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THE BURALI-FORTI PARADOX

BARKLEY ROSSER

In the system presented by Quine in his book *Mathematical logic*,[1] one can derive the Burali-Forti Paradox. It is the purpose of this paper to present the details of this derivation.[2] For the derivation, only familiarity with Quine's book is assumed.

The present derivation is based on the derivations given by Hobson[3] and by Whitehead and Russell.[4]

In both these sources, the primary interest is the theory of ordinals, and the Burali-Forti Paradox is of interest only as something to be avoided. In the present paper the primary interest is the Burali-Forti Paradox, so that much of the theory of ordinals is absent from this paper and only those details remain which are relevant to the derivation of the paradox. We now give a brief verbal summary of the argument. The references D806, etc., refer to the formal definitions and theorems below.

For a serial relation we take the relation "$\leqq$" between real numbers as a model. PM takes "$<$" as a model, but in modern treatments, such as the theory of lattices, "$\leqq$" is preferred. A serial relation (D806) must have properties corresponding to the familiar properties:

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[1] W. V. Quine, ***Mathematical logic,*** New York 1940.

[2] It has come to my attention that there is a question of priority connected with the discovery that Quine's system admits the Burali-Forti Paradox. Quine tells me that a former student of his, Mr. Roger C. Lyndon, while studying the theory of ordinals in Quine's system, came upon the Burali-Forti Paradox. This happened in the latter half of October. Mr. Lyndon's first reaction to his discovery was that it must be the result of an error on his part. After an unsuccessful effort to find such an error, Mr. Lyndon sent his proof to Quine in December. Quine describes the proof as painstaking, detailed, and correct. Hence Mr. Lyndon certainly deserves credit for independent discovery of the paradox.

In my case, the circumstances are as follows. Toward the end of September, I wrote Quine to the effect that I had been unable to convince myself that his system did not admit the Burali-Forti Paradox, and suggested that he look into the matter. Somewhat later, Quine wrote back that he was busy, and requested me to investigate carefully. Still later, I sent Quine an earlier draft of the present paper. I happen to have kept his reply, which was fairly prompt, and was dated October 24, and stated that my manuscript undoubtedly established the presence of the paradox.

Obviously, we must fix the date of my discovery of the paradox as being the date on which I prepared that manuscript, rather than the earlier date on which I suspected the presence of the paradox.

On the basis of the above data, it would seem equitable to say that my discovery of the paradox and Mr. Lyndon's discovery were simultaneous.

[3] E. W. Hobson, ***The theory of functions of a real variable,*** first edition, Cambridge, England, 1907, see Chapter III; second edition, 1921, and third edition, 1927, see Chapter IV of the first volume.

[4] A. N. Whitehead and Bertrand Russell, ***Principia mathematica,*** volumes 2 and 3, see *150–*152, *154, *155, *160, *161, *180, *181, *200–*202, *204–*208, *210–*214, *250–*256.

1. If $a \leqq b$ and $b \leqq c$, then $a \leqq c$.
2. If $a \leqq b$ and $b \leqq a$, then $a = b$.
3. Either $a \leqq b$ or $b \leqq a$.
4. For any a, $a \leqq a$.

We express these by saying that a serial relation is transitive (D802), antisymmetric (D803), connected (D804), and reflexive (D805). A series is well-ordered (D808) if every non-empty set of terms of the series has a minimum term (D807). Two series P and Q are ordinally similar under the correspondence R (D811) if R is a one-to-one correspondence (D810) of the series which preserves order. A segment of a well-ordered series (D813) consists of all terms which precede a given term. A basic result is that no segment of a well-ordered series can be ordinally similar to the whole series (†812). Every well-ordered series determines an ordinal number (D815 and D816). A shorter well-ordered series determines a smaller ordinal (D817 and D818). One might expect us to say that R is shorter than S if R is ordinally similar to a segment of S. However, for ease in proving theorems, it is more convenient to say that R is as short as S (D814) if every segment of R is ordinally similar to a segment of S. The equivalence of the two definitions is a consequence of †822. A key theorem is that the series of ordinals is a well-ordered series under the relation "$\leqq$" (†835).

All the material up to and including †835 belongs to the standard theory of ordinals, and should be valid in any system which makes pretensions of being adequate for mathematics. It is from here on that the theorems become questionable. So it is from here on that the derivation of the Burali-Forti paradox should break down for a system with the proper inhibitions.

Now consider the series of all ordinals less than the ordinal α. This is well-ordered (†835 and †809), hence has an ordinal number. By †841, that ordinal number is α itself. We are now ready for the paradox. By †835, the series of all ordinals is well-ordered, so it has an ordinal number, N. By †841, the series of all ordinals less than N also has the number N. So the series of ordinals less than N is ordinally similar to the series of all ordinals. That is, we have a segment of a well-ordered series ordinally similar to the whole series, which contradicts †812.

For the so-called systems of set theory (See Bernays, Fraenkel, Neumann, Zermelo in the Bibliography to Quine's *Mathematical logic*), †842 fails to hold. For PM, †839 holds only if one attaches different type subscripts to the NO's occurring therein. This means that the two supposedly contradictory results deduced in †844 would not really be contradictory because of the different way in which type subscripts would have to be attached in them. See PM, vol. 3, p. 80, on this point. In systems with a simple theory of types a similar restriction holds.

In *New foundations for mathematical logic*[5] and in *On the theory of types*,[6]

[5] W. V. Quine, *New foundations for mathematical logic*, ***The American mathematical monthly***, vol. 44 (1937), pp. 70–80. We shall refer to this paper as Quine I.

[6] W. V. Quine, *On the theory of types*, this Journal, vol. 3 (1938), pp. 125–139. We shall refer to this paper as Quine II.

Quine proposes a number of ways of weakening the theory of types. It will be instructive to consider the status of the Burali-Forti Paradox under these different proposals. For definiteness, let us take our formulas in the symbolism of Quine I. A formula is stratified if one can attach a subscript to each occurrence of a variable so that for each occurrence of $x \in y$, the subscript attached to y is exactly one more than the subscript attached to x, and provided certain restrictions are fulfilled. Quine sets down the two restrictions:

1. In every part $(x)\phi$, all occurrences of x shall have the same subscript attached.

2. For each free variable, all of its free occurrences must have the same subscript attached.

On p. 78 of Quine I and in Section 4 of Quine II, it is proposed that we use only stratified formulas which satisfy both restrictions. Then the formula of †841 is not stratified, and the Burali-Forti Paradox fails. In Section 5 of Quine II, it is proposed that we use only stratified formulas which satisfy the first restriction, and we no longer ask that the second restriction be satisfied. If we take †841 in the form

$$\vdash \alpha \in \mathrm{NO} \,.\supset.\, \mathrm{Nr}(\mathrm{seg}_\alpha \leqq) = \alpha,$$

then it is stratified. However the A used in the proof of †836, the A used in the proof of †841, and the Q used in the alternative proof of †841 are unstratified, and the proofs of †841 fail. Finally, on p. 79 of Quine I, Quine suggests using stratification with both restrictions, but only insisting on its use in R3 (see Quine I, p. 77). This amounts to using stratification with only the first restriction and only insisting on its use in R3. Hence this is the least restriction of all. Nevertheless the A used in the proof of †841 and the Q used in the alternative proof of †841 are unstratified, and the proofs of †841 still fail.

Quine has let me see the manuscript of a paper in which he proposes to remove the Burali-Forti Paradox from his *Mathematical logic* by replacing the questionable axiom *200 by three weaker axioms:[7]

(i) (x) $\quad \iota x \in \mathrm{V}.$

(ii) $(x)(y)$ $\quad x,y \in \mathrm{V} \,.\supset.\, \bar{x} \cap \bar{y} \in \mathrm{V}.$

(iii) (z) $\quad z \in \mathrm{V} \,.\supset.\, \hat{x}(\exists y)(y \in x \,.\, x \cap \overline{\iota y} \in z) \in \mathrm{V}.$

Under this proposal, all theorems of *Mathematical logic* remain provable, but †837 to †844 below will fail. However †801 to †836 below will all go through, and hence these thirty-six theorems may be considered as an introduction to the theory of ordinal numbers for *Mathematical logic*. Therefore the theorems and definitions have been numbered as though they constituted an eighth chapter of the (revised) *Mathematical logic*. The notation has not been completely subordinated to that of *Mathematical logic*, but uses capital italic letters for relations (except A and B in the proofs of †835, †836, and †841), small italic letters for individuals, and small Greek letters for classes and ordinals (ordinals being

[7] This paper has since appeared, in this JOURNAL, vol. 6 (1941), pp. 135–149. *Editor.*

classes of relations). Other minor variations will be observed, but they should cause no confusion.

I wish to thank Prof. W. V. Quine for making a critical reading of the manuscript of this paper, and for the suggestions which he has made in connection with it.

