The Burali-Forti Paradox

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Any article about the Burali-Forti paradox has to begin with an apology. There has been far too much written about the Burali-Forti paradox and much of what has been written is *terrible*. So why should this article be any different? The answer is that readers may find its approach to be more in harmony with the modern appreciation of the fertility of type-theory.

The Burali-Forti paradox, like lots of other paradoxes, is a proof that something is impossible. (By some definitions a paradox is a nothing more than a proof of $\perp$). The point I wish to emphasise below is that there may be lots of different ways of describing the impossibility, so that if one is to fully understand the situation, one needs to engage with all those different ways. The thought here is that the different ways are not competing resolutions of the contradiction—Mathematics is not firefighting after all—but rather are *different takes on the underlying mathematics*.

There is a famous trope about the five blind men and the elephant. My friend Kelsang Rabten, a Buddhist monk, tells me that in his tradition the meaning of this parable is taken to be that if you have only one teacher you will have only one insight.

The Burali-Forti paradox is of course an elephant, if a rather abstract one, and there are plenty of abstractions groping it. One of them is a set-theoretic foundationalist. . If you are such a person (as far too many people are), so that for you—in the final analysis—everything is a set, you will have no option but to see Burali-Forti as a theorem about sets; you will see the elephant as a proof that the collection of ordinals (however implemented) cannot be a set. This is in all the textbooks, and it's true—in some particularly thin sense of 'true'. The problem for us here and now is not that it isn't *true*; the problem is that it isn't particularly *enlightening*. It's not so much an explanation as an artefact of the equipment used to contrive the explanation. And—since that equipment is set-theory—it tells you more about set theory than it does about the Burali-Forti paradox: artefacts never tell you about the world, they only tell you about the equipment that you are using to investigate that world.

The particular blind man talking to you now is a set theorist who—because of his peculiar early conditioning history—was compelled to think about (and think *through*) the Quine systems, which are—admittedly—set theories, but are set theories that are sufficiently different from ZF-like theories to bring one up short with the thought that *There Might Be Another Way*. He was also exposed to the stimulus of Theoretical Computer Science through being a postdoc in a Computer Science department. Anyone who has been through that has a chance of groping parts of the elephant that set-theoretic foundationalism cannot reach. One is given a chance to reëxamine the Mathematics to which the Burali-Forti paradox beckons us.

Ordinals are a kind of generalisation of natural numbers; natural numbers with an additional `limit` constructor. One can usefully think of the ordinals as an end-extension of the naturals. What do natural numbers do? What are they *for*? They measure the lengths of lists. In most typed programming languages `list`s are a polymorphic data type. For two distinct types `a` and `b` the types `a-list` and `b-list` are distinct. We say that the data type `list` is *polymorphic*. However the datatype of natural numbers that measure the lengths of those lists is always taken to be *monomorphic*. *Prima facie*, I suppose, naturals ought to be polymorphic too—since they arise by abstracting away from a polymorphic datatype—but it seems pretty clear that it is actually safe to take them to be monomorphic. The natural numbers that measure the lengths of `a-list`s are the same natural numbers that measure the lengths of `b-list`s. Take-home thought: naturals *prima facie* ought to be polymorphic but it turns out to be OK to take them to be monomorphic. Keep this thought in mind when approaching ordinals.

So: what about ordinals? Are they monomorphic too? Well, finite ordinals are natural numbers and *they* are monomorphic. So far so good. Right from the dawn of ordinal arithmetic Cantor knew that the natural order relation on ordinals is wellfounded, so that, for any ordinal α, the ordinals below α form a wellordering and that wellordering will have an ordinal. Now you don't have to be a paid-up type theorist to think that perhaps—whatever the ADT of an ordinal α—the ADT of the ordinal of the wellordering of ordinals-below-α might be distinct from the ADT of the ordinal α . . . an ADT somehow *derived* from the ADT of α, and with an intimate relation to that ADT, but not actually *identical* to it. Thus ordinals (like naturals) are in principle polymorphic, and for the same reason: they *start off* polymorphic, just as natural numbers did. Might it be safe to take them, too, to be monomorphic? It does seem to be perfectly safe to think of countable ordinals as monomorphic . . . however the Burali-Forti paradox tells us that the answer is "no"! In particular the ordinal of the wellordering of *all* ordinals of type `a` cannot itself be of type `a`.

We shouldn't expect the BF paradox to tell us *at what point* ordinals cease to be monomorphic, merely to tell us that there is such a point. This question of quite where ordinals cease to be monomorphic is a good one to think about. BF tells us that if the monomorphic ordinals form an initial segment then it is a *proper* initial segment, so it is natural to ask what operations that proper initial segment is closed under. . . `successor` for one. The thought that it might be closed under some other operations is a rich source of axioms. (Does every normal function from ordinals $\to$ ordinals have a monomorphic fixed point, for example?) It seems to me that this is how we should understand large cardinal axioms: as assertions that the initial segment of monomorphic ordinals is closed under ever more operations.

So: there we have another take on the elephant, one that I hope may be helpful to other blind men engaging with it. This is not in competition or contradiction with the set theoretic analysis; it's merely a shot from another angle. I am not objecting to the ZF-iste take on Burali-Forti; all I object to is the mistake of thinking that it is the last word on the matter. The Burali-Forti paradox is a birthmark on the skin of Mathematics and we pick at it to get at the Mathematics beneath it. Set theory is merely one way of doing it among many, and we handicap ourselves if it is the only weapon we use.