AC Fails in the Natural Analogues of V and L That Model the Stratified Fragment of ZF

Thomas Forster

ABSTRACT[1]

If G is a group of permutations of V_ω it has countably many different actions on V, since for each $n < \omega$ it can move x by permuting the elements of $\bigcup^n x$ of finite rank and fixing the rest. A set that is fixed by everything in G under the nth action of G is said to be n-**symmetric**; if it is n-symmetric for all sufficiently large n it is just plain **symmetric**. The class of hereditarily symmetric sets is a model for the stratified axioms of ZF but contains no wellordering of V_ω! The other structure is a stratified analogue of L, but the construction is extremely fragile.

I am indebted to Randall Holmes, Adrian Mathias and Philip Welch for helpful discussions.

CONTENTS

Introduction

This paper is a report on work-in-progress. I am very grateful to Ali Enayat for inviting me to present it at the Baltimore Joint Meeting, and to The American University for financial support associated with that presentation. I am grateful to Ali Enayat also (and to the American Mathematical Society) for the opportunity to present this snapshot to the public in their volume of proceedings of that meeting.

There are two well-known ways of using permutations of the carrier set of a model of set theory to produce new models of set theory, namely the Fraenkel-Mostowski and the Rieger-Bernays constructions. This is a third method which has affinities with both but should not be confused with either.

[1]AMS Subject classifications: 03E25 03E35 03E45

1. Definitions

(x, y) is the transposition swapping x and y; $\langle x, y\rangle$ is the ordered pair of x and y. Lower case Greek letters will be used to range over permutations.

(Readers familiar with the basics of the literature on NF can skip the rest of this section.)

A formula of set theory is **stratified** iff by assigning type subscripts to its variables we can turn it into a well-formed formula of simple type theory. That is to say, a wff ϕ is stratified iff we can find a **stratification** for it, namely a map f from its variables (after relettering where appropriate) to $\mathbb{N}$ such that if the atomic wff '$x = y$' occurs in ϕ then $f(\text{'}x\text{'}) = f(\text{'}y\text{'})$, and if '$x \in y$' occurs in ϕ then $f(\text{'}y\text{'}) = f(\text{'}x\text{'}) + 1$. Variables receiving the same integer in a stratification are said to be of the same **type**. If n successive naturals are used, the formula is said to be n-**stratified**. A stratified formula with a free variable is said to be an n-**formula** iff there is a stratification giving that variable the label 'n', and gives at least one variable the label '0' and no variable a negative label. A *function* is said to be stratified iff it is represented by a stratified expression ϕ such that $\forall x_1 \ldots x_n \exists! y \phi$. This idea is less natural than one might think, for the class of stratified functions is not closed under composition: singleton and binary union are both stratified, but their composition, $x \mapsto x \cup \{x\}$, is not. The largest class of stratified functions of unbounded arity closed under composition is the class of **homogeneous** functions, and the smallest class of functions closed under composition and containing all stratified functions is the class of **weakly stratified** functions. It is simple to check that a function is homogeneous iff there is a stratified expression ϕ such that $\forall x_1 \ldots x_n \exists! y \phi$ wherein all the $\vec{x}$ and y have the same type, and equally simple to check that a function is weakly stratified iff there is an expression ϕ such that $\forall x_1 \ldots x_n \exists! y \phi$ wherein such failures of stratification as there may be involve the $\vec{x}$ only. Thus we can apply the adjectives 'homogeneous' and 'weakly stratified' to formulæ as well as to functions. The **height** of a word W will be the number of types needed to stratify it.

2. HS: the hereditarily symmetric sets

The key lemma in understanding weakly stratified descriptions is Coret's lemma.

A permutation σ can act on a set x in countably many (natural, set-theoretic) ways. It can send x to $\sigma(x)$, or to $\sigma``x$, which is $\{\sigma(y) : y \in x\}$ (this, too, is sometimes written '$\sigma(x)$'!) and furthermore, for any n, it can act on $\bigcup^n x$. Clearly $x \in y$ iff $\sigma(x) \in \sigma(y)$ where the action on x is at top level and the action on y is one level down.

