \wedge \varphi^*(t',\ldots)$. We conclude : $t \in^* a \wedge \varphi^*(t,\ldots)$, by proposition 2 (chapter 4).

Conversely, if $t \in^* a \wedge \varphi^*(t,\ldots)$, then $\exists t' \in a\ t' \sim^* t$; so $\varphi^*(t',\ldots)$ by proposition 2 (chapter 4) ; this implies $t' \in X$, and so $t \in^* X$.

A × 7 : $(EXT)^*$ results trivially form Contr $(\sim^*)$.

Axiom "CL" is Z''-preserved too : in fact, by the preserving properties (chapter 4) it is even $\emptyset$-preserved : $CL \equiv \forall x \ \exists t\ (x \in t \wedge \forall b \in t \ \ \forall a \in b \ \ a \in t)$.

2) Gandy's result :

Let ZF be the Zermelo-Fraenkel set theory ([1], Appendix A) whose axioms are : the axioms A × 1 to A × 7 of Z and the following axiom scheme :

A × 8 : for each formula $\varphi(X,Y,\vec{a})$ not containing C as a free variable.

$$\forall X \ \exists y\ (\varphi(X,Y,\vec{a}) \wedge \forall z(\varphi(X,z,\vec{a}) \rightarrow y = z))$$
$$\Rightarrow \forall b \ \exists C \qquad C = \{y | \ \exists x \in b \quad \varphi(x,y,\vec{a})\}$$

$ZF\lambda$ is the following version of ZF : first introduce a new symbol λ (abstract operator) which means in fact : $(\lambda t)\,\varphi(t,\ldots)$ is a set y such that $y = \{t|\varphi(t\ldots)\}$. This new symbol allows to form terms of the kind : $(\lambda t)\varphi(t\ldots)$ The formulas are built up using $\mathcal{L}_{ZF}$ and such terms. The new language is called $\mathcal{L}_{ZF\lambda}$. In a precise way, the axioms of $ZF\lambda$ are : $A \times 1$ to $A \times 5$ as in ZF ; the shemes $A \times 6$ and $A \times 8$ are genralized to $\mathcal{L}_{ZF\lambda}$; at last, there is an axiom scheme defining the behaviour of "λ" : $A \times 9$:
$(\exists x \;\; x = \{t \mid \Psi(t\ldots)\}) \rightarrow \forall t(t \in (\lambda t) \;\; \Psi(t\ldots) \leftrightarrow \Psi(t\ldots))$.

Remark : as EXT is not an axiom of $ZF\lambda$, the formula "$x = \{t|\Psi(t\ldots)\}$" has to be understood as being $\forall t\ (t \in x \leftrightarrow \Psi(t\ldots))$ (as for Scott's result).

Gandy's result [4] shows that ZF and $ZF\lambda$ are equivalent for relative interpretability. W e give now a proof by contractions :

Proof : As $ZF\lambda$ can be interpreted trivially in ZF (take λ defined by : $(\lambda t)\varphi = \{t|\varphi\}$; $\{t|\varphi\}$ is uniquely determined), it suffices to give interpretation of ZF in $ZF\lambda$. First define in $ZF\lambda$ what we mean by"chosen by λ" :

Definition : x is chosen by λ iff $x = (\lambda t)\ (t \neq t)\ \vee\ (\exists t\ t \in x \wedge x = (\lambda t)(t \in x))$

The set $(\lambda t)\ (t \neq t)$ will be "the" empty set ($\emptyset$). A transitive set x will be called "hereditarily chosen" iff (x is chosen by λ and $\forall t \in x \;\; t$ is chosen by λ).

Using these definitions, we can construct in $ZF\lambda$ ordinals having the usual properties :

Definition : α is an ordinal iff (1) α is a transitive set
(2) α is a hereditarily chosen
(3) $\in$ is a (strict) well-ordering on α

Using the operator λ and the ordinals so constructed, we can define : the pair ; the couple ; the power set ; relations ; functions; the usual sets R_α : $R_0 = \emptyset$, $R_{\alpha+1} = \mathcal{P} R_\alpha$, $R_\gamma = \bigcup_{\alpha<\gamma} R_\alpha$ (γ limit ordinal).

Our second step will be to show that ZFλ and ZFλ + $\forall x \; \exists \alpha$ (ordinal) $x \in R_\alpha$ (this axiom can be written : $V = \bigcup_\alpha R_\alpha$) are equivalent.