Definitions.

D801. $\mathrm{C}(R)$ *for* $(R``\mathrm{V}) \cup (\breve{R}``\mathrm{V})$.

D802. $\mathrm{trans}(R)$ *for* $(x,y,z) \,.\, R(x,y)R(y,z) \supset R(x,z)$.

D803. $\mathrm{antisym}(R)$ *for* $(x,y) \,.\, R(x,y)R(y,x) \supset x = y$.

D804. $\mathrm{connex}(R)$ *for* $(x,y) : x,y \in \mathrm{C}(R) \,.\!\supset\!.\, R(x,y) \vee R(y,x)$.

D805. $\mathrm{ref}(R)$ *for* $(x) \,.\, x \in \mathrm{C}(R) \supset R(x,x)$.

D806. $\mathrm{Ser}(R)$ *for* $\mathrm{trans}(R) \,.\, \mathrm{antisym}(R) \,.\, \mathrm{connex}(R) \,.\, \mathrm{ref}(R) \,.\, R \subset \dot{\mathrm{V}}$.

D807. $y \min_R \alpha$ *for* $y \in \alpha \cap \mathrm{C}(R) : (z) \,.\, z \in \alpha \cap \mathrm{C}(R) \supset R(y,z)$.

D808. $\mathrm{Bord}(R)$ *for* $(\alpha,x) . x \in \alpha \cap \mathrm{C}(R) \supset (Ey)\, y \min_R \alpha$.

D809. $\Omega(R)$ *for* $\mathrm{Ser}(R) \,.\, \mathrm{Bord}(R)$.

D810. $\text{1-1}(R)$ *for* $(x,y,z) \,.\, R(x,z)R(y,z) \supset x = y : (x,y,z) \,.\, R(x,y)R(x,z) \supset y = z : R \subset \dot{\mathrm{V}}$.

D811. $P \operatorname{smor}_R Q$ *for* $\text{1-1}(R) \,.\, P = R|Q|\breve{R} \,.\, Q = \breve{R}|P|R$.

D812. $P \operatorname{smor} Q$ *for* $(ER)\, P \operatorname{smor}_R Q$.

D813. $\mathrm{seg}_x R$ *for* $\hat{y}\hat{z}(R(y,z) \,.\, R(z,x) \,.\, z \neq x)$.

D814. $\mathrm{LE}(R, S)$ *for* $(x) : x \in \mathrm{C}(R) \,.\!\supset\!.\, (Ey) \,.\, y \in \mathrm{C}(S) \,.\, (\mathrm{seg}_x R) \operatorname{smor} (\mathrm{seg}_y S)$.

D815. $\mathrm{Nr}(P)$ *for* $\hat{R}(P \operatorname{smor} R)$.

D816. NO *for* $\hat{\gamma}(EP) \,.\, P \in \mathrm{V} \,.\, \Omega(P) \,.\, \gamma = \mathrm{Nr}(P)$.

D817. $\leqq$ *for* $\hat{\alpha}\hat{\gamma}(ER,S) \,.\, \alpha,\gamma \in \mathrm{NO} \,.\, \alpha = \mathrm{Nr}(R) \,.\, \gamma = \mathrm{Nr}(S) \,.\, \mathrm{LE}(R,S)$.

D818. $\alpha \leqq \gamma$ *for* $\leqq(\alpha,\gamma)$.

Theorems.

†801. $\vdash (R,x,y) \,.\, R(x,y) \supset x,y \in \mathrm{C}(R)$.

Obvious.

†802. $\vdash (R,x,y,\alpha) \,.\, \mathrm{antisym}(R) \,.\, x \min_R \alpha \,.\, y \min_R \alpha \,.\!\supset\!.\, x = y$.

Obvious.

†803. $\vdash (R,x,y,z) \,.\, \mathrm{antisym}(R) \,.\, R(x,y) \,.\, R(y,z) \,.\, y \neq z \,.\!\supset\!.\, x \neq z$.

Proof. Since $\vdash R(x,y) \,.\, x = z \,.\!\supset\!.\, R(z,y)$, we have from D803 that $\vdash \mathrm{antisym}(R) \,.\, R(x,y) \,.\, R(y,z) \,.\, x = z \,.\!\supset\!.\, y = z$.

†804. $\vdash (R,S) : \Omega(R) \,.\, \mathrm{Ser}(S) \,.\, S \subset R \,.\!\supset\!.\, \Omega(S)$.

Proof. Assume the hypothesis. Clearly it suffices to prove $\mathrm{Bord}(S)$. So assume $x \in \alpha \cap \mathrm{C}(S)$. Then $x \in (\alpha \cap \mathrm{C}(S)) \cap \mathrm{C}(R)$. So by $\mathrm{Bord}(R)$, there is a y such that

$$y \min_R (\alpha \cap \mathrm{C}(S)). \tag{1}$$

We wish to show that $y \min_S \alpha$. We have from (1) that $y \in \alpha \cap \mathrm{C}(S)$. Let $z \in \alpha \cap \mathrm{C}(S)$. Then $z \in (\alpha \cap \mathrm{C}(S)) \cap \mathrm{C}(R)$. So by (1),

$$R(y,z). \tag{2}$$

Since connex(S) and $y,z \epsilon \mathrm{C}(S)$, we have $S(y,z) \vee S(z,y)$. If $S(z,y)$, then $R(z,y)$, whence by (2) and antisym(R), $y = z$, whence $S(y,z)$. So in any case $S(y,z)$. So $y \min_S \alpha$.

†805. $\vdash (P) . P \subset \dot{\mathrm{V}} \supset P \operatorname{smor}_I P$.

Obvious by †557, †559, and †560.

†806. $\vdash(P,Q,R,S) : P \operatorname{smor}_R Q . S = \breve{R} .\supset. Q \operatorname{smor}_S P$.

Obvious by †461 and †491.

†807. $\vdash(P,Q,R,S,T,U) : P \operatorname{smor}_S Q . Q \operatorname{smor}_T R . U = S|T .\supset. P \operatorname{smor}_U R$.

Obvious by †491 and †494.

†808. $\vdash (P,Q) : \Omega(P) . P \operatorname{smor} Q .\supset. \Omega(Q)$.

Proof. Assume $\Omega(P)$ and $P \operatorname{smor}_R Q$. Let $Q(x,y)Q(y,z)$. Then $R(R\text{‘}x,x) . P(R\text{‘}x,R\text{‘}y) . R(R\text{‘}y,y)$ and $R(R\text{‘}y,y) . P(R\text{‘}y,R\text{‘}z) . R(R\text{‘}z,z)$ by D811. So since trans(P), $R(R\text{‘}x,x)P(R\text{‘}x,R\text{‘}z)R(R\text{‘}z,z)$. That is, $Q(x,z)$. So trans(Q). The proofs of antisym(Q), connex(Q), and ref(Q) are similar. Also $Q \subset \dot{\mathrm{V}}$ follows from $Q = \breve{R}|P|R$. So

$$(1) \qquad \mathrm{Ser}(Q).$$

Now let $x \epsilon \alpha \cap \mathrm{C}(Q)$. Then $R\text{‘}x \epsilon (R\text{“}\alpha) \cap \mathrm{C}(P)$. So there is a y such that

$$(2) \qquad y \min_P R\text{“}\alpha.$$

Since $y \epsilon \mathrm{C}(P)$ by (2) and since $P = R|Q|\breve{R}$, we get $y \epsilon R\text{“}\mathrm{V}$. So let $R(y,z)$. Then $z \epsilon \mathrm{C}(Q)$. Also, since $y \epsilon R\text{“}\alpha$ and 1-1(R), $z \epsilon \alpha$. Suppose further that $w \epsilon \alpha \cap \mathrm{C}(Q)$. Then $R\text{‘}w \epsilon (R\text{“}\alpha) \cap \mathrm{C}(P)$. So by (2), $P(y,R\text{‘}w)$. So $\breve{R}(z,y) . P(y,R\text{‘}w) . R(R\text{‘}w,w)$. So $Q(z,w)$, since $Q = \breve{R}|P|R$. So $z \min_Q \alpha$. So Bord(Q).

†809. $\vdash (R,S,x) :. \Omega(R) :\supset: S = \operatorname{seg}_x R .\equiv. S \subset R . \Omega(S) . \mathrm{C}(S) = \hat{y}(R(y,x) . y \neq x)$.