It is customary in most areas of mathematics to use the same notation for both these first two actions, and it's usually easy to disambiguate the notation. In set theory, where it isn't, we would often write $\sigma``y$ for the action of σ at the lower of these two levels. However, for the programme of this paper an *ad hoc* pragmatic strategy for distinguishing between merely *two* of these actions is not going to be enough: we need a consistent uniform notation that covers those countably many actions. This is because the same biconditional (namely $x \in y$ iff $\sigma(x) \in \sigma(y)$) holds if the action on y is $n + 1$ levels down and the action on x is n levels down. So let $j^n(\sigma)$ be that permutation which sends x to the result of moving members of $\bigcup^n x$ according to σ. This immediately gives us that, for each n,

$$(\forall xy)(x \in y \longleftrightarrow (j^n\sigma)(x) \in (j^{n+1}\sigma)(y)). \tag{2.1}$$

Now let's take a formula in the language of set theory and manipulate it by replacing some of its atomic subformulæ by substitutivity of the biconditional using equation 2.1. Notice that if all occurrences of a variable have the same prefix, and that variable is bound, we can delete the prefix (since it's a permutation). When can we do this? Precisely when there is no conflicting information about which level a variable lives at; and when is that? When the formula is stratified!

This proves

LEMMA 2.1. *Coret's lemma (Coret* [**1**], [**2**]*) If Φ is weakly stratified then*

$$\Phi(x_1, \ldots, x_k) \longleftrightarrow \Phi((j^{n_1}\sigma)(x_1), \ldots, (j^{n_k}\sigma)(x_k)),$$

for any permutation σ, where n_k is the integer assigned to the variable 'x_k' in some fixed stratification.

On the face of it this proof should work for stratified formulæ only, not for all weakly stratified formulæ as well. However multiple occurrences of a single free variable in a formula at different types can be given different prefixes. ■

In particular Coret's lemma gives us the following useful corollary:

COROLLARY 2.2. If W is a weakly stratified function and n is bigger than the height of W then

$$(j^n\sigma)W(x_1 \ldots x_k) = W(y_1 \ldots y_k),$$

where the terms y_i are the corresponding variables x_i with prefixes derived from σ. Occurrences of an x variable at different types are treated as different variables and receive different prefixes.

Exploiting the notation '$[x/y]W$' for the result of substituting the variable 'x' for 'y' in the word W (and '$[x/y; x'/y']W$' for the result of substituting the variables 'x' for 'y' and 'x'' for 'y'' in the word W) we could also write this as

$$(j^n\sigma)W = [\sigma_1(x_1)/x_1; \sigma_2(x_2)/x_2; \ldots]W,$$

where the $\vec{x}$ are the free variables of W. σ_i is of course $j^{n_i}(\sigma)$ where n_i is the type of x_i in W.

That is to say, when $n >$ height(W) we can **import** the permutation. For example $(j^2\pi)\mathcal{P}(x) = \mathcal{P}((j\pi)x)$.

Another corollary of Coret's lemma is

$$(\forall\sigma)(\forall x)(\phi(x) \longleftrightarrow \phi((j^n\sigma)(x)),$$

as long as $n \geq$height(ϕ). x and $(j^n\sigma)(x)$ resemble each other for "the top n levels" so this is telling us that stratified formulæ "look only finitely levels down" and express properties that are in some obscure set-theoretic sense *local*. Obscure this sense may be, but it is an important intuition to develop in connection with stratified formulæ.

We can see further that (ignoring parameters for the moment) by Coret's lemma any set that has a unique stratified description must be fixed by $j^n(\sigma)$ for all permutations σ where $n >$ height of the formula uniquely describing it. We will say—for the moment—that a set is n-**symmetric** iff it is fixed by every permutation that is j^n of something, and **symmetric** if it is n-symmetric for some n. I say "for the moment" because this definition will have to be torn up and the word "symmetric" recycled.

Let's have a brief reality check. Any permutation that is j of anything will fix the empty set, and any permutation that is j^2 of anything will fix the empty set and its singleton, and any permutation that is j^n of anything will fix everything of rank $< n$. So every hereditarily finite set (of rank n) is (n-)symmetric, and everything in V_ω is symmetric. Of course one cannot infer that a set is symmetric if all its members are symmetric, though it will of course be $n+1$-symmetric as long as all its members are n-symmetric.

