A compact proof that choice from well-ordered partitions fails in NF, following Elliot Glazer

Lavinia Randall Holmes

August 13, 2026

This is a working note and could easily have mistakes in it. The results in it are derived from work originally done by Elliot Glazer and ChatGPT Sol, his AI sidekick. I have used my own expertise to give a faster way to prove that it is false in NF: the combinatorics are lifted entirely from Glazer, and the final argument in terms of functions not increasing faster than T in the ordinals is mine (with a debt ultimately to Specker and C. Ward Henson).

The axiom of well-ordered choice ($AC_{\texttt{wo}}$) is the assertion that every well-ordered partition has a choice set. Given an arbitrary ordinal indexed sequence of sets X_α, we can construct a sequence of disjoint sets $Y_{T^3\alpha} = \iota^3``X_\alpha \times \{\alpha\}$ (if s is a well-ordering, we define s_α as the item in the well-ordering with segment before it of order type α: s_α is of type one lower than that of the segment, which is of type one lower than the ordinal index α). A choice set from the well-ordered partition obtained by dropping empty sets from the range of this sequence readily gives us a well-orderable set which meets each nonempty set in the original sequence.

For any cardinal κ, we define $\exp(\kappa)$ as the image under T^{-1} of the cardinality of the power set of any $X \in \kappa$. We define $\aleph(\kappa)$ as the image under T^{-2} of the cardinality of the set of order types of well-orderings of subsets of any $X \in \kappa$. These are precise analogues of cardinal operations in ordinary set theory and have familiar properties mod stratification considerations. We do point out that we are using a type level ordered pair: otherwise the exponent in the definition of $\aleph$ would be different.

We compute a bound on $\aleph(\exp(\kappa))$ for any cardinal κ. Suppose that $X \in \kappa$. Let $Y \subseteq \mathcal{P}(X)$ have a well-ordering. Then $Y \times Y$ has a well-ordering. Let $C_\alpha = (A_\alpha, B_\alpha)$ be an ordinal indexing of $Y \times Y$. We consider the ordinal indexed sequence $D_\alpha = A_\alpha - B_\alpha$. By $\mathrm{AC}_{\mathtt{wo}}$ there is a well-orderable set E which meets each nonempty D_α. The map $(A \in Y \mapsto A \cap E)$ is an injection, because if A and B are distinct elements of Y, there is an element of $A \Delta B$ in E by construction. We see that any well-ordering of a set of subsets of X is isomorphic to a well-ordering of subsets of a well-ordered subset E of X: the cardinality of E is bounded by $\aleph(\kappa)$. So $\aleph(\exp(\kappa)) \leq \aleph(\exp(\aleph(\kappa)))$.

This implies further that $\aleph(\exp(\kappa)) < \aleph(\aleph(\exp(\aleph(\kappa))))$.

For sufficiently large concrete n (large enough that all computations on both sides succeed) we have

$\aleph(T^{n-1}(|V|)) = \aleph(\exp(T^n(|V|))) < \aleph(\aleph(\exp(\aleph(T^n(|V|)))))$.

And this is impossible, for reasons which can be stated in very general terms.

Define a special function on cardinals of well-ordered sets as a function F which commutes with T, is nondecreasing, and satisfies $F(\alpha) > \alpha$.

For no special function can we have $F(T(\alpha)) > \alpha$, if $T(\alpha) < \alpha$ and α is not dominated by a cantorian $\beta = T(\beta)$.

We prove this. Suppose otherwise for a particular special F and cardinal α of a well-ordered set.

Let γ be smallest such that for some n, $F^n(\gamma) \geq \alpha$.

Note that $T(\gamma)$ is smallest such that for some n, $F^n(T(\gamma)) \geq T(\alpha)$, and by hypothesis this is the same cardinal: clearly $F^{n+1}(T(\gamma)) \geq \alpha$ so $T(\gamma) \geq \gamma$, and we have m such that $F^m(\gamma) \geq \alpha > T(\alpha)$, so $T(\gamma) \leq \gamma$.

Now let N be smallest such that $F^N(\gamma) \geq \alpha$. $N > 0$ since $F^0(\gamma) = \gamma$, being cantorian, is certainly less than α.

Certainly $T(N)$ is smallest such that $F^{T(N)}(\gamma) \geq T(\alpha)$.

We have $F^{N-1}(\gamma) < \alpha$ and $F^N(\alpha) \geq \alpha$.

We have $F^{T(N)-1}(\gamma) < T(\alpha) < \alpha$, $F^{T(N)}(\gamma) \leq T(\alpha)$, and by hypothesis $F^{T(N)+1}(\gamma) \leq \alpha$.

But this means that N must be either $T(N)$ or $T(N) + 1$.

We cannot have $N = T(N)$ because this would make $F^{TN}(\gamma) = F^N(\gamma)$ cantorian and greater than α.

We cannot have $N = T(N)+1$ because N and $T(N)$ must have the same parity (odd or even).

$F(\alpha) = \aleph(\aleph(\exp(\alpha)))$ is a special function on cardinals of well-ordered sets.

$\alpha = \aleph(T^{n-1}(|V|))$ satisfies $T(\alpha) < \alpha$ and is not dominated by any cantorian cardinal of well ordered sets.

So we have arrived at a contradiction. So our original assumption of choice from well-ordered partitions is inconsistent with NF.

Note that this is a disproof of the Axiom of Choice itself, and might reveal more about what is really going on than the original proof.