A Permutation Method Yielding Models of the Stratified Axioms of Zermelo Fraenkel Set Theory

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1 Introduction

This paper endeavours to survey some of the recent work investigating models of the stratified axioms of Zermelo-Fraenkel Set Theory (ZF). Specifically we will focus on a hybridisation of the Fraenkel-Mostowski-Specker and Rieger-Bernays permutation techniques developed by Thomas Forster to produce models of the stratified fragment of ZF refuting certain unstratified axioms. The major motivation is the hope that models of fragments of stratified comprehension will shed some light on the consistency strength of Quine's "New Foundations" (NF). The job of completely understanding and exploring generalisations of Forster's technique is nowhere near completion, by any stretch of the imagination. We will paint the partial picture representing our present understanding as an overture to discussing some of the important and interesting open problems in this research. Any results discussed in the paper that are attributed to an individual, but not referenced to a published text, have emerged in discussions on this subject held at Cambridge University.

2 Forster's Hereditarily Symmetric Sets

In [Forster 2003] Thomas Forster builds an inner model by inductively collecting sets fixed by the action of a group of finitely supported permutations of V_ω acting at some level down. He demonstrates that this class models the stratified axioms of Zermelo-Fraenkel set theory, axiomatised with replacement, while refuting the axiom of choice. It is important to note that when we restrict our attention to the stratified axioms of ZF set theory we need to be explicit about exactly which axiomatisation of this theory we are considering. While, in the presence of foundation and separation, full replacement is equivalent to collection, the restriction of our attention to stratified formulae renders the collection scheme strictly stronger than the replacement scheme. This can be seen by considerring Zermelo set theory (Z). An old result of Coret's [Coret 1970] demonstrates that Z proves every instance of stratified replacement, while Adrian Mathias [Mathias 2000] has observed that it is easy to find models of this theory in which stratified collection fails — consider for example a model that contains no infinite set of infinite sets each of different cardinality. In the other direction a simple application of stratified comprehension to the stratified collection schema yields stratified replacement. Later in this paper we will appeal to stratified collection in order to show that Forster's model interprets ZF set theory. For this reason we will prove that the models we are considering satisfy stratified collection wherever it is possible. In order to review Forster's construction and further examine some of the features of this model, we first need to recall some basic definitions. These definitions will be familiar to the reader acquainted with NF.

Definition 2.1 *We use ι to denote the operation of taking the singleton of a set.*

$$I.e.\ \iota x = \{x\}.$$

For a set x we also define

$$\mathcal{P}_1(x) = \{\iota y \mid y \in x\}.$$

Definition 2.2 *If ϕ is a formula in the language of set theory, then a **stratification** for ϕ is an assignment of natural numbers to variables such that if x and y are variables and $x \in y$ occurs in ϕ then x is assigned the natural number n if and only if y is assigned the natural number $n+1$, and if $x = y$ occurs in ϕ then x is assigned the natural number n if and only if y is assigned the natural number n. A sentence or formula, ϕ, in the language of set theory, that admits a stratification is said to be **stratified**. If $\phi(x_0, \ldots, x_{n-1})$ is a formula admitting assignment of natural numbers to variables that fails to be stratification only by virtue of the natural numbers assigned to some of the $x_0, \ldots, x_{n-1}$, then we say that ϕ is **weakly stratified**. By writing a function, $F(\vec{x})$, as $\phi(y, \vec{x})$ where ϕ is a formula such that $\forall \vec{x} \exists! y \phi(y, \vec{x})$, we can extend this notion of stratification and weak stratification to functions. It should be noted that in order for a function $F(\vec{x})$ to be considered to be weakly stratified the formula $\phi(y, \vec{x})$ witnessing that function must be weakly stratified with y receiving a well-defined type.*

For the purposes of this paper we will work in Zermelo-Fraenkel set theory making it explicit whenever we appeal to the Axiom of Choice. Being creatures of habit we have taken ordered pairs to be represented by the Wiener-Kuratowski ordered pair, that is to say $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$ [1]. If $\langle M, \in \rangle$ is some model of a set theory (it is at least a model of extensionality), then a permutation, σ, is a bijection from M onto M. The support of a permutation, σ, is the subset of M that is moved by σ. We often describe a permutation, σ, by only specifying σ restricted to its support.

Definition 2.3 *We define the operation j as $\lambda f \lambda x.(f``(x))$. That is to say, if σ is a permutation and x is a set, then $(j^n\sigma)(x)$ is the action of σ on the n^{th} level down in x.*

As we have mentioned, we will be interested in groups of permutations of sets. Following standard notion, if H and G are groups and H is a subgroup of G, then we will write $H \leq G$. If H is a normal subgroup of G, then we will write $H \trianglelefteq G$.

Definition 2.4 *If X is a class, then we will denote the full symmetric group of permutations of X by* **Symm**(X). *The group of finitely supported permutations of X will be denoted* **FSymm**(X).

Definition 2.5 *If A is a set of set permutations, then we define $\langle A \rangle$ to be the group generated by the elements of A. In direct defiance of a popular abuse of notation we will always include the poisson brackets between the generating elements and the angle brackets. This will make it clear when we are talking about a group and when we are talking about an ordered tuple.*

The heuristic motivation underlying the construction of Forster's inner model stems from the observation that stratified formulae defining sets are only able to specify properties 'at a finite number of levels down'. This is made explicit by Coret's Lemma:

Lemma 2.1 *(Coret's Lemma) [Coret 1964, Coret 1970] If Φ is a stratified formula, then for any permutation σ,*

$$\Phi(x_0, \ldots, x_{k-1}) \iff \Phi((j^{n_0}\sigma)x_0, \ldots, (j^{n_{k-1}})x_{k-1})$$

where the n_is are obtained from a stratification of Φ. $\square$

A consequence of this fact is that sets defined by stratified set abstracts will be fixed by $j^n\sigma$ for all permutations σ and sufficiently large n. This motivates the following definitions:

Definition 2.6 *Let x be a set. For a group of permutations, G, define*

$$G_k(x) = \{\sigma \in G \mid (j^n\sigma)(x) = x \textit{ for all } n \geq k\}.$$

[1] This specific coding was proposed by Kuratowski as a simplification of an earlier representation of ordered pairs developed by Wiener. Despite being the potential cause of confusion the name Wiener-Kuratowski ordered pair has become standard.

For the purposes of this section we will fix some $G \leq \mathbf{FSymm}(V_\omega)$.

Definition 2.7 *We call a set, x, k-symmetric if and only if $G_k(x) = G$. We say that x is **symmetric** if and only if there exists a $k \in \omega$ such that $G_k(x) = G$.*

Definition 2.8 *We define HS to be the class of all hereditarily symmetric sets.*

We will show that, working within ZF set theory, $\langle HS, \in \rangle$ is a model of the stratified axioms of ZF set theory.

Lemma 2.2 *If $\phi(y, \vec{x})$ is a formula witnessing a weakly stratified function and $\vec{s}$ are symmetric, then the unique t such that $\phi(t, \vec{s})$ holds is also symmetric.*

Proof This follows from Coret's Lemma. □

In [Forster 1995, p81ff] Thomas Forster exhibits a collection of fundamental operations such that any class closed under power set and finite compositions of these operations is a model of stratified Δ_0-separation. For brevity's sake we will omit the proof of this fact here.