Proof. Assume $\Omega(R)$ and $S = \operatorname{seg}_x R$. By D813:

$$(1) \qquad S \subset \dot{\mathrm{V}}.$$

$$(2) \qquad S \subset R.$$

Suppose $S(u,v)S(v,w)$. Then $R(u,v) . R(v,x) . v \neq x . R(v,w) . R(w,x) . w \neq x$. So, since trans($R$), $R(u,w) . R(w,x) . w \neq x$. So $S(u,w)$. So

$$(3) \qquad \mathrm{trans}(S).$$

Suppose $S(u,v)S(v,u)$. Then $R(u,v)R(v,u)$ by (2). So $u = v$. So

$$(4) \qquad \mathrm{antisym}(S).$$

Suppose $S(u,v)$. Then $R(u,v) . R(v,x) . v \neq x$. Then by †803 and Ser($R$), we have $R(u,u) . R(u,x) . u \neq x$ and $R(v,v) . R(v,x) . v \neq x$. So $S(u,u)$ and $S(v,v)$. So

$$(5) \qquad S(u,v) \supset S(u,u)S(v,v).$$

From this one easily gets

$$(6) \qquad \mathrm{ref}(Q).$$

Suppose $u,v \in \mathrm{C}(S)$. Then by (6), $S(u,u)$ and $S(v,v)$. So $R(u,u) \centerdot R(u,x) \centerdot u \neq x$ and $R(v,v) \centerdot R(v,x) \centerdot v \neq x$. Since $\mathrm{Ser}(R)$, we have $R(u,v) \vee R(v,u)$.

Case 1. $R(u,v)$. Then $R(u,v) \centerdot R(v,x) \centerdot v \neq x$. So $S(u,v)$.

Case 2. $R(v,u)$. Then $R(v,u) \centerdot R(u,x) \centerdot u \neq x$. So $S(v,u)$.

So $S(u,v) \vee S(v,u)$. So

$$\text{(7)} \qquad \mathrm{connex}(S).$$

By (3), (4), (7), (6), and (1), $\mathrm{Ser}(S)$. Using this, (2), and †804, we get

$$\text{(8)} \qquad \Omega(S).$$

Now let $y \in \hat{y}(R(y,x) \centerdot y \neq x)$. Then $R(y,x) \centerdot y \neq x$. So $R(y,y) \centerdot R(y,x) \centerdot y \neq x$. So $S(y,y)$. So $y \in \mathrm{C}(S)$. So

$$\text{(9)} \qquad \hat{y}(R(y,x) \centerdot y \neq x) \subset \mathrm{C}(S).$$

Let $y \in \mathrm{C}(S)$. Then by (6), $S(y,y)$. So $R(y,y) \centerdot R(y,x) \centerdot y \neq x$. So $y \in \hat{y}(R(y,x) \centerdot y \neq x)$. So by (9),

$$\text{(10)} \qquad \mathrm{C}(S) = \hat{y}(R(y,x) \centerdot y \neq x).$$

From $\Omega(R)$ and $S = \mathrm{seg}_x R$ we have derived (2), (8), and (10). Conversely, assume $\Omega(R)$, (2), (8), and (10). If $S(u,v)$, then $v \in \mathrm{C}(S)$. So $R(v,x) \centerdot v \neq x$ by (10). Also if $S(u,v)$, then $R(u,v)$ by (2). So

$$\text{(11)} \qquad (u,v) : S(u,v) \centerdot \supset \centerdot R(u,v) \centerdot R(v,x) \centerdot v \neq x.$$

If $R(u,v) \centerdot R(v,x) \centerdot v \neq x$, then $v \in \mathrm{C}(S)$ by (10). Also $R(u,x) \centerdot u \neq x$ by $\mathrm{trans}(R)$ and †803, so that $u \in \mathrm{C}(S)$ by (10). So, by (8), $S(u,v) \vee S(v,u)$. If $S(v,u)$, then $R(v,u)$ by (2). Hence $u = v$, since $R(u,v)$ and $\mathrm{antisym}(R)$. So $S(u,v)$. So in any case $S(u,v)$. So

$$\text{(12)} \qquad (u,v) : R(u,v) \centerdot R(v,x) \centerdot v \neq x \centerdot \supset \centerdot S(u,v).$$

By (11), (12), †447, and $S \subset \dot{\mathrm{V}}$ (which comes from (8)), we have $S = \mathrm{seg}_x R$.

†810. $(R,S,x,y) : \Omega(R) \centerdot S = \mathrm{seg}_x R \centerdot y \in \mathrm{C}(S) \centerdot \supset \centerdot \mathrm{seg}_y S = \mathrm{seg}_y R$.

Proof. Assume the hypothesis. By †809:

$$\text{(1)} \qquad \mathrm{S} \subset R.$$

$$\text{(2)} \qquad \Omega(S).$$

$$\text{(3)} \qquad \mathrm{C}(S) = \hat{y}(R(y,x) \centerdot y \neq x).$$

Let $T = \mathrm{seg}_y S$. By (2) and †809:

$$\text{(4)} \qquad T \subset S.$$

$$\text{(5)} \qquad \Omega(T).$$

$$\text{(6)} \qquad \mathrm{C}(T) = \hat{z}(S(z,y) \centerdot z \neq y).$$

So by (1) and (4),

$$\text{(7)} \qquad T \subset R.$$

Let $u \in \mathrm{C}(T)$. Then by (6), $S(u,y) \,.\, u \neq y$. So by (1), $R(u,y) \,.\, u \neq y$. So $u \in \hat{z}(R(z, y) \,.\, z \neq y)$. So

(8) $$\mathrm{C}(T) \subset \hat{z}(R(z,y) \,.\, z \neq y).$$

Let $u \in \hat{z}(R(z,y) \,.\, z \neq y)$. Then $R(u,y) \,.\, u \neq y$. Also by (3) and $y \in \mathrm{C}(S)$, we have $R(y,x) \,.\, y \neq x$. Hence $R(u,y) \,.\, R(y,x) \,.\, y \neq x$. So $S(u,y)$. So by (2), $S(u,u)$. So $S(u,u) \,.\, S(u,y) \,.\, u \neq y$. So $T(u,u)$. So $u \in \mathrm{C}(T)$. So

(9) $$\hat{z}(R(z,y) \,.\, z \neq y) \subset \mathrm{C}(T).$$

By (8) and (9), $\mathrm{C}(T) = \hat{z}(R(z,y) \,.\, z \neq y)$. By this, (7), (5), and †809, $T = \mathrm{seg}_y R$. So $\mathrm{seg}_y S = \mathrm{seg}_y R$.

Cor. $(R,S,x,y) : \Omega(R) \,.\, S = \mathrm{seg}_x R \,.\, R(y,x) \,.\, y \neq x \,.\!\supset\!.\, \mathrm{seg}_y S = \mathrm{seg}_y R$.

†811. $\vdash (P,n) : 1\text{-}1(P) \,.\, n \in \mathrm{Nn} \,.\!\supset 1\text{-}1(P^n)$.

Proof by induction on n using †682 and †683.

†812. $\vdash (R,S,x) : \Omega(R) \,.\, S \subset \mathrm{seg}_x R \,.\, x \in \mathrm{C}(R) \,.\!\supset\!.\, \sim(S \text{ smor } R)$.

Proof. Assume the hypothesis, together with $S \text{ smor}_Q R$. Put $T = \mathrm{seg}_x R$. So

(1) $$S \subset T,$$

and by †809,

(2) $$T \subset R,$$

(3) $$\mathrm{C}(T) = \hat{y}(R(y,x) \,.\, y \neq x).$$

Put $\theta(n)$ for $(Q^n)`x$. Then by †682,

(4) $$\theta(0) = x,$$

and by †683, †684, and †538,

(5) $$(n) : n \in \mathrm{Nn} \,.\, x \in \mathfrak{r}(Q^n) \,.\!\supset\!.\, \theta(n+1) = Q`\theta(n).$$

Lemma. $(n) : n \in \mathrm{Nn} \,.\supset .\, x \in \mathfrak{r}(Q^n) \,.\, R(\theta(n+1),\theta(n)) \,.\, \theta(n+1) \neq \theta(n)$.