Clearly Coret's lemma will tell us that the class of symmetric sets is closed under all weakly stratified functions. This fact does not rely at all on the collection of permutations over which we quantify in the definition of 'symmetric' being the class of all permutations. We will exploit this insensitivity enthusiastically, as the idea of a symmetric set is attractive and fruitful and we do not wish to be hampered by the fact that—as we are about to find out—in ZF there are hardly any symmetric sets to speak of.

It is easy to see that if x is n-symmetric, then $\bigcup x$ is $n-1$-symmetric, and so on, so that $\bigcup^{n-1} x$ is 1-symmetric. But if y is 1-symmetric, $\pi``y = y$ *for all permutations* π. Thus y must be V or $\emptyset$. This is turn tells us that if x is n-symmetric either $\bigcup^{n-1} x$ is $\emptyset$—in which case x is in V_ω, or $\bigcup^{n-1} x = V$—in which case x is a proper class.

How did this disaster unfold? The point is that, if x is n-symmetric and π is any permutation whatever, then $(j^n\pi)(x)$ must be x. Now $j^n(\pi)$ acts on x by moving stuff in-and-out of x n levels down according to π. Since π can be any old permutation of V, it can swap any old rubbish into x, and if x is fixed by such actions it must consist, n levels down for some n, all that any-old-rubbish. Either that or be empty.

This result—that if x is symmetric either x is hereditarily finite or $TC(x) = V$—is known from work on NF (it was first remarked on by Boffa). Although it is not a disaster for NF studies[2] it certainly does mean that the idea of symmetric sets cannot be straightforwardly applied in the study of ZF, since ZF proves the existence of sets of infinite rank whose transitive closure is not the universe. Indeed, unwary readers may well be spooked by this into concluding that the idea of n-symmetry cannot be usefully applied to ZF at all: I certainly was, for years. They would be wrong. What this result is telling us is instead that our notion of symmetric is defective, and the defect is that it involves quantification over *all* permutations. If we restrict our permutations somehow then there are corresponding limits on the amount of rubbish that can be swapped in and out of sets that are n-symmetric under the new dispensation. So our new definition is: a set is symmetric if, for sufficiently large n, it is fixed by $j^n(\sigma)$ for all permutations σ of something or other. But what is this something-or-other to be? It will have to be a proper initial segment of the universe, lest n-symmetry of x permit us to swap unbounded rubbish into x as before. So let us rewrite our notion of n-symmetry to read "x is

[2]In a sense, the whole concept of "disaster for NF studies" is on hold until the question of the consistency of NF is cleared up. However, it would be very odd if NF were consistent but nevertheless proved the existence of a nonsymmetric set. After all—other than extensionality—every axiom of NF is an assertion that the universe is closed under an operation that takes symmetric sets to symmetric stes. This makes it natural to expect that if NF has models at all then it should have models in which every set is symmetric.

n-symmetric iff $x = (j^n\pi)(x)$ for all $\pi \in \Sigma(V_\kappa)$". ($\Sigma(V_\kappa)$ is the full symmetric group on V_κ.) Just as, under the old dispensation, $\bigcup^n(x) = V$ or $\emptyset$ if x is n-symmetric, we now find that $\bigcup^n x \supseteq V_\kappa$ or $\bigcup^n x = \emptyset$ if x is n-symmetric in the new sense. This would have the effect that no set of rank between ω and κ could be symmetric. And this in turn would scupper any chances of a nontrivial theory of hereditarily symmetric sets. Clearly κ will have to be ω, so we use $\Sigma(V_\omega)$ rather than $\Sigma(V)$. In fact for technical reasons which will become clearer later (lemma 2.7 and theorem 3.1) it is necessary to restrict still further to a group of permutations of finite support. It doesn't matter a great deal at this stage which group of permutations of finite support but readers who do not like loose ends may wish to have in mind the alternating group $\mathbf{Alt}(V_\omega)$ of those permutations that are the product of an even number of transpositions. It has the advantage of simplicity.

So we rewrite our notion of symmetric to

DEFINITION 2.3. .

(1) x is n-**symmetric** if $\sigma(x) = x$ for all x in j^n"G where G is a group of permutations of V_ω of finite support, unspecified for the moment;
(2) x is **symmetric** if it is n-symmetric for all sufficiently large n;
(3) HS is the class of hereditarily symmetric sets.

