NF_3 + AxInf is equivalent to Second-Order Arithmetic
Jean-François Pabion;
and
The 3-stratifiable theorems of NF_3
by
Boffa and Crabbé

Rendered into English by Thomas Forster

August 8, 2023

There are no footnotes in either of the original texts, so all the footnotes here are comments from the translator.

1 NF_3 + AxInf is equivalent to Second-Order Arithmetic Jean-François Pabion

The text was supplied to the translator by Marcel Crabbé, who fortunately had kept a photocopy, and he has helped greatly with some details of the translation. Crabbé's English is better than my French(!). This is a fairly free translation.
Boffa thought highly of this paper, and told everyone to read it.

A note by Jean-François Pabion, presented by Gustave Choquet.
Comptes Rendus Acad. Sci. Paris **290** (30/vi/80). Sér. A—1117
Submitted 2nd June 1980, accepted 16th June 1980.

TST_3 + AxInf is the theory TST of simply typed set theory with three levels augmented by the axiom of Infinity. Boffa has shown that TST_3 + AxInf interprets elementary arithmetic. We show that this interpretation extends to a conservative interpretation[1] of second-order arithmetic PA_2.

[1] 'Conservative' here represents an attempt to translate 'conservatrice'. I am not yet clear what is going on, tho' i *think* that what is being claimed is that TST_3 + AxInf does not prove any more arithmetic than is in the range of the interpretation we are about to see. If one could only state this properly it would probably become obvious!

1.1 Interpretation of PA_2 in TST_3 + AxInf

TST_3 is the theory TST of simply typed set theory with three levels, with extensionality and comprehension. It proves the existence of a set F_2 whose members are the finite sets of elements of level 0. This enables us to formulate an axiom of infinity:

$$(\exists x_1)(x_1 \notin F_2) \tag{I}$$

We briefly review Boffa's [?] interpretation of arithmetic. We can capture

> *"There is a bijection between $x \setminus y$ and $y \setminus x$"*

by saying

> *"There is a set P of (unordered) pairs such that every $p \in P$ has one member in $x \setminus y$ and one in $y \setminus x$, and everything in x `XOR` y belongs to a unique pair in P."*

We write this last as $x \sim y$. This relation $\sim$ *between members of* F_2 is precisely equipollence[2].

We make the following identifications:

Natural Number	=	$\sim$ - equivalence class of a finite set of atoms;
Family of naturals	=	set of finite families of atoms closed under $\sim$;
Membership	=	inclusion[3].

Since we have the axiom I of infinity this gives us an interpretation of PA_2.

Proposition 1 *The above interpretation of PA_2 into TST_3 + AxInf is conservative.*

What exactly does this mean?

1.2 Plan of the Proof

We will describe an interpretation of TST_3 + AxInf into PA_2 for which the reconstruction of PA_2 into TST_3 + AxInf is demonstrably isomorphic to the structure with which we started. The technique is reminiscent of the Fraenkel-Mostowski construction in set theory.

Let $\mathfrak{N} = \langle N, O, S, +, \cdots \mathcal{D}\rangle$ be a model of PA_2. ($\mathcal{D}$ is a family of subsets of N). In $\mathfrak{N}$ "finite set" corresponds naturally to "bounded subset". To each $n \in N$ we associate the element X_n of $\mathcal{D}$ given by

- if $n = 2m$ then $X_n = \{x$: the x^{th} bit in the binary representation of $m = 1\}$;
- if $n = 2m + 1$ then $X_n = N \setminus X_{2m}$.

This bijects N with the set of finite-or-cofinite[4] sets of natural numbers.

We will say a permutation σ of N is *internal* if its graph is in $\mathcal{D}$. (We fix an arbitrary coding of pairs). Any internal σ is defined in a natural way on $\mathcal{D}$.

[2] Notice that this relation uses only three levels.

[4] This version of the Ackermann bijection is due to Oswald and (arguably) Church [2].

Lemma 1
Suppose $A \in \mathcal{D}$; A is finite-or-cofinite iff there is $n \in N$ such that A is fixed by every internal permutation fixing every $i \leq n$.