Proof by induction on n. Let $n = 0$. Then $Q^n = I$, so that $x \in \mathfrak{r}(Q^n)$ by †551. Since $x \in \mathrm{C}(R)$, $R(x,x)$. So $(Eu,v) \,.\, Q(u,x) \,.\, S(u,v) \,.\, Q(v,x)$, because $R = \breve{Q}|S|Q$. Remembering that $1\text{-}1(Q)$, this gives $S(Q`x,Q`x)$. However $x = \theta(0)$, so that by (5), $\theta(1) = Q`\theta(0) = Q`x$. So $S(\theta(1),\theta(1))$. So by (1), $T(\theta(1),\theta(1))$. So by (3), $R(\theta(1),x) \,.\, \theta(1) \neq x$. Hence $R(\theta(1),\theta(0)) \,.\, \theta(1) \neq \theta(0)$ by (4). Now assume the lemma for n. So $x \in \mathfrak{r}(Q^n) \,.\, R(\theta(n+1),\theta(n)) \,.\, \theta(n+1) \neq \theta(n)$. Since $R = \breve{Q}|S|Q$, we have $(Eu,v) \,.\, Q(u,\theta(n+1)) \,.\, S(u,v) \,.\, Q(v,\theta(n))$. So $(Ev)Q(v,\theta(n))$. So $\theta(n) \in \mathfrak{r}Q$ since $1\text{-}1(Q)$. So $Q(Q`\theta(n),\theta(n))$. Also $Q^n(\theta(n),x)$, since $x \in \mathfrak{r}(Q^n)$. So $\theta^{n+1}(Q`\theta(n),x)$. Hence by †811, $x \in \mathfrak{r}Q^{n+1}$. Also, since $R(\theta(n+1),\theta(n))$, $R = \breve{Q}|S|Q$, and $1\text{-}1(Q)$, we have $Q(Q`\theta(n+1),\theta(n+1)) \,.\, S(Q`\theta(n+1),Q`\theta(n)) \,.\, Q(Q`\theta(n),\theta(n))$. So, since $\theta(n+1) \neq \theta(n)$, we have $S(Q`\theta(n+1),Q`(n)) \,.\, Q`\theta(n+1) \neq Q`\theta(n)$. So by (5), $S(\theta(n+2),\theta(n+1)) \,.\, \theta(n+2) \neq \theta(n+1)$. So by (1) and (2), $R(\theta(n+2),\theta(n+1)) \,.\, \theta(n+2) \neq \theta(n+1)$. This completes the proof of the lemma.

Now put $\beta = \hat{y}(En) \,.\, n \epsilon \mathrm{Nn} \,.\, y = \theta(n)$. By (4), $x \epsilon \beta$. So $x \epsilon \beta \cap \mathrm{C}(R)$. So by D808 there is a y such that $y \min_R \beta$. That is, there is an n such that

$$(6) \qquad n \epsilon \mathrm{Nn} \,.\, y = \theta(n) \,.\, y \epsilon \mathrm{C}(R) : (z) \,.\, z \epsilon \beta \cap \mathrm{C}(R) \supset R(y,z).$$

Choose $z = \theta(n+1)$. Then by the lemma,

$$(7) \qquad R(z,y) \,.\, z \neq y.$$

So $z \epsilon \mathrm{C}(R)$. Also $z \epsilon \beta$. So by (6), $R(y,z)$. So by antisym(R) and (7), we have a contradiction.

Cor. $\vdash (R,S,x) : \Omega(R) \,.\, S \subset \mathrm{seg}_x R \,.\, x \epsilon \mathrm{C}(R) \,.\!\supset\!.\, \sim(R \text{ smor } S)$.

†813. $\vdash (R,x,y) : \Omega(R) \,.\, x,y \epsilon \mathrm{C}(R) \,.\, (\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_y R) \,.\!\supset\!.\, x = y$.

Proof. Assume the hypothesis, together with $x \neq y$. Let $S = \mathrm{seg}_x R$ and $T = \mathrm{seg}_y R$. Since $x,y \epsilon \mathrm{C}(R)$, $R(x,y) \vee R(y,x)$.

Case 1. $R(y,x)$. Then $R(y,x) \,.\, y \neq x$. So by †809:

$$(1) \qquad y \epsilon \mathrm{C}(S).$$

$$(2) \qquad \Omega(S).$$

By (1) and †810,

$$(3) \qquad \mathrm{seg}_y S = T.$$

So $T \subset \mathrm{seg}_y S$. So by (1), (2), and †812, Cor., $\sim(S \text{ smor } T)$. This contradicts $(\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_y R)$.

Case 2. $R(x,y)$. Similar contradiction.

†814. $\vdash (R,x) \,.\, \Omega(R) \supset \mathrm{LE}(\mathrm{seg}_x R, R)$.

Proof. Assume $\Omega(R)$. Let $S = \mathrm{seg}_x R$. Let $y \epsilon \mathrm{C}(S)$. Then by †810, $\mathrm{seg}_y S = \mathrm{seg}_y R$. So by †805, $(\mathrm{seg}_y S) \text{ smor } (\mathrm{seg}_y R)$. Also by †809, $S \subset R$, so that $y \epsilon \mathrm{C}(R)$. Hence $(Ez) \,.\, z \epsilon \mathrm{C}(R) \,.\, (\mathrm{seg}_y S) \text{ smor } (\mathrm{seg}_z R)$. So $\mathrm{LE}(S,R)$.

†815. $\vdash (P,R,S,x,y) : \Omega(R) \,.\, \Omega(S) \,.\, R \text{ smor}_P\, S \,.\, P(x,y) \,.\, x \epsilon \mathrm{C}(R) \,.\!\supset\!.\, y \epsilon \mathrm{C}(S) \,.\, (\mathrm{seg}_x R) \text{ smor}_P\, (\mathrm{seg}_y S)$.

Proof. Assume the hypothesis. Since $P(x,y)$ and 1-1(P),

$$(1) \qquad P(x,u) \,.\!\equiv\!.\, u = y.$$

Since $x \epsilon \mathrm{C}(R)$, therefore $R(x,x)$. So since $R = P|Q|\breve{P}$, $(Eu,v) \,.\, P(x,u) \,.\, S(u,v) \,.\, P(x, v)$. So by (1), $(Eu,v) \,.\, u = y \,.\, S(u,v) \,.\, v = y$. So $S(y,y)$. So

$$(2) \qquad y \epsilon \mathrm{C}(S).$$

Since 1-1(P),

$$(3) \qquad P(v,b) \,.\, P(v,c) \,.\!\equiv\!.\, P(v,b) \,.\, b = c.$$

Since 1-1(P), therefore $P(v,b) \,.\, P(x,y) \,.\, b = y \,.\!\equiv\!.\, P(v,b) \,.\, P(x,y) \,.\, v = x$. So $P(v,b) \,.\, P(x,y) \,.\, v \neq x \,.\!\equiv\!.\, P(v,b) \,.\, P(x,y) \,.\, b \neq y$. So, since $P(x,y)$,

$$(4) \qquad P(v,b) \,.\, v \neq x \,.\!\equiv\!.\, P(v,b) \,.\, b \neq y.$$

Now let A be $(\mathrm{seg}_x R)(u,v)$. Then

$$A \,.\!\equiv\!.\, R(u,v) \,.\, R(v,x) \,.\, v \neq x.$$

by D813. So

$$A \mathrel{.{\equiv}.} (Ea,b,c,d) \mathrel{.} P(u,a) \mathrel{.} S(a,b) \mathrel{.} P(v,b) \mathrel{.} P(v,c) \mathrel{.} S(c,d) \mathrel{.} P(x,d) \mathrel{.} v \neq x,$$

since $R = P|Q|\breve{P}$. So

$$A \mathrel{.{\equiv}.} (Ea,b,c,d) \mathrel{.} P(u,a) \mathrel{.} S(a,b) \mathrel{.} P(v,b) \mathrel{.} b = c \mathrel{.} S(c,d) \mathrel{.} d = y \mathrel{.} v \neq x,$$

by (3) and (1). So

$$A \mathrel{.{\equiv}.} (Ea,b) \mathrel{.} P(u,a) \mathrel{.} S(a,b) \mathrel{.} S(b,y) \mathrel{.} P(v,b) \mathrel{.} v \neq x,$$

by *234. So

$$A \mathrel{.{\equiv}.} (Ea,b) \mathrel{.} P(u,a) \mathrel{.} S(a,b) \mathrel{.} S(b,y) \mathrel{.} b \neq y \mathrel{.} P(v,b),$$

by (4). So

$$A \mathrel{.{\equiv}.} (Ea,b) \mathrel{.} P(u,a) \mathrel{.} (\mathrm{seg}_y S)(a,b) \mathrel{.} P(v,b),$$

by D813. So

$$\mathrm{A} \mathrel{.{\equiv}.} (P|(\mathrm{seg}_y S)|\breve{P})(u,v).$$

Remembering what A is, we have proved

(6) $$\mathrm{seg}_x R = P|(\mathrm{seg}_y S)|\breve{P}.$$

In a similar manner we prove

(7) $$\mathrm{seg}_y S = \breve{P}|(\mathrm{seg}_x R)|P.$$

So $(\mathrm{seg}_x R)$ smor_P $(\mathrm{seg}_y S)$. This with (2) gives the desired result.

†816. $\vdash (R,S) : \Omega(R) \mathrel{.} \Omega(S) \mathrel{.} R \text{ smor } S \mathrel{.{\supset}.} \mathrm{LE}(R,S) \mathrel{.} \mathrm{LE}(S,R)$.