We will want to ensure that any set that is n-symmetric is also m-symmetric for all $m > n$. In the (original) NF context one defined a set to be n-symmetric. iff it was fixed by everything in j^n"$(\Sigma(V))$. It so happens that j^{n+1}"$(\Sigma(V)) \subseteq j^n$"$(\Sigma(V))$, so in the Old Dispensation it happened automatically that every n-symmetric sets was m-symmetric for $m \geq n$. Here we have write it in explicitly: were we to take for G a group that is not closed under j then we would have to redefine n-symmetric to be m-symmetric (in the old sense) for all $m \geq n$. This is another desideratum for G to be borne in mind!

Notice that although we call these sets symmetric, they differ from the symmetric sets in Fränkel-Mostowski models in that they are wellfounded and extensional: HS is a substructure of the cumulative hierarchy.

REMARK 2.4. Suppose $(\exists! x)\phi(x, \vec{s})$ where the $\vec{s}$ are all symmetric and ϕ is weakly stratified. Then the unique witness is also symmetric.

Proof: By Coret's lemma $V \models \phi((j^n\pi)(x), \ldots (j^{n_i}\pi)(s_i) \ldots)$ for any π, with n and the n_i depending only on ϕ. But since the witness is unique, all these $(j^n\pi)x$ are identical and the witness must be n-symmetric. ■

COROLLARY 2.5. The class of symmetric sets is closed under application of stratified functions.

One might have expected to be able to prove a completeness-like result to the effect that a set can be n-symmetric iff it is uniquely described by an n-formula. But any hope of that was lost when we loosened the definition of n-symmetric by trimming the bundle of permutations by which a set would have to be fixed. It is this loosening that makes the following two lemmas possible.

LEMMA 2.6. *For all ordinals α, V_α is symmetric.*

Proof: In fact V_α is n-symmetric for all $n \geq 1$. If σ is a permutation that moves hereditarily finite sets only, and x is a set of infinite rank, what can σ move x to by acting on $\bigcup^n x$? It clearly cannot move x to anything of different rank. That is to say, it fixes V_α setwise, as desired. ■

Lemma 2.7. *For all ordinals α, $HS \cap V_\alpha \in HS$.*

Proof: This is the first place where we (appear to) need the fact that our group is a group of permutations of finite support. That fact is enough to ensure that all the permutations in it are—considered as their graphs—symmetric.

The key fact is that, for each n, there is a stratified binary operation ("n-application") that takes a permutation σ and a set x and returns the result of σ acting on $\bigcup^n x$. In our notation this is $(j^n\sigma)(x))$. So if x is symmetric, *and σ is symmetric* then $(j^n\sigma)(x)$ is symmetric too. This will show that if x is hereditarily symmetric then so is $(j^n\sigma)(x)$. But everything inside $TC((j^n\sigma)(x))$ is either (i) the same as something inside $TC(x)$ (if it was more than n levels down) or (ii) is obtained from a *symmetric* object *at most* n levels down inside x by k-application for some $k \leq n$. All such objects in $TC(x)$ are symmetric (x is hereditarily symmetric) and are moved to objects inside $(j^n\sigma)(x)$ by this stratified operation which, as we have seen, preserves symmetry. Everything in $TC((j^n\sigma)(x))$ is obtained by one of these two processes, both of which preserve symmetry, so $(j^n\sigma)(x)$ is hereditarily symmetrical as desired.

This means that $V_\alpha \cap HS$ is hereditarily symmetric, since all that the nth action of any σ can do is permute members of $V_\alpha \cap HS$. ■

For this proof of lemma 2.7 to work we need the permutations to be symmetric, and one way of ensuring this is to take them to be of finite support. For all I know there may be a (presumably entirely different) proof that doesn't assume that the permutations are symmetric.