Definition 2.9 *We define the following **rudimentary stratified functions**:*

$$J_1(R) = \{\langle \iota x, \iota y \rangle \mid \langle x, y \rangle \in R\}$$

$$J_2(x, y) = x \backslash y$$

$$J_3(x, y) = \{x, y\}$$

$$J_4(x) = \{\iota y \mid y \in x\}$$

$$J_5(x) = \bigcup x$$

$$J_6(X) = \{x \mid \langle x, y \rangle \in X\}$$

$$J_7(X, Y) = \{\langle x, y, z \rangle \mid \langle x, y \rangle \in X \wedge z \in Y\}$$

$$J_8(x) = \{\langle \iota u, v \rangle \mid u \in v \in x\}$$

$$J_9(x, y) = \{u \in y \mid x \in u\}$$

$$J_{10}(X, Y) = \{\langle x, y, z \rangle \mid \langle x, z \rangle \in X \wedge y \in Y\}$$

$$J_{11}(X) = \{\langle x, y \rangle \mid \langle y, x \rangle \in X\}$$

In should be noted that in order to ensure that J_7 and J_{10} are stratified functions, the definition of ordered triple needs to be homogeneous. To achieve this we replace the standard definition of ordered triple as nested Wiener-Kuratowski ordered pairs by the Hailperin ordered triple [Hailperin 1944]:

$$\langle x, y, z \rangle = \langle \iota^2 x, \langle y, z \rangle \rangle.$$

To extend Δ_0-stratified separation in $\langle HS, \in \rangle$ to full stratified separation we need the following consequences of our choice of G:

Lemma 2.3 *If x is a set of rank $\alpha \geq \omega$ then for all $\sigma \in G$ and $n \in \omega$, $(j^n\sigma)(x)$ has rank α.*

Proof The proof procedes by induction on rank. If x is a set of rank ω then for all $\sigma \in G$ and $n \in \omega$, $(j^n\sigma)(x)$ has rank ω since $G \leq \mathbf{FSymm}(V_\omega)$. Now, assume that α is the least ordinal greater than ω such that for some $n \in \omega$ and $\sigma \in G$, the rank of $(j^n\sigma)(x)$ differs from α. Since $\alpha > \omega$, we know that $n > 0$. Therefore there must be a $y \in x$ with rank $\beta \geq \omega$ such that $j^{n-1}\sigma$ changes that rank of y. Since $y \in x$, $\beta < \alpha$ which contradicts minimality. □

Lemma 2.4 *For all ordinals α, V_α is symmetric.*

Proof If $\alpha < \omega$, then for all $n \geq \alpha + 1$ and all $\sigma \in G$, $(j^n\sigma)(x) = x$ for every $x \in V_\alpha$. If $\alpha \geq \omega$, then Lemma 2.3 demonstrates that for all $\sigma \in G$ and $n \in \omega$, $(j^n\sigma)(V_\alpha) = V_\alpha$. □

Lemma 2.5 *For all $\sigma \in G$, $x \in HS$ and $n \in \omega$, $(j^n\sigma)(x) \in HS$.*

Proof Every $\sigma \in G$ has finite rank, so $\sigma \in HS$. Observing that the j operation can be defined by a stratified formula we can apply Lemma 2.2 to conclude that, for all $\sigma \in G$, $x \in HS$ and $n \in \omega$, $(j^n\sigma)(x) \in HS$. □

It should be noted that Lemma 2.5 relies heavily on the fact that G was chosen to be a subgroup of the group of finitely supported permutations of finite rank. Using this lemma we can show that every initial segment of HS is a set in HS.

Lemma 2.6 *For all ordinals α, $HS \cap V_\alpha \in HS$.*

Proof For $n < \omega$, V_n is finite with finite rank, and therefore $V_n \cap HS = V_n \in HS$. Now, assume $\alpha \geq \omega$ and let $x \in V_\alpha \cap HS$. Lemma 2.5 shows that for all $n \in \omega$ and $\sigma \in G$, $(j^n\sigma)(x) \in HS$. Lemma 2.3 demonstrates that for all $n \in \omega$ and $\sigma \in G$, $(j^n\sigma)(x) \in V_\alpha$. Therefore for all $n \in \omega$ and $\sigma \in G$, $(j^n\sigma)(V_\alpha \cap HS) = V_\alpha \cap HS$, and so $V_\alpha \cap HS \in HS$. □

It should be noted that the proof of Lemma 2.6 actually shows us that if α is infinite, then $V_\alpha \cap HS$ is 1-symmetric. These results allow us to prove the following:

Theorem 2.7 *$\langle HS, \in\rangle$ is a model of the stratified axioms of ZF.*

Proof The fact that $\langle HS, \in\rangle$ satisfies Extensionality follows from the fact that HS is transitive. Observing that HS is a subclass of the cumulative hierarchy immediately implies that $\langle HS, \in\rangle$ is well-founded.

$$\text{Consider } \mathbb{N} = \{V_\alpha \cap HS \mid \omega \leq \alpha < \omega + \omega\}.$$

Since each initial segment of HS is 1-symmetric, $\mathbb{N} \in HS$. Moreover, $\mathbb{N}$ is well-orderred by inclusion and demonstrates that HS satisfies the Axiom of Infinity.
We check that the Power set axiom holds. Let $x \in HS$. In V we know that $\mathcal{P}(x) \subseteq V_\alpha$ for some α. Therefore, using Lemma 2.2,

$$\mathcal{P}^{HS}(x) = \mathcal{P}(x) \cap V_\alpha \cap HS \in HS.$$

Routine checks reveal that $\langle HS, \in\rangle$ is closed under rudimentary stratified operations (Definition 2.9). This demonstrates that $\langle HS, \in\rangle$ satisfies the Axiom of Pairing, the Axiom of Union and Δ_0-Separation. Now, let $A \in HS$ and assume that $\phi(x, y, \vec{z})$ is a formula and $\vec{c} \in HS$. Assume that for all $y \in HS$, $\exists x \phi^{HS}(x, y, \vec{c})$.

$$\text{Let } \psi(x, y) \iff \phi^{HS}(x, y) \wedge x \in HS \wedge \text{ the rank of } x \text{ is minimal.}$$

Let X be such that, for all $y \in A$, there exists $x \in X$ such that $\psi(x, y)$. The existence of such a X is guaranteed by collection in the original model. Now $X \subseteq V_\alpha$ for some ordinal α. Therefore $V_\alpha \cap HS \in HS$ and, for all $y \in A$, there exists a $x \in V_\alpha \cap HS$ such that $\phi(x, y)$. Therefore $\langle HS, \in\rangle$ satisfies full collection. Specifically this demonstrates that $\langle HS, \in\rangle$ satisfies every stratified instance of collection and replacement. □

Forster has also shown that the Axiom of Choice fails badly in $\langle HS, \in\rangle$. Closer examination reveals that HS fails to linearly order any set unless that set is symmetric by virtue of the fact that all its elements are k-symmetric for some $k \in \omega$. A similar observation was made by Andre Pétry in the context of NF. In [Pétry 1974] Pétry demonstrates that if $F(x, y)$ is a stratified homogeneous function, then it cannot be a linear ordering of the universe.