Proof. Assume $\Omega(R)$, $\Omega(S)$, and R smor_P S. If $x \in \mathrm{C}(R)$, then $\mathrm{R}(x,x)$. So $(Eu,v) \mathrel{.} P(x,u) \mathrel{.} S(u,v) \mathrel{.} P(x,v)$. Hence there is a y such that

(1) $$P(x,y),$$

(2) $$y \in \mathrm{C}(S).$$

So by †815, $(\mathrm{seg}_x R)$ smor_P $(\mathrm{seg}_y S)$. So $(Ey) \mathrel{.} y \in \mathrm{C}(S) \mathrel{.} (\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_y S)$. We have now proved

(3) $$\Omega(R) \mathrel{.} \Omega(S) \mathrel{.} R \text{ smor } S \mathrel{.{\supset}.} \mathrm{LE}(R,S).$$

However if R smor S, then S smor R by †806. So $\Omega(R) \mathrel{.} \Omega(S) \mathrel{.} R \text{ smor } S \mathrel{.{\supset}.} \mathrm{LE}(S,R)$.

†817. $\vdash (R,S,x,y,z) : \Omega(R) \mathrel{.} \Omega(S) \mathrel{.} (\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_y S) \mathrel{.} R(z,x) \mathrel{.} y \in \mathrm{C}(S) \mathrel{.} \mathrel{{\supset}.} (Eu) \mathrel{.} S(u,y) \mathrel{.} (\mathrm{seg}_z R) \text{ smor } (\mathrm{seg}_u S)$.

Proof. Assume the hypothesis. By †816,

(1) $$\mathrm{LE}(\mathrm{seg}_x R, \mathrm{seg}_y S).$$

Case 1. $z = x$. Take $u = y$.

Case 2. $z \neq x$. Then by †809, $z \in \mathrm{C}(\mathrm{seg}_x R)$. So by (1) there is a u such that

(2) $$u \in \mathrm{C}(\mathrm{seg}_y S),$$

(3) $$(\mathrm{seg}_z(\mathrm{seg}_x R)) \text{ smor } (\mathrm{seg}_u(\mathrm{seg}_y S)).$$

By †810 and (2), $\mathrm{seg}_u(\mathrm{seg}_y S) = \mathrm{seg}_u S$, and by †810 and $z \in \mathrm{C}(\mathrm{seg}_x R)$, $\mathrm{seg}_z(\mathrm{seg}_x R) = \mathrm{seg}_z R$. So $(\mathrm{seg}_z R)$ smor $(\mathrm{seg}_u S)$. Also by (2) and †809, $S(u,y)$.

†818. $\vdash (R,S) \centerdot \Omega(R) \centerdot \Omega(S) \centerdot \mathrm{LE}(R,S) \centerdot \mathrm{LE}(S,R) \centerdot\!\supset\!\centerdot\, R$ smor S.

Proof. Assume the hypothesis. Put

(1) $$P = \hat{u}\hat{s}(u \in \mathrm{C}(R) \centerdot s \in \mathrm{C}(S) \centerdot (\mathrm{seg}_u R) \text{ smor } (\mathrm{seg}_s S)).$$

If $P(u,v)$ and $P(u,w)$, then $u \in \mathrm{C}(R) \centerdot v \in \mathrm{C}(S) \centerdot w \in \mathrm{C}(S) \centerdot (\mathrm{seg}_u R)$ smor $(\mathrm{seg}_v S) \centerdot (\mathrm{seg}_u R)$ smor $(\mathrm{seg}_w S)$. So by †806 and †807, $v,w \in \mathrm{C}(S) \centerdot (\mathrm{seg}_v S)$ smor $(\mathrm{seg}_w S)$. So by †813, $v = w$. Similarly from $P(u,w)$ and $P(v,w)$, one would get $u = v$. Obviously $P \subset \dot{\mathrm{V}}$. So

(2) $$1\text{-}1(P).$$

Suppose $P(x,u) \centerdot S(u,v) \centerdot P(y,v)$. Then:

(3) $$(\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_u S).$$

(4) $$(\mathrm{seg}_y R) \text{ smor } (\mathrm{seg}_v S).$$

(5) $$x,y \in \mathrm{C}(R).$$

(6) $$u,v \in \mathrm{C}(S).$$

(7) $$S(u,v).$$

By (5), $R(x,y) \vee R(y,x)$.

Case 1. $R(x,y)$. Then $R(x,y)$.

Case 2. $R(y,x)$. Then by (3), (6), and †817, there is a w such that $S(w,u) \centerdot (\mathrm{seg}_y R)$ smor $(\mathrm{seg}_w S)$. So by (4), †806, and †807, $w \in \mathrm{C}(S) \centerdot (\mathrm{seg}_v S)$ smor $(\mathrm{seg}_w S)$. So by (6) and †813, $v = w$. This with $S(w,u)$ gives $S(v,u)$. So by (7), $u = v$. So by (2), $x = y$. So $R(x, y)$. So in either case, $R(x,y)$. So we have proved

(8) $$P(x,u) \centerdot S(u,v) \centerdot P(y,v) \centerdot\!\supset\!\centerdot\, R(x,y).$$

So

(9) $$P|S|\breve{P} \subset R.$$

Now let $R(x,y)$. Then $x,y \in \mathrm{C}(R)$. So, since $\mathrm{LE}(R,S)$, there are u and v such that $u \in \mathrm{C}(S) \centerdot (\mathrm{seg}_x R)$ smor $(\mathrm{seg}_u S)$ and $v \in \mathrm{C}(S) \centerdot (\mathrm{seg}_y R)$ smor $(\mathrm{seg}_v S)$. So:

(10) $$P(x,u).$$

(11) $$P(y,v).$$

Since $u,v \in \mathrm{C}(S)$, $S(u,v) \vee S(v,u)$.

Case 1. $S(u,v)$. Then $S(u,v)$.

Case 2. $S(v,u)$. Then $R(y,x)$ by (8), (10), and (11). So $x = y$, since we are assuming $R(x,y)$. So $u = v$ because of (2). So $S(u,v)$.

So $S(u,v)$ in either case. So by (10) and (11), $P(x,u) \mathbin{.} S(u,v) \mathbin{.} P(y,v)$. So $(P|S|\breve{P})(u,v)$. So $R \subset P|S|\breve{P}$. So

$$(12) \qquad R = P|S|\breve{P}.$$

Let $(\breve{P}|R|P)(u,v)$. Then $(Ea,b) \mathbin{.} P(a,u) \mathbin{.} R(a,b) \mathbin{.} P(b,v)$. So by (12), $(Ea,b,c,d) \mathbin{.} P(a,u) \mathbin{.} P(a,c) \mathbin{.} S(c,d) \mathbin{.} P(b,d) \mathbin{.} P(b,v)$. So by (2), $S(u,v)$. So

$$(13) \qquad \breve{P}|R|P \subset S.$$

Let $S(u,v)$. Then $u,v \in \mathrm{C}(S)$. So, since $\mathrm{LE}(S,R)$, there are x and y such that $x \in \mathrm{C}(R) \mathbin{.} (\mathrm{seg}_u S)$ smor $(\mathrm{seg}_x R)$ and $y \in \mathrm{C}(R) \mathbin{.} (\mathrm{seg}_v S)$ smor $(\mathrm{seg}_y R)$. So by †806, $P(x,u)$ and $P(y,v)$. So by (8), $R(x,y)$. So $P(x,u) \mathbin{.} R(x,y) \mathbin{.} P(y,v)$. So $(\breve{P}|R|P)(u,v)$. So $S \subset \breve{P}|R|P$. So with (13), $S = \breve{P}|R|P$. So with (2) and (12), R smor$_P$ S.

†819. $\vdash (R,S,T) : \mathrm{LE}(R,S) \mathbin{.} \mathrm{LE}(S,T) \mathbin{.\supset.} \mathrm{LE}(R,T)$.

Proof straightforward, using †807.

†820. $\vdash (R,S) : \Omega(R) \mathbin{.} \Omega(S) \mathbin{.} \mathrm{LE}(R,S) \mathbin{.} \sim\mathrm{LE}(S,R) \mathbin{.\supset.} (Ey) \mathbin{.} y \in \mathrm{C}(S) \mathbin{.} R$ smor $(\mathrm{seg}_y S)$.

