Sur les Cas Stratifiés du Schéma de Remplacement

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June 27, 2020

C.R. Acad. Sc. Paris, t 271 (15 juillet 1970) Série A pp. 57–60
A note (*) by Jean Coret, presented by Mr. Jean Leray
(presented at the session held on 29/vi/1970)

We will establish the following result: every stratified instance of the replacement scheme is provable in the set theory of Zermelo (Z^-) with neither the axiom of choice nor foundation.

Upper-case Roman letters F, G... will denote operators definable in Z^-; $F \circ G$ is the composition of F and G; $F^{(k)}$ is the operator F iterated k times, with $F^{(0)} = \mathcal{I}$, the identity. If F is a permutation of the universe, F^{-1} is the inverse of F.

Definition 1 *F is* **permitted** *if*

$$(\forall a)(\exists b)(\forall y)(y \in b \longleftrightarrow (\exists x \in a)(y = F(x)))$$

If F is permitted, let F^* be the operator

$$a \mapsto \{Fa : x \in a\}$$

For example, in Z^-, the operators $\bigcup$ (sumset) and $\mathcal{P}$ (power set) and $\{,\}$ (singleton) are permitted. If F and G are permitted so is $F \circ G$. If F is permitted, so is F^*, and we will write as $F^{[k]}$ for the suite of operations defined by

$$F^{[0]} =: \mathcal{I}; \qquad F^{[1]} =: F; \qquad F^{[k+1]} =: (F^{[k]})^*.$$

Definition 2 *F is an* **admissible** *permutation if both F and F^{-1} are permitted.*

We can now define the suite F_k by:

$$F_0 =: \mathcal{I}; \qquad F_1 = F; \qquad F_{k+1} =: (F_k)^* \circ F_1.$$

By induction on k one proves immediately: if F is an admissible permutation, then the F_k are all admissible permutations. See note 1.

Lemma 3 *If F is an admissible permutation, Φ a stratified formula with free variables $x_1 \dots x_n$, $\sigma = (\sigma_1 \dots \sigma_n)$ be a suite of integers stratifying Φ, Φ' the result of replacing $x_i \in x_j$ by $x_i \in F(x_j)$ throughout, then*

$$(\forall x_1 \dots x_n)[\Phi(x_1 \dots x_n) \longleftrightarrow \Phi'((F_{\sigma_1})^{-1} x_1 \dots (F_{\sigma_n})^{-1} x_n)]$$

Proof:
We prove lemma 1 by induction on the complexity of Φ. ■

We can infer from it results analogous to those in reference [1].

Lemma 4 *Let F be an admissible permutation. Then the following hold for every integer k.*

1. *If F is the identity on $\bigcup^k x$, then $F^{[k+1]} x = x$;*
2. $\bigcup^k \circ F^{[k+1]} = F^* \circ \bigcup^k$

Lemma 5 *With the notations of lemma 4, if Φ is stratified then for all integers m*

$$(\forall x_1 \dots x_n)[\Phi(x_1 \dots x_n) \longleftrightarrow \Phi((F^{[\sigma_1+m]} x_1 \dots (F^{[\sigma_n+m]} x_n)]$$

Proof:
We prove lemma 5 by noting that if σ stratifies Φ, so does the sequence $\sigma + m$. ■

Definition 6 *Let*

- *Φ be a formula with free variables among $x_1 \dots x_n$;*
- *K be a subset of $\{1, 2 \dots n\}$;*
- *n_i be the number of occurrences of x_i in Φ;*
- *$y_{j,k}$, for $j = 1, \dots n_k$ and $k \notin K$ a suite of variables not occurring in Φ.*

Then Φ_K is the expression obtained from Φ by replacing the jth occurrence of x_k—for $k \notin K$—by $y_{j,k}$. If Φ_K is stratified we will say that Φ is stratified on the set *K, or—further—that the variables x_i with $i \in K$* are stratified in *K. If K is empty we will say that Φ* is pseudo-stratified.

See note 5.

For example:

$$\Phi(x_1) : (\forall t)(\forall t')((t \in t' \wedge t' \in x) \rightarrow t \in x)$$

Now for $K = \emptyset$ we have

$$\Phi_\emptyset(y_{1,1}, y_{2,1}) : (\forall t)(\forall t')((t \in t' \wedge t' \in y_{1,1}) \rightarrow t \in y_{2,1})$$

The expression "x is a transitive set" is accordingly pseudo-stratified.

Corollary 7 *If Φ is stratified in K and F is an admissible permutation then, for every integer m*

$$\forall x_1 \dots x_n[\Phi(x_1 \dots x_n) \longleftrightarrow \Phi_K(\dots F^{[\sigma_i+m]}x_i \dots F^{[\sigma_{j,k+m}]}x_k \dots)]$$

for $i \in K$, $k \notin K$, $j \in \{1 \dots n_k\}$ and σ a stratification of Φ_K. See note 2.

Lemma 8 *If the variable x_{n+1} is stratified in Φ, and σ stratifies $\Phi_{\{n+1\}}$ then the following holds:*

if

$$(\Phi(x_1 \dots x_n, x_{n+1}) \wedge \Phi(x_1 \dots x_n, x'_{n+1})) \to x_{n+1} = x'_{n+1}$$

then $\Phi(x_1 \dots x_n, x_{n+1})) \to x_{n+1} \in \mathcal{P}^{\sigma_{n+1}+1}(v_1 \cup \dots v_n)$,

where

$$v_i =: \bigcup_{j=1 \dots n_i} \bigcup^{\sigma_{j,i}} x_i; \qquad i = 1, \dots n$$

Proof:

For every integer q we have $x \subseteq \mathcal{P}^q \circ \bigcup^q x$, and because $\mathcal{P}^q$ is $\subseteq$-increasing it will suffice to prove $u \subseteq v$ where See note 7

$$u =: \bigcup^{\sigma_{n+1}} x_{n+1} \quad \text{and} \quad v =: v_1 \cup \dots \cup v_n.$$

If $u \not\subseteq v$ then we can find a and b such that $a \in u$, $a \notin v$, and $b \notin u \cup v$. See note 8.