Definition 2.10 *We say that a set x is **uniformly symmetric** if and only if there exists a $k \in \omega$ such that every $y \in x$ is k-symmetric.*

Theorem 2.8 *If x is not uniformly symmetric, then there is no linear ordering of x in $\langle HS, \in \rangle$.*

Proof Let x be a set in $\langle HS, \in \rangle$ that is not uniformly symmetric. Assume that $L \subseteq x \times x$ is a linear ordering of x in HS. Therefore, L is k-symmetric for some $k \in \omega$. Since x is not uniformly symmetric we can find a $m > k$, $y \in x$ and $\sigma \in G$ such that $(j^m\sigma)(y) \neq y$. Since σ is a permutation with finite support it can be written as a finite product of disjoint cycles. Therefore there exists an $n \in \omega$ such that σ^n is the identity. Since L is a linear ordering we know that either $\langle (j^m\sigma)(y), y \rangle \in L$ or $\langle y, (j^m\sigma)(y) \rangle \in L$. Assume firstly that $\langle y, (j^m\sigma)(y) \rangle \in L$. Since L is k-symmetric we can conclude that $\langle (j^m\sigma)(y), (j^m\sigma)(j^m\sigma)(y) \rangle = \langle (j^m\sigma)(y), (j^m\sigma^2)(y) \rangle \in L$, therefore by transitivity $\langle y, (j^m\sigma^2)(y) \rangle \in L$. Inductively applying this argument we can see that $\langle y, (j^m\sigma^{n-1})(y) \rangle = \langle y, (j^m\sigma^{-1})(y) \rangle \in L$. Therefore, applying $j^m\sigma$ again, we get $\langle (j^m\sigma)(y), y \rangle \in L$, which is a contradiction. A symmetrical argument shows that $\langle (j^m\sigma)(y), y \rangle \in L$ is also impossible. Therefore L cannot be a linear ordering. □

The proof of this strong result relies heavily on the fact that $G \leq \mathbf{FSymm}(V_\omega)$. Later in this paper we will generalise Forster's technique by relaxing this constraint on G. In order to anticipate these discussions we record here a slightly weaker result that does not make any reference to specific properties of the generating group.

Theorem 2.9 *If x is not uniformly symmetric, then there is no well-ordering of x in $\langle HS, \in \rangle$.*

Proof Assume that x is not uniformly symmetric. Let $R \subseteq x \times x$ be a well-ordering of x such that $R \in HS$. Therefore R is k-symmetric. Let $n \geq k$ and let $\sigma \in G$ be such that $(j^n\sigma)$ moves some $y \in x$. Let $z \in x$ be the R-minimal element of x moved by $(j^n\sigma)$. Therefore $\langle z, (j^n\sigma)(z) \rangle \in R$. Since R is k-symmetric, $(j^{n+3}\sigma^{-1})(R) = R$, so $\langle (j^n\sigma^{-1})(z), z \rangle \in R$. But this contradicts the minimality of z. □

The validity of the Axiom of Choice in the model from which we construct HS renders the relationship between uniformly symmetric sets and the existence of well orderings exact.

Corollary 2.10 *(Axiom of Choice) If the Axiom of Choice holds in the original model and x is a set in $\langle HS, \in \rangle$, then x can be well ordered if and only if x is uniformly symmetric.*

Proof Assume that the Axiom of Choice holds in the original model and $x \in HS$. Theorem 2.9 gives us the forward direction of the bi-conditional. Assume that, for all $y \in x$, y is k-symmetric. Using the Axiom of Choice let $R \subset x \times x$ be a well ordering of x. It is clear that R is $k+3$-symmetric. □

One major difference between ZF and set theories equipped with full stratified comprehension, such as NF, is the ability of the latter to speak coherently about very large sets such as the set of all ordinals (NO) or the set of all sets (V). These large sets display behaviour that contradicts some of the provable properties of sets in the domain of discourse of ZF set theory. For example, $V = \mathcal{P}(V)$ and so Cantor's Theorem fails. Closer inspection reveals that this breakdown is due to the failure of the unstratified assertion that a set is the same size as its set of singletons. In fact, in NF, V is too big to be the same size as *any* set of singletons. This has motivated the investigation of the status of these unstratified properties of sets in the model $\langle HS, \in \rangle$. As an *entrée* to these discussions we recall another definition familiar to the NFists:

Definition 2.11 *We say that a set x is **cantorian** if and only if there is a bijection between x and $\mathcal{P}_1(x)$. If the the restriction of ι to x is a set, then we call x **strongly cantorian**.*

In $\langle HS, \in \rangle$ there is an exact correspondence between uniformly symmetric and strongly cantorian sets.

Theorem 2.11 *(Forster) Let x be a set in $\langle HS, \in \rangle$; x is uniformly symmetric if and only if x is strongly cantorian.*

Proof Assume that x is uniformly symmetric with each $y \in x$ k-symmetric.

$$\text{Let } S = \{\langle y, \iota y\rangle \mid y \in x\}.$$

Let $\sigma \in G$ and let $n \in \omega$.

$$(j^{k+4+n}\sigma)(S) = \{\langle (j^{k+1+n}\sigma)(y), \iota(j^{k+n}\sigma)(y)\rangle \mid y \in x\} = S.$$

Therefore $S = \iota \restriction x$ is $k+4$-symmetric.
Conversely assume that x is strongly cantorian. Therefore $S = \iota \restriction x$ is k-symmetric for $k \in \omega$. Let $n \in \omega$ and $\sigma \in G$. Now, $(j^{k+4+n}\sigma)(S) = \{\langle (j^{k+1+n}\sigma)(y), \iota(j^{k+n}\sigma)(y)\rangle \mid y \in x\} = S$. This shows that for all $n \in \omega$ and $\sigma \in G$, if $y \in x$, then $(j^{k+n}\sigma)(y) = (j^{k+1+n}\sigma)(y)$. Now, assume that x is not uniformly symmetric. Let $n \in \omega$ and let $y \in x$ and $\sigma \in G$ be such that $(j^{k+n}\sigma)(y) \neq y$. Now y must itself be m-symmetric for some $m > k+n$. But, since $(j^{k+n}\sigma)(y) = (j^{k+1+n}\sigma)(y)$, we can prove by induction that $(j^m\sigma) \neq y$, which contradicts the fact that y is m-symmetric. Therefore x is uniformly symmetric. □

This raises the question: are there any sets in HS which behave like V in NF in the sense that they cannot be placed in bijection with any set of singletons?

Definition 2.12 ***IO*** *is the assertion that every set is the same size as a set of singletons.*

We will see in the proceding section that the validity of this principle allows the model $\langle HS, \in\rangle$ to interpret full unstratified ZF set theory. Another motivation for studying IO is that it represents a natural point of difference between ZF and Quine's NF. It is hoped that demonstrating the consistency of the negation of IO relative to the stratified fragment of ZF will give us some clue as to how to produce models of Quine's system. Nathan Bowler has shown that if we assume the Axiom of Choice in the original model, then IO holds in $\langle HS, \in\rangle$.

Definition 2.13 *Let H be a group of permutations. We will write $J_n(H)$ for the group generated by $\{j^k\sigma \mid \sigma \in H \wedge k \geq n\}$.*

Definition 2.14 *We define the* ***stabiliser*** *of a set x above $k \in \omega$ to be:*

$$\mathrm{stab}_k(x) = \{\sigma \in J_k(G) \mid \sigma(x) = x\}.$$

Lemma 2.12

$$\mathbf{FSymm}(V_\omega) \trianglelefteq J_0(\mathbf{FSymm}(V_\omega)).$$

Proof Let $\tau \in \mathbf{FSymm}(V_\omega)$ and let $\sigma \in J_0(\mathbf{FSymm}(V_\omega))$. Now, any thing that is not moved to within the range of τ by σ will be fixed by $\sigma^{-1}\tau\sigma$. Therefore $\sigma^{-1}\tau\sigma$ is a permutation with finite support. □

Theorem 2.13 *(Axiom of Choice) (Bowler)*
Every set in $\langle HS, \in\rangle$ is the same size as a set of singletons.