Proof:
Internal permutations preserve finiteness/cofiniteness. Thus, to any internal permutation σ, one can associate a new permutation σ^* defined by

$$X_{\sigma^*(n)} = \sigma``X_n$$

We also have

$$2^{\sigma(n)+1} = \sigma^*(2^{n+1})$$

Now, for $A \in \mathcal{D}$ we will say A is *invariant* if there is $n \in N$ such that A is fixed by every internal permutation fixing all $i \leq n$. We can now define a new structure $\mathfrak{M} = \langle M_0, M_1, M_2, \epsilon \rangle$ for the language of TST_3:

- $M_0 = N$;
- $M_1 = \{X_n : n \in N\}$;
- $M_2 = \{X \in \mathcal{D} : X \text{ is invariant}\}$

with the membership relation ϵ defined thus:

If $a \in M_0$ and $A \in M_1$ then $a \, \epsilon \, A$ iff $a \in A$;
If $a \in N$ and $A \in M_2$ then $X_a \, \epsilon \, A$ iff[5] $a \in A$.

It is then easy to check that $\mathfrak{M}$ satisfies extensionality. Verifying the axioms of comprehension needs lemma 1 and the following

Lemma 2 *Let σ be an internal permutation of N. By having σ act as itself on M_0 and M_1, and as σ^* on M_2 we have an automorphism of $\mathfrak{M}$.*

Proof:
Let $\mathcal{F}$ be $\{2n : n \in N\}$. $\mathcal{F}$ is in M_2, and clearly $\mathcal{F}$ is the set of finite sets of atoms (in the sense of $\mathfrak{M}$). Therefore $\mathfrak{M}$ and $\mathfrak{N}$ have the same notion of finiteness. Indeed, for $X, Y \in \mathcal{F}$, we have $\mathfrak{M} \models X \sim Y$ iff X and Y are equinumerous according to $\mathfrak{N}$. In $\mathfrak{N}$, let us declare $|x|$ to be the least y such that $\{0, \ldots, y-1\}$ is equipollent to X_{2x} (or 0). Wossat?
An isomorphism between $\mathfrak{N}$ and the arithmetic of $\mathfrak{M}$ is now given by

For $n \in N$, $n \mapsto \{2m : |m| = n\}$;
For $A \in \mathcal{D}$, $A \mapsto \{2m : |m| \in A\}$.

[5] I found this made more sense when i thought of it as:
If $X_a \in M_1$ and $A \in M_2$ then $X_a \, \epsilon \, A$ iff $a \in A$.

1.3 Remarks

The interpretation we have given from TST_3 + AxInf into PA_2 is not conservative. For example it verifies $(\forall x_1)(x_1$ is finite or cofinite).

Of course we can construct lots of other interpretations: all one needs is a subalgebra of $\mathcal{D}$ which can be coded and which contains all singletons. However we have not so far found an interpretation $\mathrm{TST}_3 \hookrightarrow \mathrm{PA}_2$ which is conservative.

Boffa-Crabbé [1] have shown that NF_3 + AxInf (NF *sans* axioms that need three types to stratify them, plus the axiom of infinity) is a conservative extension of TST_3 + AxInf.

References

[1] M Boffa and M Crabbé "Les théorèmes 3-stratifiés de NF_3" Comptes Rendus hebdomadaires des séances de l'Académie des Sciences de Paris (série A) **280** (1975), pp. 1657-1658.

[2]

[3]

2 Remarks by the Translator

I have changed the notation from TT_3I etc. to TST_3 + AxInf to comply with modern practice and also to avoid a collision with the notation that uses an 'I' suffix to denote a system with predicative restrictions on its set abstraction scheme.

3 Boffa-Crabbé on NF_3 and TST_3

C.R. Acad Sc. Paris **280** (23 Juin 1975) Série A — 1657

The 3-stratifiable theorems of NF_3

NF_3 is that fragment of NF axiomatised by the 3-stratifiable axioms of NF. We characterise the 3-stratifiable theorems of NF_3 (and of NF_3 + AxInf) in terms of TST, the theory of types[6].

Let k be a natural number ≥ 2 TST_k is TST restricted to levels 0 to $k-1$. NF_k is the theory axiomatised by the axiom of NF that can be k-stratified. Let TST_k^∞ is TST_k be TST_k plus axioms saying that level 0 contains $\geq n$ things for every concrete n. TST_k^+ is TST_k plus all k-stratifiable expressions of the form $A \longleftrightarrow A^+$. Grishin [1], [2], [3] proves the consistency of NF_3 in arfithmetic and proves NF = NF_4.