Proof. Assume the hypothesis. Put

$$(1) \qquad \beta = \hat{y}\sim(Ex) \mathbin{.} x \in \mathrm{C}(R) \mathbin{.} (\mathrm{seg}_y S) \text{ smor } (\mathrm{seg}_x R).$$

Since $\sim\mathrm{LE}(S,R)$, $(Ey) \mathbin{.} y \in \mathrm{C}(S) \mathbin{.} \sim(Ex) \mathbin{.} x \in \mathrm{C}(R) \mathbin{.} (\mathrm{seg}_y S)$ smor $(\mathrm{seg}_x R)$. That is, $(Ey) \mathbin{.} y \in \beta \cap \mathrm{C}(S)$. So, since $\mathrm{Bord}(S)$, there is a y such that

$$(2) \qquad y \min_S \beta.$$

Put $T = \mathrm{seg}_y S$. Let $z \in \mathrm{C}(T)$. Then by †809, $S(z,y) \mathbin{.} z \neq y$. So $z \in \mathrm{C}(S)$. If also $z \in \beta$, then by (2), $S(y,z)$. This with $S(z,y) \mathbin{.} z \neq y$ would give a contradiction. So $z \in \mathrm{C}(T) \supset \sim(z \in \beta)$. So $z \in \mathrm{C}(T) \mathbin{.\supset.} (Ex) \mathbin{.} x \in \mathrm{C}(R) \mathbin{.} (\mathrm{seg}_z S)$ smor $(\mathrm{seg}_x R)$. However, by †810, $z \in \mathrm{C}(T) \mathbin{.\supset.} \mathrm{seg}_z S = \mathrm{seg}_z T$. So

$$(3) \qquad \mathrm{LE}(T,R).$$

Assume

$$(4) \qquad x \in \mathrm{C}(R) \mathbin{.} z \in \mathrm{C}(S) \mathbin{.} (\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_z S).$$

If $S(y,z)$, then by †806 and †817, we get $(Eu) \mathbin{.} R(u,x) \mathbin{.} (\mathrm{seg}_y S)$ smor $(\mathrm{seg}_u R)$. By †806, this would contradict $y \in \beta$, which follows by (2). So

$$(5) \qquad \sim S(y,z).$$

However, since $y \in \mathrm{C}(S)$ by (2), we have by (4), $S(y,z) \vee S(z,y)$. So

$$(6) \qquad S(z,y).$$

Also by (4) and (5), we have $z \neq y$. So by †809,

$$(7) \qquad z \in \mathrm{C}(T).$$

Hence by †810, $\mathrm{seg}_z S = \mathrm{seg}_z T$. Combining these results, we have proved

$x \epsilon \mathrm{C}(R) \,.\, z \epsilon \mathrm{C}(S) \,.\, (\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_z S) \,.\!\supset\!.\, z \epsilon \mathrm{C}(T) \,.\, (\mathrm{seg}_x R) \text{ smor } (\mathrm{seg}_z T)$. By use of this and LE$(R,S)$, one can readily prove LE(R,T). By (3) and †818, R smor T. Since $y \epsilon \mathrm{C}(S)$ follows from (2), we have the desired result.

†821. $\vdash (R,S) : \Omega(R) \,.\, \Omega(S) \,.\!\supset\!.\, \mathrm{LE}(R,S) \vee \mathrm{LE}(S,R)$.

Proof. Assume $\Omega(R)$, $\Omega(S)$, $\sim$LE(R,S), and $\sim$LE(S,R). Put

(1) $$\beta = \hat{y}\sim(Ex) \,.\, x \epsilon \mathrm{C}(R) \,.\, (\mathrm{seg}_y S) \text{ smor } (\mathrm{seg}_x R).$$

As in the proof of †820, we infer that there is a y such that

(2) $$y \min_S \beta.$$

Put $T = \mathrm{seg}_y S$. As in the proof of †820, we infer

(3) $$\mathrm{LE}(T,R).$$

Now suppose LE(R,T). By †814, LE(T,S). So by †819, LE(R,S). This contradicts $\sim$LE(R,S), which we assumed. So

(4) $$\sim\mathrm{LE}(R,T).$$

Then by †820, $(Ex) \,.\, x \epsilon \mathrm{C}(R) \,.\, T \text{ smor } (\mathrm{seg}_x R)$. This contradicts $y \epsilon \beta$.

†822. $\vdash (R,S) :. \Omega(R) \,.\, \Omega(S) :\supset: \sim\mathrm{LE}(S,R) \,.\!\equiv\!.\, (Ey) \,.\, y \epsilon \mathrm{C}(S) \,.\, R \text{ smor } (\mathrm{seg}_y S)$.

Proof. Assume $\Omega(R)$, $\Omega(S)$. If $\sim$LE(S,R), then LE(R,S), by †821. So $(Ey) \,.\, y \epsilon \mathrm{C}(S) \,.\, R$ smor $(\mathrm{seg}_y S)$ by †820. This proves half the equivalence. Now let $y \epsilon \mathrm{C}(S) \,.\, R$ smor $(\mathrm{seg}_y S)$ and LE(S,R). From R smor $(\mathrm{seg}_y S)$ we get LE$(R,\mathrm{seg}_y S)$ by †816 and †809. By †814, LE$(\mathrm{seg}_y S,S)$. So by †819, LE(R,S). This with LE(S,R) and †818 gives S smor R. By †807, we have S smor $(\mathrm{seg}_y S)$. This contradicts †812, Cor.

†823. $\vdash (R,S) : R \text{ smor } S \,.\!\supset\!.\, \mathrm{Nr}(R) = \mathrm{Nr}(S)$.

Proof. Let R smor S. If $Q \epsilon \mathrm{Nr}(R)$, then $Q \epsilon \mathrm{V} \,.\, R$ smor Q. Hence $Q \epsilon \mathrm{V} \,.\, S$ smor Q by †806 and †807. So $Q \epsilon \mathrm{Nr}(S)$. Similarly, if $Q \epsilon \mathrm{Nr}(S)$, then $Q \epsilon \mathrm{Nr}(R)$.

†824. $\vdash (P) : P \epsilon \mathrm{V} \,.\, \Omega(P) \,.\!\supset\!.\, P \epsilon \mathrm{Nr}(P)$.

Proof. Assume $P \epsilon \mathrm{V} \,.\, \Omega(P)$. By †805, P smor P. So $P \epsilon \mathrm{Nr}(P)$.

†825. $\vdash (P,Q) : \mathrm{Nr}(P) \epsilon \mathrm{NO} \,.\, \mathrm{Nr}(P) = \mathrm{Nr}(Q) \,.\!\supset\!.\, P \text{ smor } Q \,.\, \Omega(P) \,.\, \Omega(Q) \,.\, \mathrm{LE}(P,Q) \,.\, \mathrm{LE}(Q,P)$.

Proof. Assume the hypothesis. Then by D816 there is an R such that $R \epsilon \mathrm{V} \,.\, \Omega(R) \,.\, \mathrm{Nr}(P) = \mathrm{Nr}(R)$. Hence by †824, $R \epsilon \mathrm{Nr}(P)$ and $R \epsilon \mathrm{Nr}(Q)$. So P smor R and Q smor R. So by †806 and †807, P smor Q. Also $\Omega(P)$ and $\Omega(Q)$ by †806 and †808. So by †816, LE$(P,Q) \,.\,$ LE(Q,P).

†826. $\vdash (\alpha) : \alpha \epsilon \mathrm{NO} \,.\!\supset\!.\, \alpha \leqq \alpha$.

Proof. Assume $a \epsilon \mathrm{NO}$. Then by D816, there is a P such that

$$\alpha \epsilon \mathrm{NO} \,.\, \Omega(P) \,.\, \alpha = \mathrm{Nr}(P).$$

By †805 and †816, LE(P,P). So $\alpha \epsilon \mathrm{NO} \,.\, \alpha = \mathrm{Nr}(P) \,.\, \mathrm{LE}(P,P)$. So $\alpha \leqq \alpha$ by D817 and D818.

†827. $\vdash \mathrm{C}(\leqq) = \mathrm{NO}$.

Proof. By D817, $\mathrm{C}(\leqq) \subset \mathrm{NO}$. By †826, $\mathrm{NO} \subset \mathrm{C}(\leqq)$.

†828. $\vdash \text{trans}(\leqq)$.

Proof. Assume $\alpha \leqq \beta \,.\, \beta \leqq \gamma$. Then by D818 and D817 there are P, Q, R, S such that

$$\alpha,\beta \in \text{NO} \,.\, \alpha = \text{Nr}(P) \,.\, \beta = \text{Nr}(Q) \,.\, \text{LE}(P,Q),$$

$$\beta,\gamma \in \text{NO} \,.\, \beta = \text{Nr}(R) \,.\, \gamma = \text{Nr}(S) \,.\, \text{LE}(R,S).$$

By †825, LE(Q,R). So by †819, LE(P,S). So $\alpha \leqq \gamma$.

†829. $\vdash \text{antisym}(\leqq)$.

Proof. Assume $\alpha \leqq \beta$, $\beta \leqq \alpha$. Then there are P, Q, R, S such that

(1) $$\alpha,\beta \in \text{NO} \,.\, \alpha = \text{Nr}(P) \,.\, \beta = \text{Nr}(Q) \,.\, \text{LE}(P, Q),$$

(2) $$\beta,\alpha \in \text{NO} \,.\, \beta = \text{Nr}(R) \,.\, \alpha = \text{Nr}(S) \,.\, \text{LE}(R, S).$$

By †825, $\Omega(P)$, $\Omega(R)$, LE(Q,R), and LE(S, P). So by †819,

(3) $$\Omega(P) \,.\, \Omega(R) \,.\, \text{LE}(P,R) \,.\, \text{LE}(R,P).$$

So by †818, P smor R. So by †823, $\alpha = \beta$.