Proof Let $P = \{\langle \iota x, y\rangle \mid x \in y \in HS \cap V_\omega\}$.
We begin by claiming that $\mathrm{stab}_3(P) = J_4(G)$. Let $\sigma \in J_4(G)$. Therefore $\sigma = j^4\tau$ for some $\tau \in J_0(G)$. Now, $\sigma(P) = \{\langle \iota\tau(x), j\tau(y)\rangle \mid x \in y \in HS \cap V_\omega\} = P$ since $x \in y \iff \tau(x) \in j\tau(y)$.
Conversely, let $\sigma \in \mathrm{stab}_3(P)$. Therefore $\sigma = j^3\tau$ where $\tau \in J_0(G)$ and

$$\sigma(P) = \{\langle \tau(\iota x), \tau(y)\rangle \mid x \in y \in HS \cap V_\omega\} = P.$$

Assume that $\sigma \notin J_4(G)$. Therefore $\sigma = j^3\tau$ where $\tau \in J_0(G)$ and $\tau \notin J_1(G)$.

$$\text{Therefore } \tau = \prod_{0 \leq i \leq k} j^{m_i}\tau_i \text{ where each } \tau_i \in G \text{ and } m_i \in \omega.$$

Since $G \subseteq \mathbf{FSymm}(V_\omega)$ we can apply Lemma 2.12 to obtain:

$$\tau = \pi \cdot \prod_{1 \leq i \leq k} j^{m_i}\tau_i, \text{ where } \pi \in \mathbf{FSymm}(V_\omega) \text{ and } \tau_i \in G \text{ with } m_i > 0.$$

By our preceding observation this means that we can assume without loss of generality that $\tau \in G$ and τ is not the identity permutation. Now, let $z \in HS \cap V_\omega$ such that $\tau(z) \neq z$ and let $y \in z$ such that $y \notin \tau(z)$. Therefore $\tau(\iota y) \neq \iota y$, so say that $\tau(\iota y) = \iota w$. But this means that τ must move every $q \in HS \cap V_\omega$ such that $y \in q$ and $w \notin q$. But there are infinitely many such q, which contradicts the assumption that $\tau \in G$. Therefore $\sigma \in J_4(G)$, which completes the proof of the claim.

Specifically this demonstrates that $P \in HS$. We will show that this P can be used to construct sets in HS whose elements are the union of specific orbits of j^k"G for k large enough. We claim that, for a given $H \leq J_5(G)$, if there exists a $k \in \omega$ such that $J_{6+k}(G) \leq H$, then there exists an $R_H \in HS$ such that $\mathrm{stab}_5(R_H) = H$. To prove this we let $k \in \omega$ be such that $J_{6+k}(G) \leq H \leq J_5(G)$.

$$\text{Let } Q_n = \{\iota^{i}\text{`}P \mid i \leq n\}.$$

Now, by the preceding claim, $\mathrm{stab}_3(\iota^i P) \cap J_{3+i}(G) = J_{4+i}(G)$. Therefore,

$$\mathrm{stab}_4(Q_n) = j\text{``}\bigcap_{i \leq n} \mathrm{stab}_3(\iota^i P) = J_{5+n}(G).$$

Specifically this demonstrates that for all $n \in \omega$, $Q_n \in HS$. Now, let $R_H = \{\pi(Q_k) \mid j\pi \in H\}$. Since each $Q_k \in HS$, $R_H \subseteq HS$. Now we need to show that $\mathrm{stab}_5(R_H) = H$. Let $\sigma \in H$. Therefore $\sigma = j\tau$ where $\tau \in J_4(G)$. Now, $\sigma(R_H) = \{\tau\pi(Q_k) \mid j\pi \in H\} = R_H$. Conversely, let $\sigma \in \mathrm{stab}_5(R_H)$. Therefore $\sigma = j\tau$ where $\tau \in J_4(G)$, and $\sigma(R_H) = \{\tau\pi(Q_k) \mid j\pi \in H\}$. Since j of the identity is in H, there is a $j\pi \in H$ such that $\tau\pi(Q_k) = Q_k$. Therefore $\tau\pi \in \mathrm{stab}_4(Q_k) = J_{5+k}$. Since $J_{6+k} \leq H$, $j\tau j\pi \in H$. Therefore since $j\pi \in H$, $\sigma = j\tau \in H$. Therefore $\mathrm{stab}_5(R_H) = H$.

Note that since $\mathrm{stab}_5(R_H) = H \geq J_{6+k}(G)$ and $\{j^{6+k}\sigma \mid \sigma \in G\} \subseteq J_{6+k}(G)$, $G_{6+k}(R_H) = G$ and so $R_H \in HS$. And this completes the proof of the claim.

Now, let $X \in HS$. Therefore $G_n(X) = G$ for some $n > 7$. Now, let A be the set of orbits of $J_6(G)$ acting on X. Therefore,

$$A = \{A_y \mid y \in X\} \text{ where } A_y = \{\sigma(y) \mid \sigma \in J_6(G)\}.$$

Using the Axiom of Choice in the original model, index the set of orbits by ordinal numbers:

$$A = \{A_\alpha \mid \alpha \in \xi\}.$$

Using choice again, let $x_\alpha \in A_\alpha$ for each $\alpha \in \xi$. And let $l_\alpha = HS \cap V_{\omega+\alpha}$ for each $\alpha \in \xi$. Note that $G_1(l_\alpha) = G$ for each $\alpha \in \xi$.

$$\text{Let } y_\alpha = R_{\mathrm{stab}_6(x_\alpha)} \cup \iota l_\alpha \text{ for each } \alpha \in \xi.$$

Therefore $\mathrm{stab}_5(y_\alpha) = \mathrm{stab}_6(x_\alpha)$.

Let $Z = \{\iota y_\alpha \mid \alpha \in \xi\}$. Define $f : X \longrightarrow Z$ by $f(\sigma(x_\alpha)) = \sigma(\iota y_\alpha)$ for each $\sigma \in J_6(G)$. Since each x_α is a representative of an orbit, f is a well-defined function. Now, assume that $f(x) = f(y)$. Our construction of the y_αs clearly prohibits $\tau(\iota y_\alpha) = \sigma(\iota y_\beta)$ for $\tau, \sigma \in J_6(G)$ and $\beta \neq \alpha$. Therefore $\sigma(\iota y_\alpha) = \tau(\iota y_\alpha)$ and, $x = \tau(x_\alpha)$ and $y = \sigma(x_\alpha)$ for $\sigma, \tau \in J_6(G)$. Therefore $\tau^{-1}\sigma \in \mathrm{stab}_6(\iota y_\alpha)$. Therefore $\tau^{-1}\sigma \in \mathrm{stab}_6(x_\alpha)$, and so $x = y$. It is clear that f is surjective by definition. Now, let $\sigma \in J_9(G)$. Therefore $\sigma = j^3\tau$ for some $\tau \in J_6(G)$ and

$$\sigma(f) = \{\langle \tau\pi(x_\alpha), \tau\pi(\iota\text{`}y_\alpha)\rangle \mid (\alpha \in \xi) \wedge (\pi \in J_6(G))\} = f.$$

Therefore $f \in HS$ and f is a bijection between X and a set of singletons. □

This result demonstrates that assuming the Axiom of Choice in the original model places a limitation on the extent to which $\langle HS, \in \rangle$ can exhibit failures of unstratified properties of models of ZF. It is still an open question whether or not a failure of the Axiom of Choice in the original model can be turned into a failure of IO in $\langle HS, \in \rangle$.

We round off this section by making a few observations about the relationship between the class HS and the group used to generate this class.

Definition 2.15 *Let G be a group of set permutations. When ambiguity threatens we will write $HS[G]$ for the class of hereditarily symmetric sets generated by G.*

The following observations follow immediately from our construction of the class of hereditarily strongly symmetric sets.

Theorem 2.14 *Let $G \leq \mathbf{FSymm}(V_\omega)$.*

(i) If $H \leq G$, then $HS[G] \subseteq HS[H]$.

(ii) If X is a set (in the original model) and $G \leq \mathbf{FSymm}(V_\omega \backslash \mathrm{TC}(X))$, then $X \in HS[G]$ and X is uniformly symmetric.