Proposition 2 *The 3-stratifiable theorems of NF_3 are precisely the theorems of TST_3^∞.*

Proof:

By using [] one can see that the 3-stratifiable theorems of NF_3 are precisely the theorems of TST_3^+. It remains to be shown that $TST_3^+ = TST_3^\infty$. This reduces to the problem of of showing that every infinite model of TST_3 satisfies $A \longleftrightarrow A^+$ for every 2-stratifiable formula A. Let B be the formula obtained from A by replacing in A every atomic subformula of the form $x_0 \in x_1$ by $x_0 \leq x_1$ and restricting to atoms every quantifier ranging over variables of type 0. Let M_1 be the boolean algebra of elements of type 1 and M_2 be the boolean algebra of elements of type 2. It is evident that $M \models A$ iff $M_1 \models B$ and that $M \models A^+$ iff $M_2 \models B$. Since M is infinite we know that M_1 and M_2 are both infinite atomic boolean algebras, so we know from [5] section 5.5 that they are elementarily equivalent ... which implies that $M \models A \longleftrightarrow A^*$.

■

An Aside: By drawing inspiration from [6] and quantifier elimination for separable Boolean rings (see [7] p 62) we can even give an effective procedure for transforming a proof of a 3-stratifiable theorem of NF_3 into a proof in TST_3^∞ of the correrspondin formula of the language of TST.

Corollary 1

(i) Every 3-stratifiable theorem of NF_3 is true in almost all finite models of TST_3;

(ii) Every 3-stratifiable expression true in infinitely many finite models of TST_3 is consistent with NF_3;

(iii) The set of 3-stratifiable expressions true in almost all finite models of TST_3 is consistent with NF_3;

[6] There are no footnotes in the original text, so all the footnotes here are comments from the translator.

(iv) There is not finite extension of TST_3 whose theorems are precisely the 3-stratifiable theorems of NF_3;

(v) If AI *is a 3-stratifiable version of the axiom of infinity*[7] *(for example, axiom C of [8]) then the 3-stratifiable theorems of NF_3 +* AI *coincide with the theorems of NF_3 + AI.*

Remark:

For each 2-stratifiable expression A, let B be the formula in the langauge of boolean algebras obtained as above. It is easy to see that A is a theorem of NF_2 iff B is a theorem of the theory of infinite atomic boolean algebras. This remains true even if we replace NF_2 by the theory T whose axioms are: extensionality, existence of singletons, binary unions ($x \cup y$) and complements. This means that $\mathrm{T} = \mathrm{NF}_2$. Thus the models of NF_2 are precisely the structures $\langle M, \in \rangle$ where M is a boolean algebra with a bijection i to its set of atoms, and $x \in y \longleftrightarrow i(x) \leq y$.

References

[1] Grishin, V.N. "Consistency of a fragment of Quine's NF system" Soviet. Math. Doklady, **10**, 1969, p' 1387-1390'

[2] Grishin, V.N. "Concerning some fragments of Quine's NF system" (in Russian). Issledovania po matematicheskoy lingvistike, matematicheskoy logike i informatsionym jazykam (Moscow), pp. 200-212.

[3] Grishin, V.N. "The equivalence of Quine's NF system to one of its fragments" (in Russian). Nauchno-tekhnicheskaya Informatsiya (series 2) 1, pp. 22-24. (1972) pp 22–24.

[4] Specker, "Typical Ambiguity", Logic, Methodology and Philosophy of Science (Proc 1960 intern' congr.), Stanford, 1962, pp. 116–124.

[5] Chang et Keisler Model Theory, North-Holland' 1973'

[6] Crabbé, M. "Types ambigus" Comptes Rendus hebdomadaires des séances de l'Académie des Sciences de Paris (série A) 280, pp. 1-2. Comptes rendus, 28O, série A, 1975, pp 1–2 1967'

[7] Kreisel et Krivine, Eléments de Logique Mathématique, Dunod, Paris,

[8] Gödel, K, The Consistency of the Continuum Hypothesis Princeton 1940.

[7] For example: say that a set is *even* if it has a partition into pairs. The axiom of infinity will now say that there are sets $x \in y$ with both y and $y \setminus \{x\}$ even.

4 On first looking into Pabion's "NF_3 + AxInf is equivalent to Second-Order Arithmetic"

Pabion is interested in the relation between NF_3 and PA_2, second-order arithmetic. It is evident that there is a close connection between the two, and Pabion has some useful things to say about it.

At the very least, one expects the two theories to be mutually interpretable, at least once one haas augmented TT_3 with AxInf, the axiom of infinity. So there are two directions to be studied: interpret PA_2 into TT_3 + AxInf, and *vice versa*.