Indeed, ZFλ is trivially interpretable in ZFλ + $V = \bigcup_\alpha R_\alpha$. Conversely, take in ZF$\lambda$ the class $H = \{x| \; \exists\alpha$ (ordinal) $x \in R_\alpha\} = \bigcup_\alpha R_\alpha$; then $<H,\in,=>$ is an interpretation of ZFλ + $V = \bigcup_\alpha R_\alpha$.

Now it suffices to give an interpretation of ZF in ZFλ + $\forall x \; \exists\alpha \; x \in R_\alpha$. Our inter pretation will be defined as in part 1 of this chapter :

$x \sim^* y$ iff $\exists t \, (x \in t \wedge y \in t \wedge t$ is transitive $\wedge \, x \sim_t y)$

(where $\sim_t$ is the maximum contraction on $<t,\in>$).

The interpretation is obtained by replacing $\in$ by $\in^*$ $(x \in^* y \leftrightarrow \exists x' \sim^* x \, \exists y' \sim^* y x' \in y')$ and = by $\sim^*$. The proof of Scott's result shows that this gives an interpretation of Z. So we have just to verifiy that the axioms A × 8 are well interpreted. Suppose (in ZFλ + $V = \bigcup_\alpha R_\alpha$) that φ has the property : $\forall x \; \exists y [\varphi^*(x,y,\ldots) \wedge \forall y' (\varphi^*(x,y',\ldots) \Rightarrow y' \sim^* y)]$. We have to show :

$$\exists \cup \; \forall y \, [y \in^{*} \cup \leftrightarrow \exists \, x \in^{*} a \; \varphi^{*}(x,y)] .$$

The problem now is that if we take some x such that $x \in^{*} a$, there is not a unique y such that $\varphi^{*}(x,y,\ldots)$ but a class of such y (all equivalent for $\sim^{*}$). So, for each $x \in a$, take $\alpha_x = (\mu\alpha)(\exists \, y \in R_{\alpha} \, \varphi^{*}(x,y,\ldots))$.

[$\mu\alpha$ = the smallest ordinal α such that] .

Define : $A_x = (\lambda y)(y \in R_{\alpha_x} \wedge \varphi^{*}(x,y,\ldots))$. A_x is the <u>set</u> of all y satisfying $\varphi^{*}(x,y,\ldots)$ such that their rank is minimal.

Take $\cup_o = \bigcup_{x \in a} A_x$. Then $\cup_o$ is the set $\cup$ we search ;

1) if $y \in^{*} \cup_o$, then $\exists \, y' \in \cup_o \; y' \sim^{*} y$.

So $\exists \, x \in a \; \varphi^{*}(x,y',\ldots)$. By proposition 2 (chapter 4) we have : $\exists \, x \in a \; \varphi^{*}(x,y,\ldots)$. So $\exists x \in^{*} a \; \varphi^{*}(x,y,\ldots)$.

2) if $\exists x \in^{*} a \; \varphi^{*}(x,y,\ldots)$, then $\exists x' \in a \; (x' \sim^{*} x \wedge \varphi^{*}(x,y,\ldots))$ (by proposition 2 (chapter 4)). As we have $\varphi^{*}(x',y,\ldots)$ there must be some $y' \in R_{\alpha_{x'}}$ with $\varphi^{*}(x',y',\ldots)$ (by definition of α_x). So $y' \in A_{x'}$ and $y' \in \cup_o$. From $\varphi^{*}(x',y',\ldots) \wedge \varphi^{*}(x',y,\ldots)$ we deduce : $y' \sim^{*} y$. So $y \in^{*} \cup_o$.

This achieves our proof.

CHAPTER 7 : Application to NF.

The axioms of NF (="New Foundations" of Quine ; cf [5]) are :

1) EXT : $\forall t\ (t \in x \Leftrightarrow t \in y) \Rightarrow X = Y$

2) $\exists x\ \forall t\ (t \in x \Leftrightarrow \varphi)$ for each stratified formula φ not containing "x" free.

(Remember that a stratified formula is one which can be written in the language of the simple theory of types).

<u>Theorem 1</u> : Let the theory T be some extension of NF' = NF without EXT. Suppose there is a stratified formula $\Theta(x,y)$ with same type for "x" and "y" such that, if $x \sim^{*} y$ means $\Theta(x,y)$, we have : $T \vdash \mathrm{Contr}(\sim^{*})$. Then there is an interpretation of NF in T.