We proved above that V_α is symmetric, in fact that it is 1-symmetric. Under the old dispensation the only 1-symmetric set was the empty set (and possibly the universe, if it is a set). The familiar connections between logic and algebra would lead us to expect that a set should be n-symmetric if and only if it has a unique description by an n-formula. (I'll leave out the small print, because in this case it doesn't work anyway). This expectation is not met, because the condition on n-symmetry has been weakened. This wrinkle (things lacking stratified descriptions nevertheless turning out to be symmetric because we have cut down the family of permutations that we require them to be fixed by) to a certain extent counteracts another wrinkle, which arises when we procede (as we do next) to consider which axioms hold in HS. In the standard cases where we are trying to prove that $H_\phi \models$ replacement the task is made easy by the fact that the ϕ in question is preserved under surjection. (Hereditarily finite sets, hereditarily countable sets, sets hereditarily of size less than $\beth_\omega$, all routinely used in proofs of independence of the axioms, all have this feature.) However the surjective image of a symmetric set need not be symmetric, and this blocks the usual proof that the hereditarily-something structure is a model of replacement. We need something extra, and that indispensible something extra is lemma 2.7. The two wrinkles annihilate one another in a flash of illumination!

Theorem 2.8. *HS is a model of the axioms of extensionality, pairing, power set, sum set, infinity, and the stratified instances of replacement and separation.*

Proof: HS satisfies extensionality because it is transitive. It satisfies pairing and sumset because (by corollary 2.5) it is closed under the stratified operations corresponding to those axioms. (The power set operation is not Δ_0 so we cannot despatch the axiom of power set in this way: we will deal with it later!)

It satisfies infinity because V_ω is hereditarily symmetric. There are complications about verifying the axiom of infinity in this context because in the absence of unstratified separation various ZF-equivalent versions of the axiom cease to be equivalent. We will verify two versions. (i) There is a version that says that there is a nonempty set closed under $\lambda x.x \cup \{x\}$. This is clearly satisified. However the relevance of this version to the development of arithmetic is tied to the availability of the Von Neumann implementation of ordinals, and we are clearly not going to be using that here. (There are immense difficulties in the way of manipulating Von Neumann ordinals if one has only stratified replacment not full replacement). So we need version (ii), that says there is a set with a wellordering with no last element and no limit point. It is not clear that HS thinks that V_ω has a countably infinite subset, but it does at least know that V_ω is not inductively finite. And, given an infinite (hereditarily symmetric) dedekind-finite set X we can construct a genuinely countable set by a standard method that uses only stratified machinery: the set of equivalence classes under equipollence of X's inductively finite subsets is a (hereditarily symmetric) subset of $\mathcal{P}^2(X)$ with a natural wellordering to length ω and this set will give rise to an implementation of arithmetic to accommodate the most exacting tastes.

It is slightly trickier to verify power set than it was to verify sumset and pairing. If $x \in HS$ then the collection of HS subsets of x is a subset of HS but is it symmetric? The collection of symmetric subsets of x is the intersection of $\mathcal{P}(x)$ (which is symmetric since it is obtained from a symmetric set by a stratified operation) and $V_\alpha \cap HS$ which is hereditarily symmetric by lemma 2.7, and the intersection of two symmetric sets is symmetric.

We now verify stratified replacement: The image of a hereditarily symmetric set in a stratified function with hereditarily symmetric parameters must be symmetric, and, since all its members are hereditarily symmetric, it will be too, so that should be the end of the matter, but we have to be careful. Even if the function is apparently defined by a stratified formula, it won't be stratified once we have restricted all the bound variables to HS, because the definition of HS is not stratified! However, we can use reflection to cut down all quantifiers to suitable initial segments of V, and we can then exploit the fact (lemma 2.7) that for any ordinal α, $HS \cap V_\alpha$ is hereditarily symmetric, so the formula defining the function is equivalent to a stratified formula where all parameters are terms denoting members of HS. ■

So what is HS a model of? The stratified fragment of ZF, as promised in the title? Here we have to be very careful about what we mean by the phrase: "the stratified fragment of ZF". Do we mean the theory axiomatised by the stratified axioms of ZF? If so, then yes, for that is what we have just shown. If we mean the theory axiomatised by the stratified *theorems* of ZF then the answer might be 'no', since these theories are not the same. As Mathias ([**5**], the beginning of section 9, on page 217.) has noted, the assertion "there is an infinite set of infinite sets all of different sizes" is a stratified theorem of ZF but doesn't follow from the stratified axioms of ZF, since, by Coret [**1**], all those axioms are theorems of Zermelo set theory,[3] whereas the Mathias assertion is not. It seems fairly clear that the Mathias formula is satisfied in HS but there may be stratified theorems of ZF that aren't.