†830. $\vdash \text{connex}(\leqq)$.

Proof. Assume $\alpha,\beta \in \text{C}(\leqq)$. Then $\alpha,\beta \in \text{NO}$ by †827. So by D816 there are P and Q such that

$$P \in \text{V} \,.\, \Omega(P) \,.\, \alpha = \text{Nr}(P),$$

$$Q \in \text{V} \,.\, \Omega(Q) \,.\, \beta = \text{Nr}(Q).$$

So by †821, LE(P,Q) **v** LE(Q,P). So by D817 and D818, $\alpha \leqq \beta \,.\!\vee\!.\, \beta \leqq \alpha$.

†831. $\vdash \text{ref}(\leqq)$.

Proof. Use †827 and †826.

†832. $\vdash \text{Ser}(\leqq)$.

Proof. Clearly $\leqq\ \subset \dot{\text{V}}$. Use †828, †829, †830, and †831.

†833. $\vdash (\alpha, \beta, P) : \beta \leqq \alpha \,.\, \beta \neq \alpha \,.\, \alpha = \text{Nr}(P) \,.\!\supset\!.\, (Eu) \,.\, u \in \text{C}(P) \,.\, \beta = \text{Nr}(\text{seg}_u P)$.

Proof. Assume the hypothesis. Then by D818 and D817, there are Q and R such that

$$\beta,\alpha \in \text{NO} \,.\, \beta = \text{Nr}(Q) \,.\, \alpha = \text{Nr}(R) \,.\, \text{LE}(Q, R).$$

Then by $\text{Nr}(Q) = \text{Nr}(Q)$ and †825, we get $\Omega(Q)$, $\Omega(P)$, and LE(R,P). So by †819,

(1) $$\text{LE}(Q,P).$$

Now assume LE(P,Q). Then by †818, P smor Q. So by †823, $\alpha = \beta$. This is a contradiction. So

(2) $$\sim\text{LE}(P,Q).$$

So by †820, there is a u such that $u \in \text{C}(P) \,.\, Q$ smor $(\text{seg}_u P)$. So by †823, $u \in \text{C}(P) \,.\, \beta = \text{Nr}(\text{seg}_u P)$.

†834. $\vdash (\alpha,\beta,P,u,v) : \Omega(P) \,.\, P(u,v) \,.\, \alpha = \mathrm{Nr}(\mathrm{seg}_u P) \,.\, \beta = \mathrm{Nr}(\mathrm{seg}_v P) \,.\, \alpha,\beta \in \mathrm{NO} \,.\supset.\, \alpha \leqq \beta.$

Proof. Assume the hypothesis.

Case 1. $u = v$. Then $\alpha = \beta$, and we can use †826.

Case 2. $u \neq v$. Then by †810, Cor., $\mathrm{seg}_u P = \mathrm{seg}_u(\mathrm{seg}_v P)$. So by †809 and †814, $\mathrm{LE}(\mathrm{seg}_u P, \mathrm{seg}_v P)$. Use D817 and D818.

†835. $\vdash \Omega(\leqq)$.

Proof. By †832, it suffices to prove $\mathrm{Bord}(\leqq)$. So let

$$\alpha \in A \cap \mathrm{C}(\leqq). \tag{1}$$

Case 1. $(\beta) : \beta \in A \cap \mathrm{C}(\leqq) \,.\supset.\, \alpha \leqq \beta$. Then $\alpha \min_{\leqq} A$ by D807. So $(Ey) \,.\, y \min_{\leqq} A$.

Case 2. $\sim(\beta) : \beta \in A \cap \mathrm{C}(\leqq) \,.\supset.\, \alpha \leqq \beta$. Then there is a β such that

$$\beta \in A \cap \mathrm{C}(\leqq) \,.\, \sim(\alpha \leqq \beta). \tag{2}$$

So by †830 and †831,

$$\beta \leqq \alpha \,.\, \beta \neq \alpha. \tag{3}$$

Then by †827 and D816, there is a P such that

$$P \in \mathrm{V} \,.\, \Omega(P) \,.\, \alpha = \mathrm{Nr}(P). \tag{4}$$

So by (3) and †833, there is a w such that

$$w \in \mathrm{C}(P) \,.\, \beta = \mathrm{Nr}(\mathrm{seg}_w P). \tag{5}$$

Put

$$B = \hat{u}(\mathrm{Nr}(\mathrm{seg}_u P) \in A \cap \mathrm{C}(\leqq)). \tag{6}$$

Then by (2) and (5)

$$w \in B \cap \mathrm{C}(P). \tag{7}$$

So by (4) and D808, there is a u such that

$$u \min_P B. \tag{8}$$

Then by (6)

$$\mathrm{Nr}(\mathrm{seg}_u P) \in A \cap \mathrm{C}(\leqq). \tag{9}$$

Also by (8), (7), and D807, $P(u,w)$. So by (4), (9), (5), (2), and †834,

$$\mathrm{Nr}(\mathrm{seg}_u P) \leqq \beta. \tag{10}$$

We will now prove that $\mathrm{Nr}(\mathrm{seg}_u P) \min_{\leqq} A$. To this end, assume

$$\gamma \in A \cap \mathrm{C}(\leqq). \tag{11}$$

Subcase I. $\beta \leqq \gamma$. Then $\mathrm{Nr}(\mathrm{seg}_u P) \leqq \gamma$ by (10) and †828.

Subcase II. $\sim(\beta \leqq \gamma)$. Then by †830 and †831 and (2) and (11),

$\gamma \leqq \beta \,.\, \gamma \neq \beta$. Then by †833 and (5), there is a v such that $v \,\epsilon\, \mathrm{C}(\mathrm{seg}_w P) \,.\, \gamma = \mathrm{Nr}(\mathrm{seg}_v(\mathrm{seg}_w P))$. So by †809 and †810 and (4),

(12) $$v \,\epsilon\, \mathrm{C}(P) \,.\, \gamma = \mathrm{Nr}(\mathrm{seg}_v P).$$

So by (11) and (6), $v \,\epsilon\, B \cap \mathrm{C}(P)$. So by (8), $P(u,v)$. So by (4), (12), (9), (11), and †834, we get $\mathrm{Nr}(\mathrm{seg}_u P) \leqq \gamma$.

So in any case, $\mathrm{Nr}(\mathrm{seg}_u P) \leqq \gamma$. So $\mathrm{Nr}(\mathrm{seg}_u P) \min_{\leqq} A$.

†836. $\vdash (\alpha,P) : \alpha \,\epsilon\, \mathrm{NO} \,.\, \alpha = \mathrm{Nr}(P) \,.\supset.\, \mathrm{LE}(\mathrm{seg}_\alpha \leqq, P)$.

Proof. Put

(1) $$A = \hat{\alpha}(EP) \,.\, \alpha = \mathrm{Nr}(P) \,.\, \sim\mathrm{LE}(\mathrm{seg}_\alpha \leqq, P).$$

Assume the theorem false. Then $(E\alpha)(EP) \,.\, \alpha \,\epsilon\, \mathrm{NO} \,.\, \alpha = \mathrm{Nr}(P) \,.\, \sim\mathrm{LE}(\mathrm{seg}_\alpha \leqq, P)$. So by (1) and †827, $(E\alpha) \,.\, \alpha \,\epsilon\, A \cap \mathrm{C}(\leqq)$. So by †835 and D808, there is an α such that

(2) $$\alpha \min_{\leqq} A.$$

By D807 and †827, $\alpha \,\epsilon\, A \cap \mathrm{NO}$. So by (1), there is a P such that

(3) $$\alpha \,\epsilon\, \mathrm{NO} \,.\, \alpha = \mathrm{Nr}(P) \,.\, \sim\mathrm{LE}(\mathrm{seg}_\alpha \leqq, P).$$

Hence by †825 and $\mathrm{Nr}(P) = \mathrm{Nr}(P)$, we get $\Omega(P)$. So by †835, †809, and †822, there is a β such that $\beta \,\epsilon\, \mathrm{C}(\mathrm{seg}_\alpha \leqq) \,.\, P \text{ smor } (\mathrm{seg}_\beta(\mathrm{seg}_\alpha \leqq))$. So by †809 and †810,

(4) $$\beta \leqq \alpha \,.\, \beta \neq \alpha \,.\, P \text{ smor } (\mathrm{seg}_\beta \leqq).$$

Since $\beta \leqq \alpha$, $\beta \,\epsilon\, \mathrm{NO}$. So there is a Q such that

(5) $$Q \,\epsilon\, \mathrm{V} \,.\, \Omega(Q) \,.\, \beta = \mathrm{Nr}(Q).$$

Case 1. $\sim\mathrm{LE}(\mathrm{seg}_\beta \leqq, Q)$. Then by (1), (4), and (5), $\beta \,\epsilon\, A \cap \mathrm{C}(\leqq)$. So by (2), $\alpha \leqq \beta$.