We start by thinking about how to interpret PA_2 into TT_3 + AxInf. On the face of it this there is a huge obstacle. Level 0 of a model of TT_3 + AxInf contains atoms, level 1 contains sets (finite sets indeed) and level 2 contains sets of finite sets, which will do duty as natural numbers. To get PA_2 we need *sets* of numbers, and that would involve level 3, which in our case we do not have. However Boffa has a clever idea that gets past this *impasse*.

As long as x and y are finite, then $|x| = |y|$ is equivalent to there being a bijection between $x \setminus y$ and $y \setminus x$, and the existence of such a bijection can be stated without using ordered pairs, by saying *"There is a set P of (unordered) pairs such that every $p \in P$ has one member in $x \setminus y$ and one in $y \setminus x$, and everything in x `XOR` y belongs to a unique pair in P."* So we can assert bijectivity inside three types.

> In fact we can do this anyway—even without the assumption of finiteness—using a device of Henrard, but Boffa's device is simpler and does what we need.

Next we record that we can say that x is finite in a formula using three types where the variable 'x' occurs at the middle type. So in TST_3 (levels labelled 0, 1 and 2) natural numbers appear at the top level, as equivalence classes of sets of atoms. The next clever idea is to think of *sets* of natural numbers as their *sumsets*. This succeeds beco's $\bigcup$ is injective on sets of naturals. That way we get second order arithmetic inside three levels!

The other direction we want is an interpretation of TST_3 + AxInf in PA_2. Here too we seem to run out of sky, since PA_2 has only two levels while the TST_3 we are trying to shoehorn into it (with or without AxInf, it matters not) has three. For this we need ideas going back to Ackermann and Oswald. We start with a model $\mathfrak{M}$ of PA_2, and obtain from it a model $\mathfrak{N}$ of TST_3 + AxInf. The atoms of $\mathfrak{N}$ are going to be the natural numbers of $\mathfrak{M}$. The sets of atoms of $\mathfrak{N}$, too, are going to be the natural numbers of $\mathfrak{M}$, by means of a Ackermann/Oswald coding. The top level of $\mathfrak{N}$ is going to be the top level of $\mathfrak{M}$.

Next we have to ensure that we code (in the naturals of $\mathfrak{M}$) all the sets-of-atoms that the axioms of TST_3 + AxInf allege to exist. Fortunately for us, TST_3 + AxInf is not very demanding. All it can say is that the sets of atoms in $\mathfrak{N}$ form an infinite atomic boolean algebra; so it suffices to ensure that: V exists, every atom has a singleton and that sets are closed under $\setminus$, $\cup$ and $\cap$. The basic Oswald construction gives us this much, and so do lots of others.

Perhaps we should insist on a CO construction that gives us a boolean algebra with the splitting property. It is evident from Pabion's paper that his CO-style construction gives models of TST_3 + AxInf in which the boolean algebra that is level 1 does *not* have the splitting property, whence we can infer that TST_3 + AxInf does not prove that the boolean algebra that is level 1 has the splitting property.

Here is a fact that might come in useful. If $\mathfrak{M}$ is a countable model of PA_2 (second order arithmetic) then level 2 of $\mathfrak{M}$—the family of subsets of $\mathbb{N}$—is a countable atomic boolean algebra *with the splitting property*.

Let A be an infinite member of the top layer of $\mathfrak{M}$. $A = \langle a_i : i \in \mathbb{N}\rangle$ divides naturally into $A_{even} = \langle a_{2i} : i \in \mathbb{N}\rangle$ and $A_{odd} = \langle a_{2i} : i \in \mathbb{N}\rangle$. It will suffice to show that these are both sets of $\mathfrak{M}$. We will exploit to the utmost the fact that in any coding system we might be using any finite subset of $\mathbb{N}$ can be coded by a member of $\mathbb{N}$. So we can say of any finite subset A' of A that it can be split into pairs (possibly discarding the top element) of adjacent elements... and we can say this while talking only about finite sets of naturals. We then say an element of A' is *odd* if it only ever appears as the smaller element of a pair from such a decomposition, and *even* otherwise. Thus naturally A_{even} and A_{odd} are sets of $\mathfrak{M}$ that split A into two.

When defining the model of TST_3 + AxInf starting from the model of PA_2 why do we not set M_2 to be the whole of $\mathcal{D}$? This is a good question. There is a roadblock in the form of Cantor's theorem. We can have a bijection σ between the set of naturals and what the model believes to be its power set but $y = \sigma(x)$ cannot be equivalent to an expression in the language of PA_2 lest we get $\{n : n \in \sigma(n)\}$. Duh.