Because G contains transpositions moving finite ordinals to things that are not finite ordinals, no infinite ordinal can be symmetric, so HS contains no infinite Von Neumann ordinals. Readers should not infer from this that AC fails, for the lack of infinite Von Neumann ordinals might be attributable to nothing more than the failure of Mostowski collapse for functions defined by unstratified formulæ. Indeed, as Serge Grigorieff pointed out to me—and as we shall explore later in more detail—if we take G to be a group of finite permutations of $V_\omega \setminus \omega$ (the von Neumann ω) this difficulty goes away: the absence of infinite von Neumann ordinals is a red herring. Nevertheless choice *does* fail, and it fails badly.

THEOREM 2.9. *HS does not contain any total order of V_ω.*

Proof: Let $X \subseteq V_\omega \times V_\omega$ be n-symmetric, and suppose it were the graph of a total order of V_ω, Since X embodies a total order, it must contain either the ordered pair $\langle \iota^n(\emptyset), \iota^{n+1}(\emptyset)\rangle$ or the ordered pair $\langle \iota^{n+1}(\emptyset), \iota^n(\emptyset)\rangle$ but not both. ($\iota^n(x)$ is the n-times singleton of x). Now let σ be the transposition $(\emptyset, \{\emptyset\})$. Then, by n-symmetry of X, $(j^n\sigma)(X) = X$ so if X contains one of these ordered pairs, it must contain the other, contradicting antisymmetry of X. ■

This proof goes back to an observation of André Pétry from many years ago: [**6**]). (We should flag here the warning that this proof of theorem 2.9 works only for versions of 'symmetric' that use groups that contain all single transpositions of elements of V_ω. If we take G to be, say, $\mathbf{Alt}(V_\omega)$ we have to complicate the argument by considering 3−cycles instead of transpositions.)

This does not mean that HS does not contain wellordered infinite sets: it certainly contains the set of equivalence classes of finite subsets of V_ω under equipollence, and this is assuredly a set of size $\aleph_0$ as we have seen, and for the usual reasons. However it does not contain any bijection between this set and V_ω.

Finally we can note that theorem 2.9 records not only a failure of choice but of course also a failure of unstratified separation or comprehension, since there is a known wellordering of V_ω definable by an unstratified formula. Probably no reader had doubted that HS falsified some instances of unstratified comprehension, but it is as well to have a proof.

3. V = S: an axiom for stratified constructibility

In [**4**] I exhibited a finite set of stratified rudimentary functions with the feature that any set closed under them and under power set is closed under stratified Δ_0-separation. They are:

$\{\langle\{x\}, \{y\}\rangle : \langle x, y\rangle \in R\}$; $x \setminus y$; $\{x, y\}$;
$\{\{y\} : y \in x\}$;
$\bigcup x$;
$dom(x)$;
$\{\langle x, y, z\rangle : \langle x, y\rangle \in A_1 \wedge z \in A_2\}$;
$\{\langle\{u\}, v\rangle : u \in v \in x\}$;
$\{u \in y : x \in u\}$;
$\{\langle x, y, z\rangle : \langle x, z\rangle \in A_1 \wedge y \in A_2\}$;

[3] At lweast if we axiomatise ZF with replacement rather than collection: thanks to Mathias for this subtle but potentially important qualification.

$\{\langle y,x\rangle : \langle x,y\rangle \in A\}$.

I will spare readers the proof, since nothing in what follows will depend on the set of operations being this particular set rather than any other. The analogy with the J hierarchy that constructs L invites us to define a stratified analogue of L which I shall call Sr. These are the **stratrud** functions, and $stratrudclos(x)$ is of course the closure of x under them.

$Sr_0 =: \emptyset$; $Sr_{\alpha+1} =: stratrudclos(Sr_\alpha \cup \{Sr_\alpha\})$; $Sr_\lambda =: \bigcup_{\alpha<\lambda} Sr_\alpha$ for λ limit. $S =: \bigcup_{\alpha\in On} Sr_\alpha$.