□

3 Interpreting ZF in HS

Research being conducted by Vu Dang has shown that despite examples of the failure of unstratified separation being interspersed within the structure $\langle HS, \in \rangle$, this model is capable of interpreting full ZF. The idea is that it is possible to define a membership and equality relation on well-founded extensional relations equipped with a greatest element to make them look like sets. This idea was first employed by Roland Hinnion [Hinnion 1975] in an NF context to show that a strengthening of NF proves the consistency of Zermelo Set Theory.

We will work in a theory $\mathcal{T}$ whose axioms are the stratified axioms of ZF axiomatised using collection and including a stratified version of the axiom of infinity. We will use $\mathbb{N}$ to denote the object representing the set of natural numbers in our theory $\mathcal{T}$.

Definition 3.1 *Following [Hinnion 1975] we say that a relation $R \subseteq X \times X$ is a **BFEXT** if and only if R is a well-founded extensional relation with a greatest element. That is to say there exists $y \in X$ such that for all $z \in X$, there is an $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that $z = x_1$ and $y = x_n$ and $\langle x_i, x_{i+1} \rangle \in R$ for $1 \leq i \leq n-1$.* [2] *If R is a BFEXT, then the greatest element with respect to R is unique. In light of this we will use $\mathbb{1}_R$ to denote the unique greatest element of a BFEXT R* [3].

Definition 3.2 *Let $R \subseteq X \times X$ be a BFEXT. Define*

$$\mathrm{domain}(R) = \{x \in X \mid (\exists y \in X)(x, y) \in R \vee (\exists y \in X)(y, x) \in R\}.$$

Definition 3.3 *We use* BF *to denote the class of all BFEXTs.*

Definition 3.4 *We define **Weak IO** to be the assertion that every set equipped with a well-founded extensional relation is the same size as a set of singletons.*

The discussions in the previous section show that $\langle HS, \in \rangle \models \mathcal{T} + \text{Weak IO}$. We start by making the following observations within $\mathcal{T}$.

[2] In [Hinnion 1975] a BFEXT is defined as a well-founded extensional relation. In this paper we will only have cause to consider relations endowed with a maximal element. For this reason we have added this condition to the definition of BFEXT.

[3] We should comment that this is a slight departure from the notation in [Hinnion 1975] where ω_R is used to denote the greatest element.

Lemma Scheme 3.1 *(For the theory $\mathcal{T}$ + Weak IO) For each standard natural number* $\mathbf{n}$*, if* $R \subseteq X \times X$ *is a well-founded extensional relation, then* X *is the same size as a set of the form* $\iota^{\mathbf{n}}``Y$. □

Lemma 3.2 *(In the theory $\mathcal{T}$) If* $R \subseteq X \times X$ *is a well-founded relation, then there exists a transitive well-founded relation* $R^\star \supseteq R$ *such that* $R^\star$ *is the unique minimal transitive well-founded relation containing* R. □

We are now in a position to define a membership relation on BF.

Definition 3.5 *Let* $R \subseteq X \times X$ *be a well-founded relation and* $x \in X$*, define* $R \upharpoonright x$ *to be the restriction of* R *to* $\{y \mid \langle y, x\rangle \in R^\star\} \cup \{x\}$.

Definition 3.6 *For all* $R, S \in \mathrm{BF}$*, we say that* $R \in_{\mathrm{BF}} S$ *if and only if* $R \cong S \upharpoonright y$ *for some* y *directly below* $\mathbb{1}_S$*. For all* $R, S \in \mathrm{BF}$*, we say that* $R =_{\mathrm{BF}} S$ *if and only if* $R \cong S$.

In [Hinnion 1975, pp15ff] Roland Hinnion made the following observation about formulae expressed in the language of $\langle \mathrm{BF}, \in_{\mathrm{BF}}\rangle$.

Lemma 3.3 *If* $\phi(\vec{x})$ *is a formula expressed in the language* $\mathcal{L}(\in_{\mathrm{BF}}, =_{\mathrm{BF}})$ *with* $\vec{x}$ *taken from* BF*, then the translation of* ϕ *into a formula of the language of set theory* ($\mathcal{L}(\in)$) *is a stratified formula.* □

Lemma 3.4 *(In the theory $\mathcal{T}$) If* $R \subseteq X \times X$ *is a BFEXT and* $x, y \in X$ *with* $R \upharpoonright x \cong R \upharpoonright y$*, then* $x = y$.

Proof Let $f : R \upharpoonright x \cong R \upharpoonright y$ and consider $Z = \{z \in \mathrm{dom}(R \upharpoonright x) \mid f(z) \neq z\}$. It is clear that Z is a set by stratified separation. Assume that $Z \neq \emptyset$. Let $t \in Z$ be an R-minimal element of Z. For all $\langle p, t\rangle \in R$, $p \in \mathrm{dom}(R \upharpoonright x)$. Therefore, $p = f(p)$ and $\langle p, f(t)\rangle \in R$. Conversely, for all $\langle p, f(t)\rangle \in R$, $p \in \mathrm{dom}(R \upharpoonright y)$. Therefore, $p = f^{-1}(p)$ and $\langle p, t\rangle \in R$. Therefore by extensionality $t = f(t)$ which is a contradiction. □

Lemma 3.5 *(In the theory $\mathcal{T}$ + Weak IO) (Dang) If* $R \subseteq X \times X$ *is a well-founded relation with top element* $\mathbb{1}_R \in X$*, then there is a BFEXT* $S \subseteq Y \times Y$ *and a surjection* $\pi : X \longrightarrow Y$ *with the following properties:*

(i) if $\langle x, y\rangle \in R$*, then* $\langle \pi(x), \pi(y)\rangle \in S$*,*

(ii) if $\langle p, q\rangle \in S$*, the there exists* $\langle x, y\rangle \in R$ *such that* $\pi(x) = p$ *and* $\pi(y) = q$*,*

(iii) $\pi(\mathbb{1}_R) = \mathbb{1}_S$*,*

(iv) if $Z \subseteq X$ *satisfies:*

$$(x \in Z) \wedge (\langle y, x\rangle \in R) \Rightarrow (y \in Z)$$

and R *restricted to* Z *is extensional, then* π *restricted to* Z *is an isomorphism.*

Proof Let $R \subseteq X \times X$ be a well-founded relation with a top element $\mathbb{1}_R$. We define a relation $\sim$ on $X \times X$ by well-founded induction such that for all $x, y \in X$, $\langle x, y\rangle \in\sim$ if and only if

(i) if $\langle z, x\rangle \in R$, then $(\exists w \in X)(\langle z, w\rangle \in\sim) \wedge (\langle w, y\rangle \in R)$,

(ii) if $\langle z, y\rangle \in R$, then $(\exists w \in X)(\langle w, z\rangle \in\sim) \wedge (\langle w, x\rangle \in R)$.