Now let F be the transposition (a, b) (that swaps a with b while fixing everything else). It is easy to check that F is admissible. F is the identity on v_i, so—by lemma 4 part (1)—we have

$$F^{[\sigma_{j,i}+1]}(x_i) = x_i \tag{1}$$

Further, $\bigcup^{\sigma_{n+1}} \circ F^{[\sigma_{n+1}+1]}(x_{n+1}) \neq \bigcup^{(\sigma_{n+1})}(x_{n+1}) = u$

On the one hand—by lemma 4 part (2)—we have

$$\bigcup^{(\sigma_{n+1})} \circ F^{[\sigma_{n+1}+1]}(x_{n+1}) = F^* \circ \bigcup^{(\sigma_{n+1})}(x_{n+1}) = F^*(u)$$

On the other hand $u \neq F^*(u)$ by construction of F. Therefore

$$F^{[\sigma_{n+1}]}(x_{n+1}) \neq x_{n+1} \tag{2}$$

Now, by corollary 7, we obtain

$$\phi(x_1 \dots x_n, x_{n+1}) \longleftrightarrow \phi_{\{n+1\}}(\dots F^{[\sigma_{j,i}+1]}(x_i) \dots F^{[\sigma_{n+1}+1]}(x_{n+1})).$$

Formula (1) now gives us

$$\phi(x_1 \dots x_n, x_{n+1}) \longleftrightarrow \phi(x_1 \dots x_n, F^{[\sigma_{n+1}+1]}(x_{n+1})).$$

so, by the hypothesis on ϕ, $x_{n+1} = F^{[\sigma_{n+1}]}(x_{n+1})$, giving a contradiction. ■ See note 3.

From lemma 8 we infer

Corollary 9 *In every extension of Z^- every set definable by a stratified formula without parameters is hereditarily finite.*

Corollary 10 *If the variable y is stratified in ϕ then* see note 4

$Z^- \vdash (\forall x)(\forall y)(\forall z)[\Phi(x,y) \wedge \Phi(x,y') \to y = y'] \to (\forall a)(\exists b)(\forall y)(y \in b \longleftrightarrow (\exists x \in a)\Phi(x,y))$

The axiom of replacement is therefore provable in Z^-: we use only the axiom of comprehension for Φ. See note 6.

Remark:

It is easy to find examples of pseudostratified formulæ for which replacement is not provable in Z^- + Axiom of foundation + Axiom of Choice. For example, the formula with two free variables x and y saying that x is a wellordering and y an isomorphic ordinal. See note 9.

References

[1] Coret, J. Comptes Rendues Acad. Sc. Paris, tome **264** série A, 1967, p 837.

Notes

Although in general I have adhered slavishly to Coret's notation, there are limits. Where he writes 'U' I write '$\bigcup$' since that is obviously what he meant. (Why did the *Comptes Rendues* use a printing house that lacked this symbol?) He also uses a large boldface '**C**' for complementation, which I have removed by rewriting formulæ so as not to need any notation for complementation.

I have exploited the LaTeXmachinery to provide numbers for all his definitions and lemmata, not just those that he numbers.

Coret always writes "integer" ("*entier*") even in circumstances where one would have expected him to consider only natural numbers. I have translated *entier* as *integer* throughout. I have inserted 'Proof' at the start of a proof and '■' at the end of a proof.

Note 1

In the later literature the word *setlike* is used for functions that have this nice property. Specifically a function f is 1-setlike if $f``x$ exists for all $x$. Coret presumably had in mind the observation that if $f$ is 1-setlike then so is $j(f) =: \lambda x.f``x$. This is because $\{f``y : y \in x\}$ is $\{z \in \mathcal{P}(f``\bigcup x) : (\exists w \in x)(z = f``w)\}$ and therefore exists by power set and separation applied to $f``\bigcup x$—which exists because f is 1-setlike.

A function is *setlike* if $j^n(f)$ is setlike for all n. What this argument shows is that any *definable* function is setlike. In ZF the axiom scheme of replacement makes it obvious that any definable function is setlike. What is striking about the result here is that it holds for Zermelo set theory as well.

Note 2

In the antecedent of the conditional of corollary 10 the 'Φ' is in lower case. This is presumably a missprint.

Note 3

The extra information in this corollary and the preceding definition is of no use, at least not here. It may be that this gadgetry is a good way to start thinking about degrees of dysstratification of formulæ but that is not our concern here. The point to notice for the moment is that we can prove lemma 8 by relettering followed by substitution. The meat of the construction is the transposition.

Note 4

By "y is stratified in Φ" Coret means that there is a stratification of ϕ which, although it might require distinct occurrenes of the other free variables to be given distinct types, nevertheless it treats 'y' as a bound variable, in that it gives all its occurrences the same type.

Note 5

Nowadays "pseudostratified" is universally called "weakly stratified".

Note 6

This is of course a slip of the pen. He means: the *stratified* instances.

Note 7

I think nowadays we would say *monotone* rather than *increasing*.

Note 8

This construction exploits a background assumption that there is no universal set.

Note 9

A more illuminating observation at this juncture might be that this proof works only for stratified *replacement*; Mathias' counterexample shows that stratified *collection* is not a theorem of Zermelo. ("For every $n \in \mathbb{N}$ there is a set of n infinite sets all of different sizes").