That is to say, if we turn the level consisting of the naturals into a countable atomic boolean algebra with the splitting property then there will be an isomorphism between it and level 2 but it won't be definable. But if σ is not definable there is no easy way of showing that the result is a model of TST_3 + AxInf. Another consideration is that $\mathcal{D}$ might contain too much information, with the result that the model we construct is not a model of TST_3 + AxInf. For example, suppose M_1 contains only finite-or-cofinite sets (as Pabion's model in fact does). Suppose further than $\mathcal{D}$ contains the set E of finite sets of even naturals (or, strictly, the set of naturals that code finite sets of evens). But then if our model is to satisfy TST_3 + AxInf it would have to contain—at level 1, its middle level—the set of atoms that code even numbers. This is a *moiety*—neither finite nor cofinite.

So we are in the market for a way of turning a countable model of PA_2 into a model of TST_3 + AxInf that doesn't involve discarding any sets of naturals.

Thinking aloud ... Let $\mathfrak{M}$ be a countable model of PA_2. It has two level, $\mathbb{N}$ and $\mathcal{D}$. By the above remarks $\mathcal{D}$ is a countable atomic boolean algebra with the splitting property. Then we need a bijection between $\mathbb{N}$ an $\mathcal{D}$ in the form of a CO-construction that makes the algebra coded by the naturals $\mathbb{N}$ of $\mathfrak{M}$ isomorphic to the countable atomic boolean algebra with the splitting property that is $\mathcal{D}$. Both of these things can be done by *fiat*. We cook up—any old how—

a CO-style coding that makes the bottom level into a countable atomic boolean algebra with the splitting property. That is to say, we have a function σ s.t. $\sigma(n)$ is a set of naturals. This boolean algebra $\sigma``\mathbb{N}$ is going to be isomorphic to the top level beco's any two countable atomic boolean algebras with the splitting property are isomorphic. Let $\tau$ be an isomorphism $\sigma``\mathbb{N} \to$ top level of $\mathfrak{M}$.

We now have (with any luck) a model of TST_3 + AxInf
Level 0 is level 0 of $\mathfrak{M}$;
Level 1 is level 0 of $\mathfrak{M}$;
Level 2 is level 1 of $\mathfrak{M}$.

How are we to think of an element z of the top level of $\mathfrak{M}$ as a set of sets of atoms?

We say
$x_0 \in y_1$ iff $\mathfrak{M} \models x_o \in \sigma(y_1)$;
$y_1 \in z_2$ iff $\mathfrak{M} \models \sigma(y_1) \in z_2$.

$\mathfrak{M} \models y_1 \in \tau^{-1}(z_2)$

garbled

Now we have to verify that this is a model of TST_3 + AxInf. This means that we have to choose σ and τ very carefully!!

garbled

What is the (second-order!) arithmetic of (what Boffa-Crabbé call) TST_3^∞?

Consider the following construction. Start with the algebra of finite-and-cofinite sets of naturals. Add the odds and the evens; and then, recursively given x, add the odd and the even parts of x. This gives us countably many moieties, M. M naturally presents itself as the vertices of a perfect binary tree, and we can enumerate its members as: 0 (which is $\mathbb{N}$), then 1 and 2 (the odds and the evens) then $3, 4, 5$ and 6 (the four residue classes mod 4) and so on. Then add everything that has finite symmetric difference with one of these moieties. The result is a countable atomic boolean algebra with the splitting property.

Any element of this family can be represented as an ordered pair of two finite sets S_1 and S_2 of naturals. S_1 codes up a set X of moieties, and we recover $\bigcup X$ from it. The subset of $\mathbb{N}$ encoded by the pair $\langle S_1, S_2\rangle$ is now the set $\bigcup X \ \mathtt{XOR}\ S_2$. It is (or should be) evident that any subset of $\mathbb{N}$ has a unique coding in this fashion, since for any x there is precisely one finite union of moieties in its equivalence class under finite symmetric difference (Two distinct finite unions of moieties have infinite symmetric difference.)

This coding powers a CO construction of a structure for the language of set theory with the splitting property. However this is no big deal co's we can get the same effect by contruction $B(x)$ for every x. Hmm. Have we done this anywhere...? Yes, but we didn't get a model of NF0.

Is there a natural family of moieties of $\mathbb{N}$ s.t. every b.a. generated by a finite subfamily is free?