This relation is an equivalence relation on domain(R) with the property that R modulo $\sim$ is an extensional relation. Now, using stratified separation we define the set $A \subseteq \mathcal{P}(X)$ of all $\sim$-equivalence classes. Let $\theta : A \longrightarrow \iota``Y$ be a bijection between A and a set of singletons.

$$\text{Let } \pi = \{\langle x, y\rangle \in X \times Y \mid (\exists a \in A)(x \in a \wedge y \in \theta(a))\}.$$

Define $S \subseteq Y \times Y$ such that:

$$\langle x, y\rangle \in S \text{ if and only if } (\exists \langle z, w\rangle \in R)(z \in \theta^{-1}(\iota x) \wedge w \in \theta^{-1}(\iota y)).$$

Routine checks demonstrate that π is the surjection satisfying the required properties. □

Theorem 3.6 *(In the theory $\mathcal{T}$ + Weak IO) (Dang-Forster)* $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$ *is a model of ZF Set Theory.*

Proof Let $R \subseteq X \times X$ and $S \subseteq Y \times Y$ be BFEXTs with the same members with respect to the $\in_{\mathrm{BF}}$ relation. For each $x \in X$ directly below $\mathbb{1}_R$ and $y \in Y$ directly below $\mathbb{1}_S$ define $F_{x,y}$ to be the set of all isomorphisms between $R \restriction x$ and $S \restriction y$. The existence of these sets is guaranteed by stratified separation in $\mathcal{T}$. Let F be the union of all $F_{x,y}$ for $x \in X$ directly below $\mathbb{1}_R$ and $y \in Y$ directly below $\mathbb{1}_S$. Lemma 3.4 demonstrates that F is a well-defined structure preserving function which can be extended to an isomorphism between $\langle X, R \rangle$ and $\langle Y, S \rangle$. This shows that the Axiom of Extensionality holds in $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$.

To show that $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$ satisfies Foundation let $R \subseteq X \times X$ be a BFEXT and let $x \in X$ be an R-minimal element of R^{-1}“$\{\mathbb{1}_R\}$. Assume that there is a BFEXT $S \subseteq Y \times Y$ isomorphic to both $R \restriction z$ and $R \restriction y$ for $z \in X$ directly below $\mathbb{1}_R$ and $y \in X$ directly below x with respect to R. By Lemma 3.4 this implies $z = y$, which contradicts our choice of x.

The Axiom of Union follows from the observation that this operation, when translated into the language of $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$, corresponds to operations on BFEXTs that can be performed with stratified functions and preserve BFEXTness.

Now, let $R \subseteq X \times X$ and $S \subseteq Y \times Y$ be BFEXTs. Let $z \notin \mathrm{domain}(R) \cup \mathrm{domain}(S)$. Let $Q = R \cup S \cup \{\langle z, \mathbb{1}_R \rangle, \langle z, \mathbb{1}_S \rangle\}$. It is clear that $Q \subseteq (X \cup Y \cup \iota z) \times (X \cup Y \cup \iota z)$ is a well-founded relation, therefore we can use Lemma 3.5 to produce a BFEXT witnessing the fact that the Axiom of pairing holds in $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$.

We now turn to the Power set Axiom. Let $R \subseteq X \times X$ be a BFEXT. Let $P = \mathcal{P}(R^{-1}\{\mathbb{1}_R\})$. Using Lemma 3.1 we can construct a Q such that $\mu : P \longrightarrow \iota$“$Q$ is a bijection and $Q \cap X = \emptyset$. Let $Y = X \cup Q$ and define $S \subseteq Y \times Y$ by:

$$\langle a, b \rangle \in S \iff a \in Q \wedge b = \mathbb{1}_R \text{ or } b \neq \mathbb{1}_R \wedge \langle a, b \rangle \in R \text{ or } a \in \mu^{-1}(\iota\text{‘}b).$$

A routine examination reveals that applying Lemma 3.5 to $S \subseteq Y \times Y$ yields a BFEXT representing the Power set of $R \subseteq X \times X$ in $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$.

We now show that Collection holds in $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$. Let ϕ be a formula expressed in the language of $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$ such that $(\forall P \in_{\mathrm{BF}} R)(\exists Q)\phi(R, Q)$. By Lemma 3.3, the translation of ϕ into the language of set theory yields a stratified formula. Therefore by stratified collection, there exists a set A such that $\forall P \in_{\mathrm{BF}} R \exists Q \in A \phi(R, Q)$. Using a similar technique to that which we used to prove the Power set axiom we can glue together the elements of A to form a BFEXT Q' witnessing that $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$ satisfies collection for the formula ϕ.

To see that the axiom of infinity is satisfied one only has to observe that an infinite well-order can be used to produce a BFEXT corresponding to ω in $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$. □

In the previous section we saw how permutation methods can be applied to a model $\langle M, \in \rangle$ of ZF Set Theory to yield an inner model $\langle HS, \in \rangle$ of the stratified axioms of ZF. Vu Dang's consideration of the class of isomorphism types of BFEXTs demonstrates that there is an interpretation of full ZF Set Theory inside the weaker fragment of ZF modeled by $\langle HS, \in \rangle$. This leads us to the question: what is the relationship between the model $\langle M, \in \rangle$ and the model $\langle \mathrm{BF}, \in_{\mathrm{BF}} \rangle$ inside $\langle HS, \in \rangle$?

Lemma 3.7 *(Dang) If R is a BFEXT in $\langle HS, \in \rangle$, then R and* $\mathrm{domain}(R)$ *are uniformly symmetric.*

Proof Let $R \in HS$ be a BFEXT. It is clear that $\mathrm{domain}(R) \in HS$ since it is defined by a stratified formula. Assume that $\mathrm{domain}(R)$ is n symmetric for $n \in \omega$. Assume that domain(R) is not uniformly symmetric. Therefore there is an $m \geq n$ and $\sigma \in G$ such that $j^m\sigma$ moves an element of domain(R). Consider $A = \{x \in \text{domain}(R) \mid (j^m\sigma)(x) \neq x\}$. Since the action of $j^m\sigma$ on x can be described by a stratified formula, A is a set in HS. Let $y \in A$ be an R-minimal element. If $\langle z, y\rangle \in R$, then $z = (j^m\sigma)(z)$. And $\langle (j^m\sigma)(z), (j^m\sigma)(y)\rangle \in R$, since R is n symmetric. Therefore $\langle z, (j^m\sigma)(y)\rangle \in R$. Conversely, if we assume that $\langle z, (j^m\sigma)(y)\rangle \in R$, then we can apply $j^m\sigma^{-1}$ to show that $\langle z, y\rangle \in R$. Therefore since R is extensional $y = (j^m\sigma)(y)$, which contradicts our assumption. Therefore domain(R) and R are uniformly symmetric. □

Theorem 3.8 *(Dang) If $R, S \in HS$ are BFEXTs in $\langle HS, \in\rangle$, then R and S are BFEXTs in $\langle M, \in\rangle$. Moreover if $f \in M$ is an isomorphism between R and S, then $f \in HS$.*

Proof Let $R \in HS$ be a BFEXT. The assertion that R is extensional is Δ_0, therefore R is extensional in $\langle M, \in\rangle$. Assume that R is not well-founded in $\langle M, \in\rangle$. Let $A \subseteq R$ with $A \in M$ be a set with no R-minimal element. By Lemma 3.7, R is uniformly symmetric in $\langle HS, \in\rangle$, which implies that $A \in HS$. This contradicts the fact that R is well-founded in $\langle HS, \in\rangle$. Therefore R is also a BFEXT in $\langle M, \in\rangle$ and $\text{BFEXT}^{HS} \subseteq \text{BFEXT}^{M}$.
Let $R, S \in \text{BFEXT}^{HS}$. Let $f \in M$ be an isomorphism between R and S. Lemma 3.7 shows that domain(R) and domain(S) are both uniformly symmetric. Therefore $f \in HS$. □

Definition 3.7 *Let $R \subseteq X \times X$ be well-founded relation. Define the **Mostowski Collapse** of R by well-founded induction: for all $x \in X$,*

$$\mathbb{C}(x) = \{\mathbb{C}(z) \mid \langle z, x\rangle \in R\}.$$

Moreover if R is extensional, then the mapping $\mathbb{C}$ is injective.

We recall the following observation of Mostowski:

Theorem 3.9 *(Mostowski Collapsing Theorem)*
If $R \subseteq X \times X$ is a well-founded extensional relation, then there is a transitive class N such that $\mathbb{C}$ is an isomorphism between $\langle N, \in\rangle$ and $\langle X, R\rangle$. □[4]

By considering the Mostowski collapse of top elements in BFEXTs we can embed $\langle \text{BF}, \in_{\text{BF}}\rangle$ into our original model $\langle M, \in\rangle$.