<u>Proof</u> : Interpret = by $\sim^{*}$ and $\in$ by $\in^{*}$ (defined by $x \in^{*} y \Leftrightarrow \exists x' \sim^{*} x\ \exists y' \sim^{*} y\ x' \in y'$). We have $T \vdash (\mathrm{EXT})^{*}$: this results from $T \vdash \mathrm{Contr}(\sim^{*})$ and $\vdash \mathrm{Contr}(\sim^{*}) \Rightarrow (\mathrm{EXT})^{*}$. Let φ be a stratified formula. Then φ^{*} too is a stratified formula (proof by induction on the lengh of φ). In T, take some x such that $\forall t\ (t \in x \Leftrightarrow \varphi^{*}(t,\ldots))$. Then we have : $\forall t (t \in^{*} x \Leftrightarrow \varphi^{*}(t,\ldots))$ (same proof as in chapter 6) and so :

$$T \vdash [\exists x\ \forall t\ (t \in x \Leftrightarrow \varphi)]^{*}.$$

Remark that Jensen's result [8] implies that if NF is consistent, the theory T of theorem 1 has to be a proper extension of NF, for if we had an interpretation of NF in NF', then the consistency of NF would be provable in NF. So if we want to construct a contraction of the universe in NF' (definable by a stratified formula $\Theta(x,y)$), it will be necessary to add some axioms to NF'.

First, let us look how to define final equivalences and contractions in NF'. As we avoid EXT, we will define contractions as being partitions.

In a precise way :

P is a partition of V (the universe)

iff

$$(\forall x \ \exists z \in P \ x \in z) \wedge (\forall z,z' \in P \ (\exists \ t \in z \ t \in z' \Rightarrow z \sim_{EXT} z'))$$

Define $\sim_p$ by : $x \sim_p y \Leftrightarrow \exists z \in P \ (x \in z \wedge y \in z)$.

A partition P is a contraction iff $\mathrm{Cont}(\sim_p)$. A partition P is final iff $\sim_p$ is final.. The formulas "P is a contraction" and "P is final" are not stratified. So $F = \{P \mid P$ is a final partition$\}$ is not a set but a class. Through F is a class, we can define "$\leq$" on F as in chapter 1 :

$P \leq P' \leftrightarrow \forall x,y(x \sim_p y \Rightarrow x \sim_{p'} y)$. The operation "+" can be defined too by :

$x(\sim_p)^+ \ y \leftrightarrow x \sim_{(p^+)} y \leftrightarrow (\forall t \in x \ \exists \ t' \in y \ t \sim_p t') \wedge (\forall t' \in \ y \ \exists t \in x \ t \sim_p \ t')$.

It is easy to verify that "+" has the properties described in chapter 1.

In fact $\langle F, \leq \rangle$ is an inductive ordering : by this we mean that

$\forall y \ [(y \subset F \wedge y$ is a chain for $\leq \wedge y$ is a set$) \Rightarrow \exists P \in F \ \forall P' \in y \ \ P' \leq P]$.

In a more precise way :

$\forall y \ [((\forall \ P \in y \ \ P$ is a final partition$) \wedge (\forall P_1,P_2 \in y \ \ P_1 \leq P_2 \vee P_2 \leq P_1)) \Rightarrow \exists \ P$ $(P$ is a final partition $\wedge \ \forall P' \in y \ P' \leq P)]$.

It is clear now that if $<F,\leq>$ admits a fixed point P for + , then P is a contraction.

Let σ be the following axiom :

$\sigma \equiv$ "$<F,\leq>$ admits a fixed point for +".

In fact σ is a kind of axiom of choice : it is similar to a consequence of Zorn's le lemma, saying that "each inductive ordering admits a maximal element" ; as + is increasing, each maximal element has to be a fixed point.

So we have :

Theorem 2 : There is a kind of axiom of choice σ such that NF and NF' + σ are equivalent for relative interpretability ($\sigma \equiv$ "$<F,\leq>$ has a fixed point for +")

Proof : 1) Remark that NF $\vdash \sigma$; indeed : $P = USC(V) = \{\{X\} \mid X \in V\}$ is a contraction in NF. So $<F,\leq>$ has a fixed point. So NF' + σ is trivially interpretable in NF.

2) In NF' + σ, take some P such that P is a fixed point in $<F,\leq>$ for +. Then $\sim_P$ is a definable contraction :

$x \sim_P y \leftrightarrow \Theta(X,Y) \equiv \exists z(x \in z \land y \in \land z \in P)$ and Θ is stratified. So by theorem 1, NF can be interpreted in NF' + σ.

This theorem shows, as in the case of Gandy's result, that there is some connexion between EXT and some forms of choice ; in Gandy's result, the choice is done by the abstract operator λ who picks exactly one element in each class $[X]_{\sim EXT}$.

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