Case 2. $\mathrm{LE}(\mathrm{seg}_\beta \leqq, Q)$. By (4) and †816, $\mathrm{LE}(P, \mathrm{seg}_\beta \leqq)$. So by †819, $\mathrm{LE}(P,Q)$. So by D817 and D818, $\alpha \leqq \beta$.

So in any case, $\alpha \leqq \beta$. By †829 and (4), we have a contradiction. Hence the theorem must be true.

†837. $\vdash (\alpha) : \alpha \,\epsilon\, \mathrm{NO} \,.\supset.\, (\mathrm{seg}_\alpha \leqq) \,\epsilon\, \mathrm{V}$.

Proof. Use *200.

†838. $\vdash (P) : P \,\epsilon\, \mathrm{V} \,.\, \Omega(P) \,.\supset.\, \mathrm{Nr}(P) \,\epsilon\, \mathrm{NO}$.

Proof. By *200, $\mathrm{Nr}(P) \,\epsilon\, \mathrm{V}$. So it suffices to prove $(EQ) \,.\, Q \,\epsilon\, \mathrm{V} \,.\, \Omega(Q) \,.\, \mathrm{Nr}(P) = \mathrm{Nr}(Q)$. For this purpose, take Q to be P.

†839. $\vdash (\alpha) : \alpha \,\epsilon\, \mathrm{NO} \,.\supset.\, \mathrm{Nr}(\mathrm{seg}_\alpha \leqq) \,\epsilon\, \mathrm{NO}$.

Proof. Use †837, †838, †835, and †809.

†840. $\vdash (\alpha) : \alpha \,\epsilon\, \mathrm{NO} \,.\supset.\, \mathrm{Nr}(\mathrm{seg}_\alpha \leqq) \leqq \alpha$.

Proof. Let $\alpha \,\epsilon\, \mathrm{NO}$. Then there is a P such that $P \,\epsilon\, \mathrm{V} \,.\, \Omega(P) \,.\, \alpha = \mathrm{Nr}(P)$. Then by †836, $\mathrm{LE}(\mathrm{seg}_\alpha \leqq, P)$. So by †839, D817, and D818, $\mathrm{Nr}(\mathrm{seg}_\alpha \leqq) \leqq \alpha$.

†841. $\vdash (\alpha) : \alpha \,\epsilon\, \mathrm{NO} \,.\supset.\, \mathrm{Nr}(\mathrm{seg}_\alpha \leqq) = \alpha$.

Proof. Let

(1) $$A = \hat{\alpha}(\mathrm{Nr}(\mathrm{seg}_\alpha \leqq) \neq \alpha).$$

Assume the theorem false. Then $(E\alpha) \,.\, \alpha \epsilon \mathrm{NO} \,.\, \mathrm{Nr}(\mathrm{seg}_\alpha \leqq) \neq \alpha$. So $(E\alpha) \,.\, \alpha \epsilon A \cap \mathrm{C}(\leqq)$. So by †835 and D808, there is an α such that

$$\alpha \min_{\leqq} A. \tag{2}$$

Put $\beta = \mathrm{Nr}(\mathrm{seg}_\alpha \leqq)$. By †840,

$$\beta \leqq \alpha. \tag{3}$$

Case 1. $\mathrm{Nr}(\mathrm{seg}_\beta \leqq) \neq \beta$. Then $\beta \epsilon A \cap \mathrm{C}(\leqq)$. So by (2), $\alpha \leqq \beta$. So by (3), $\beta = \alpha$.

Case 2. $\mathrm{Nr}(\mathrm{seg}_\beta \leqq) = \beta$. Then by definition of β, $\mathrm{Nr}(\mathrm{seg}_\beta \leqq) = \mathrm{Nr}(\mathrm{seg}_\alpha \leqq)$. So by †825, $(\mathrm{seg}_\beta \leqq)$ smor $(\mathrm{seg}_\alpha \leqq)$. So by †813, $\beta = \alpha$.

So in either case, $\beta = \alpha$. This contradicts (1) and (2). So our theorem must be true.

We give here an outline of an alternative proof of †841, for reference in case any one should wish to test other systems for the occurrence of the Burali-Forti Paradox.

For the alternative proof we dispense with †836 to †841. By methods similar to those used in the proofs of †837 and †838, we prove:

Lemma. $\vdash (P) : P \epsilon \mathrm{V} \,.\, \Omega(P) \,.\, u \epsilon \mathrm{C}(P) \,.\!\supset\!.\, \mathrm{Nr}(\mathrm{seg}_u P) \epsilon \mathrm{NO}$.

Then to prove †841, we assume $\alpha \epsilon \mathrm{NO}$. There is a P such that

$$P \epsilon \mathrm{V} \,.\, \Omega(P) \,.\, \alpha = \mathrm{Nr}(P). \tag{1}$$

Put:

$$Q = \hat{u}\hat{\gamma}(\gamma = \mathrm{Nr}(\mathrm{seg}_u P) \,.\, \gamma \leqq \alpha \,.\, \gamma \neq \alpha \,.\, u \epsilon \mathrm{C}(P)). \tag{2}$$

$$R = \mathrm{seg}_\alpha \leqq. \tag{3}$$

By †825 and †813,

$$1\text{-}1(Q). \tag{4}$$

By †834,

$$Q(u,\beta) \,.\, P(u,v) \,.\, Q(v,\gamma) \,.\!\supset\!.\, R(\beta,\gamma). \tag{5}$$

If $Q(u,\beta) \,.\, R(\beta,\gamma) \,.\, Q(v,\gamma)$, then $P(u,v) \vee P(v,u)$. By (4) and (5), we readily deduce $P(u,v)$. So

$$Q(u,\beta) \,.\, R(\beta,\gamma) \,.\, Q(v,\gamma) \,.\!\supset\!.\, P(u,v). \tag{6}$$

If $R(\beta,\gamma)$, then $\beta \leqq \alpha \,.\, \beta \neq \alpha$. So by †833, $(Eu) \,.\, Q(u,\beta)$. Similarly $(Ev) \,.\, Q(v,\gamma)$. So by (6), $(Eu,v) \,.\, Q(u,\beta) \,.\, P(u,v) \,.\, Q(v,\gamma)$. So $(\breve{Q}|P|Q)(\beta,\gamma)$. So by (5),

$$R = \breve{Q}|P|Q. \tag{7}$$

If $P(u,v)$, then by the lemma, $\mathrm{Nr}(\mathrm{seg}_u P) \epsilon \mathrm{NO}$. Also, by †814, $\mathrm{Nr}(\mathrm{seg}_u P) \leqq \alpha$. By †825 and †812, $\mathrm{Nr}(\mathrm{seg}_u P) \neq \alpha$. So $Q(u,\mathrm{Nr}(\mathrm{seg}_u P))$.

Similarly $Q(v,\mathrm{Nr}(\mathrm{seg}_v P))$. So by (5), $R(\mathrm{Nr}(\mathrm{seg}_u P),\mathrm{Nr}(\mathrm{seg}_v P))$. So $(Q|R|\breve{Q})(u,v)$. So with (6),

$$(8) \qquad P = Q|R|\breve{Q}.$$

So by (4), (7), and (8), P smor_Q R. So $\mathrm{Nr}(P) = \mathrm{Nr}(R)$. So $\alpha = \mathrm{Nr}(\mathrm{seg}_\alpha \leqq)$.

We now continue with the derivation of the Burali-Forti Paradox.

†842. $\vdash \leqq \epsilon \mathrm{V}$.

Proof. Use *200.

†843. $\vdash \mathrm{Nr}(\leqq) \epsilon \mathrm{NO}$.

Proof. Use †838, †842, and †835.

†844. $\vdash$ Burali-Forti Paradox.

Proof. By †843,

$$(1) \qquad \vdash \mathrm{Nr}(\leqq) \epsilon \mathrm{NO}.$$

By †841,

$$(2) \qquad \vdash \mathrm{Nr}(\mathrm{seg}_{\mathrm{Nr}(\leqq)} \leqq) = \mathrm{Nr}(\leqq).$$

By †825,

$$(3) \qquad \vdash (\mathrm{seg}_{\mathrm{Nr}(\leqq)} \leqq) \text{ smor } \leqq.$$

By (1), †827, †835, and †812,

$$(4) \qquad \vdash \sim((\mathrm{seg}_{\mathrm{Nr}(\leqq)} \leqq) \text{ smor } \leqq).$$

By (3) and (4), we have an inconsistency.

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