The critical difference between Sr and L is that although Sr has a canonical wellorder for the same reason that L has, there is no reason to expect that Sr should contain initial segments of the graph of this wellorder: the definition of this wellorder is unstratified, and Sr guarantees sethood only to graphs of stratified functions. We must brace ourselves for the possible discovery that—in Philip Welch's words—"Sr does not construct itself". We will see that this is indeed the case: not only does Sr fail to construct itself, it has no global wellordering and does not wellorder—in fact does not even *totally* order—V_ω. This will follow from the fact that $S \subseteq HS$, which we will now prove.

THEOREM 3.1. *For all ordinals α everything in Sr_α is symmetric and Sr_α itself is n-symmetric for all $n \geq 1$.*

Proof: We prove this by induction on the ordinals. We need two facts and one trick. Fact (i) arises from a helpful analogy with L: L is the rud-closure of the class of all the J_α, so everything in L is denoted by a word in the rud functions with arguments among the J_α. Analogously everything in Sr is denoted by a word in the stratrud functions with arguments among the Sr_α. Fact (ii) is remark 2.4, which tells us that anything defined by a weakly stratified word over symmetric arguments will be symmetric itself. This tells us that if Sr_β is symmetric for all $\beta < \alpha$ then everything in Sr_α is symmetric. However that fact isn't on its own enough to ensure that Sr_α itself is symmetric. For this we need to be sure that if $x \in Sr_\alpha$ then for all n and all permutations $\sigma \in G$ then $(j^n\sigma)x \in Sr_\alpha$ too. We know that for each x, $x = (j^n\sigma)x$ for sufficiently large n, but we need to deal with the case of small n too. This is where we need a trick. What we wish is that any rud-closed class should be fixed by $j^n\sigma$ for any permutation $\sigma \in G$ and all n. But every permutation in G is a product of finitely many transpositions and for any transposition (x,y) and any concrete n, and any $w \in Sr_\alpha$, $(j^n(x,y))w$ is a word in the stratrud functions over arguments x, y, w, and so is in Sr_α.

■

In fact for $(j^n\sigma)w$ to be in Sr_α whenever w is it will suffice for σ to be in Sr_β for some small β, for example Sr_1. However for the moment we are playing safe and taking G to be $\mathbf{Alt}(V_\omega)$. There are other candidates like $\mathbf{Symm}(V_\omega) \cap L$ or $\mathbf{Symm}(V_\omega) \cap Sr$. The reason for preferring the alternating group is that it is simple, and this will ensure that every member of a finite symmetric set is symmetric. This in turn ensures that Sr and HS are more nicely embedded in V than they would be otherwise, and makes it much easier to prove preservation theorems.

4. Further developments

There are numerous questions to ask about this area, some of the more pressing being:

(1) How sensitive is the development of HS to the choice of the group G of permutations of V_ω?
(2) L has two equivalent presentations, both going back to Gödel: can Sr be developed in both these ways?
(3) For what classes of formulæ are the inclusion embeddings $Sr \hookrightarrow HS \hookrightarrow V$ elementary?
(4) Does HS satisfy all the stratified theorems of ZF as well as all the stratified axioms? Specifically does HS satisfy the Mathias formula?
(5) Is there an initial segment of HS that satisfies "There is a set X such that every wellordering is isomorphic to a member of X"?

But these are topics for a later paper: this short note is intended merely to start the ball rolling.

References

[1] Coret, J. [1964] Formules stratifiées et axiome de fondation. *Comptes Rendus hebdomadaires des séances de l'Académie des Sciences de Paris série A* **264** pp. 809−12 and 837−9.
[2] Coret, J. [1970] Sur les cas stratifiés du schema de remplacement. *Comptes Rendus hebdomadaires des séances de l'Académie des Sciences de Paris série A* **271** pp. 57−60.
[3] Forster, T. E., Kaye, R. M. and Mathias, A. R. D. [20xx] More on the set theory KF.
[4] Forster, T. E. Set theory with a Universal Set. Oxford Logic Guides **31**, 1995
[5] Mathias, A. R. D. The strength of Mac Lane set theory. Annals of Pure and Applied Logic **110** (2001) pp. 107–234.
[6] Pétry, A. À propos des individus dans les "New Foundations" de Quine. *Comptes Rendus hebdomadaires des séances de l'Académie des Sciences de Paris (série A)* **279**, pp. 623-624.

DPMMS
CMS
Cambridge CB3 0WB
UK

Center for Philosophy of Science
Cathedral of Learning
Pittsburgh, PA 15260
USA