Theorem 3.10 *If $R \subseteq X \times X$ and $S \subseteq Y \times Y$ are BFEXTs, then*

(i) $R \cong S$ if and only if $\mathbb{C}(\mathbb{1}_R) = \mathbb{C}(\mathbb{1}_S)$,

(ii) $R \in_{\text{BF}} S$ if and only if $\mathbb{C}(\mathbb{1}_R) \in \mathbb{C}(\mathbb{1}_S)$.

Proof Let $R \subseteq X \times X$ and $S \subseteq Y \times Y$ be BFEXTs. To see that (i) holds observes that by Theorem 3.9, $R \cong S$ if and only if there is an $\in$-isomorphism between the Mostowski Collapse of R and the Mostowski Collapse of S. Well-founded induction shows that this can happen if and only if $\mathbb{C}(\mathbb{1}_R) = \mathbb{C}(\mathbb{1}_S)$.
To see that (ii) holds observe that $R \in_{\text{BF}} S$ if and only if $R \cong S \restriction x$ for $x \in Y$ with $\langle x, \mathbb{1}_S\rangle \in S$ if and only if $\mathbb{C}(\mathbb{1}_R) \in \mathbb{C}(\mathbb{1}_S)$. □

Assuming that the Axiom of Choice holds in the original model $\langle M, \in\rangle$, Vu Dang has shown that this embedding is in fact an isomorphism.

Theorem 3.11 *(Axiom of Choice) (Dang)*
For each $x \in M$, there is a BFEXT R in BF^{HS} such that $\mathbb{C}(\mathbb{1}_R) = x$.

[4] A good discussion of this well known result can be found can be found in [Jech 2000, p69ff].

Proof Let TC(x) denote the transitive closure of x in $\langle M, \in\rangle$. Using the Axiom of Choice in $\langle M, \in\rangle$ we can index the elements of TC(ιx) by ordinals:

$$\mathrm{TC}(\iota x) = \{y_\xi \mid \xi \in |\mathrm{TC}(\iota x)|\}.$$

Let $X = \{HS \cap V_{\omega+\xi} \mid \xi \in |\mathrm{TC}(x)| + 1\}$ and define $R \subseteq X \times X$ by

$$R = \{\langle HS \cap V_{\omega+\nu}, HS \cap V_{\omega+\xi}\rangle \mid y_\nu \in y_\xi\}.$$

It is clear that $R \in HS$ and that R is a BFEXT with $\mathbb{1}_R = x$ and $\mathbb{C}(\mathbb{1}_R) = x$. □

Theorem 3.12 *(Axiom of Choice) (Dang)*
Let $\phi(x_1, \ldots, x_n)$ be a formula in the language of set theory. For all $R_1, \ldots R_n \in \mathrm{BF}^{HS}$,

$$\langle \mathrm{BF}^{HS}, \in_{\mathrm{BF}}\rangle \models \phi(R_1, \ldots, R_n) \textit{ if and only if } \langle M, \in\rangle \models \phi(\mathbb{C}(\mathbb{1}_{R_1}), \ldots, \mathbb{C}(\mathbb{1}_{R_n})).$$

Proof The proof is by induction on the complexity of ϕ. Theorem 3.10 shows that the Theorem holds for atomic formulae and the negations of atomic formulae. This easily extends to disjunctions of atomics and their negations. To see that the theorem holds for ϕ in the form $\exists y \psi(\vec{x}, y)$ we apply Theorem 3.11 to produce canonical representatives of isomorphism classes in BF^{HS} corresponding to sets witnessing the validity of ϕ in M. □

Even without the Axiom of Choice in $\langle M, \in\rangle$ we can see that $\langle \mathrm{BF}^{HS}, \in_{\mathrm{BF}}\rangle$ is isomorphic to a transitive submodel of $\langle M, \in\rangle$, however very little is known about what properties, if any, a breakdown of Choice in M would endow upon this transitive submodel.

4 The Constructible Symmetric Sets

In [Forster 2003] Forster presents another model of the stratified axioms of ZF. He constructs the model S by forming the smallest subclass of V closed under the rudimentary operations outlined in Definition 2.9.

Definition 4.1 *Let X be a set. Define $\mathrm{Cl}(X)$ to be the closure of X under rudimentary stratified operations (Definition 2.9).*

Definition 4.2 *Define by recursion:*

$$S_0 = \emptyset$$
$$S_{\alpha+1} = \mathrm{Cl}(S_\alpha \cup \iota(S_\alpha))$$
$$S_\alpha = \bigcup_{\beta<\alpha} S_\beta \textit{ for } \alpha \textit{ a limit ordinal.}$$

We first note that S is a subclass of HS.

Theorem 4.1 *For all ordinals α, every $x \in S_\alpha$ is strongly symmetric.*

Proof This follows by transfinite induction. The key observations are that each member of $S_{\alpha+1}$ is the result of a stratified function applied to the elements of S_α, and for all $\sigma \in G$ and $n \in \omega$, if x is symmetric, then so is $(j^n\sigma)(x)$. The induction step then follows by applying Lemma 2.2. □

Observing that, by definition, S is a transitive well-founded class that is closed under power set and rudimentary stratified operations demonstrates that it models the axioms of Extensionality, Pairing, Union, Foundation and Stratified Δ_0-Separation. The fact that for all α, $S_\alpha \in S$ and for all $\alpha \geq \omega$, S_α is 1-symmetric, allow us to prove Infinity and full Stratified Replacement in $\langle S, \in\rangle$. Notice that while S has a canonical well-ordering in V, this well-ordering is defined by an unstratified formula, a fact that prevents this ordering from existing in S. To see this, observe that $V_\omega \in S$ and $S \subseteq HS$ so there is no well-ordering of V_ω in S.

Theorem 4.2 *Let L denote Gödel's constructible universe.*

$$S \subseteq L.$$

Proof This follows from the observation that S can be be constructed from L and the the contruction of S is absolute between transitive models of ZF. □

Thomas Forster has observed that if V differs from L, then S differs from HS.

Theorem 4.3 *If $V \neq L$, then $S \neq HS$.*

Proof Assume $V \neq L$ and let A be a non-constructible set of infinite ordinals. Consider

$$X = \{S_\alpha \mid \alpha \in A\}.$$

It is clear that $X \in HS$ and $X \notin S$. □

Theorem 2.14 demonstrates that it is easy to exhibit a generating group that endows HS with sets that are not definable by stratified formulae. For example if $G \leq \mathbf{FSymm}(V_\omega)\backslash\omega$, then $\omega \in HS$. This leads us to ask: if X is definable and $X \in HS$ for every $G \leq \mathbf{FSymm}(V_\omega)$, then is $X \in S$? Using Theorem 2.14 again, and assuming that $V = L$, we can see that this is equivalent to asking whether there is an $X \in HS[\mathbf{FSymm}(V_\omega)]$ not definable by a stratified formula. We will prove, using the techniques that we develop in the next section, that this is indeed the case. By restricting our attention to groups of permutations of V_ω, we always incorporate sets not definable by a stratified formula, whose elements are exclusively sets of high rank, into the class HS.

5 Generalisations of HS

The search for classes modeling fragments of stratified comprehension while refuting features of models of ZF incompatible with full stratified comprehension, such as IO, has led to the investigation of generalisations of Forster's construction.

For the purposes of this section we will relax the constraint that $G \leq \mathbf{FSymm}(V_\omega)$. We only require that G be a set in the original model.

Definition 5.1 *Let G be a group. A **Filter** on G is a set, $\mathcal{F}$, of subgroups of G such that:*

(i) if $H, K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$,

(ii) if $H \in \mathcal{F}$ and $H \leq K$, then $K \in \mathcal{F}$,

(iii) $G \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.

Using this notion we generalise definitions 2.7 and 2.8.

Definition 5.2 *Let G be a group of set permutations and $\mathcal{F}$ a filter on G. We call a set, x, k-symmetric if and only if $G_k(x) \in \mathcal{F}$. We say that x is **strongly symmetric** if and only if there exists a $k \in \omega$ such that $G_k(x) \in \mathcal{F}$. We say that x is **weakly symmetric** if and only if*

$$\bigcup_{k \in \omega} G_k(x) \in \mathcal{F}.$$

Definition 5.3 *We define $HS_{\mathcal{F}}$ to be the class of all hereditarily strongly symmetric sets. And similarly we define $WHS_{\mathcal{F}}$ to be the class of all hereditarily weakly symmetric sets.*

We will show that given a specific closure condition on G both $HS_{\mathcal{F}}$ and $WHS_{\mathcal{F}}$ model the stratified axioms of ZF. We start by making the observation that weakly stratified functions preserve weak and strong symmetry.

Lemma 5.1 *If $\phi(y, \vec{x})$ is a formula witnessing a weakly stratified function and $\vec{s}$ are weakly/ strongly symmetric, then the unique t such that $\phi(t, \vec{s})$ holds is also weakly/strongly symmetric.* $\Box$

Without any conditions being placed on the group of permutations, G, both $\langle HS_{\mathcal{F}}, \in\rangle$ and $\langle WHS_{\mathcal{F}}, \in\rangle$ satisfy all of the axioms of Kaye-Forster Set Theory (KF), introduced in [Forster and Kaye 1991], minus the power set axiom.

Theorem 5.2 *Both $\langle HS_{\mathcal{F}}, \in\rangle$ and $\langle WHS_{\mathcal{F}}, \in\rangle$ satisfy Extensionality, Pairing, Union and Δ_0-stratified Separation.*

Proof The fact that $\langle HS_{\mathcal{F}}, \in\rangle$ and $\langle WHS_{\mathcal{F}}, \in\rangle$ satisfy Extensionality follows from the fact that they are transitive classes. Observing that both $HS_{\mathcal{F}}$ and $WHS_{\mathcal{F}}$ are closed under rudimentary stratified operations (Definition 2.9) allows us to see that $\langle HS_{\mathcal{F}}, \in\rangle$ and $\langle WHS_{\mathcal{F}}, \in\rangle$ satisfy Pairing, Union and Δ_0-stratified Separation. $\Box$

In order to get stratified collection, infinity and the power set axiom to hold in these models it appears that G needs to be chosen so that an analogue of Lemma 2.5 is satisfied.

Definition 5.4 *We say that $\langle G, \mathcal{F}\rangle$ satisfies the **strong closure condition** if and only if there exists a $H \in \mathcal{F}$ such that for all $\sigma \in H$, $x \in HS_{\mathcal{F}}$ and $n \in \omega$, $(j^n\sigma)(x) \in HS_{\mathcal{F}}$. Similarly, we say that $\langle G, \mathcal{F}\rangle$ satisfies the **weak closure condition** if and only if there exists a $H \in \mathcal{F}$ such that for all $\sigma \in H$, $x \in WHS_{\mathcal{F}}$ and $n \in \omega$, $(j^n\sigma)(x) \in WHS_{\mathcal{F}}$.*

Theorem 5.3 *If $\langle G, \mathcal{F}\rangle$ satisfies the strong closure condition, then $\langle HS_{\mathcal{F}}, \in\rangle$ satisfies the Power set axiom.*

Proof Let x be a set in $HS_{\mathcal{F}}$ such that $G_n(x) \in \mathcal{F}$. Now,

$$\mathcal{P}^{HS_{\mathcal{F}}}(x) = \{y \in HS_{\mathcal{F}} \mid y \subseteq x\}.$$

For each $\sigma \in G_n(x)$,

$$(j^{n+1}\sigma)(\mathcal{P}^{HS_{\mathcal{F}}}(x)) = \{(j^n\sigma)(y) \mid y \subseteq x\}.$$

But, if $y \subseteq x$, then $(j^n\sigma)(y) \subseteq (j^n\sigma)(x) = x$. Therefore $G_{n+1}(\mathcal{P}^{HS_{\mathcal{F}}}(x)) \supseteq G_n(x)$, and so $\mathcal{P}^{HS_{\mathcal{F}}}(x) \in HS_{\mathcal{F}}$. $\Box$

It is easy to see that a slight modification of this proof yields the identical result for $\langle WHS_{\mathcal{F}}, \in\rangle$ if $\langle G, \mathcal{F}\rangle$ satisfies the weak closure condition.

Now, since G is a set, $G \in V_\alpha$ for some ordinal α. It follows that if $\langle G, \mathcal{F}\rangle$ satisfies the strong (or the weak) closure conditions and $\lambda \geq \alpha$ is the next limit ordinal, then $V_\beta \cap HS_{\mathcal{F}} \in HS_{\mathcal{F}}$ (or $V_\beta \cap WHS_{\mathcal{F}} \in WHS_{\mathcal{F}}$) for all $\beta \geq \lambda$. Moreover, these initial segments will be 1-symmetric, which allows us to construct well ordered sets in $\langle HS_{\mathcal{F}}, \in\rangle$ and $\langle WHS_{\mathcal{F}}, \in\rangle$ with no greatest member. Slight modifications of the proof of Theorem 2.7 also reveal that both closure conditions imply stratified collection in the respective generalised models.

As an example we present an answer to the question asked at the end of the previous section. Consider the analogue of $\omega \backslash \iota\emptyset$ defined as follows:

$$A_1 = \iota V_\omega,$$

$$A_{n+1} = \iota A_n \cup A_n,$$

$$\mathcal{A} = \{A_i \mid 1 \leq i < \omega\}.$$

Theorem 5.4

$$HS[\mathbf{FSymm}(V_\omega)] \neq S$$

Proof

$$\text{Consider } G = \langle(\iota^2 V_\omega, \{\iota V_\omega, V_\omega\})\rangle \leq \mathbf{FSymm}(V_{\omega+3})$$
$$\text{and } \mathcal{F} = \{G\}.$$

Observe that $V_\omega \in HS_\mathcal{F}$, and G can be defined using a stratified formula from V_ω. Therefore $HS_\mathcal{F}$ satisfies the strong closure condition. Moreover, this also demonstrates that $V_\omega \in HS_\mathcal{F}$ and for all ordinals α, $S_\alpha \in HS_\mathcal{F}$. Therefore $S \subseteq HS_\mathcal{F}$. But $\mathcal{A} \in HS[\mathbf{FSymm}(V_\omega)]$ and $\mathcal{A} \notin HS_\mathcal{F}$. Therefore $HS[\mathbf{FSymm}(V_\omega)] \neq S$. □

In light of our discussions in this section, this result is no surprise. It simply demonstrates that, by restricting our attention— as we did in the first section — to a generating group that only move elements of V_ω, we allow into the class HS sets that are not extensions of any stratified formula.

There are still many open questions relating to the properties of these generalised permutation models. Perhaps one of the most pressing is: does IO hold in all these models? A slight modification of the proof of theorem 2.13 shows that IO holds in every model $HS_\mathcal{F}$ where $\mathcal{F}$ is a principle filter over $G \leq \mathbf{FSymm}(V_\alpha)$ for some ordinal α. Is it possible to make IO fail by using a G built from permutations that are not of finite support, or by using a non-principal filter over G?

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