Quine's calculus of concepts *

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1 Introduction

In this paper, we give a complete presentation of a system, the Calculus of Concepts, sketched by Quine in [?] and [?]. Quine's concern was to present a system with the full expressive power of first-order logic which made no use of bound variables. In this he succeeded, but he did not equip the system with axioms or rules of inference, so the system as it stands can only be described as a sketch.

We will refer to our full system as CC in the sequel. Its notation and certain decisions about default values differ inessentially from Quine's approach. The system is more general than Quine's; it will handle multi-sorted first-order logic, for example.

2 The Natural Semantics of CC

We follow Quine's example in first presenting CC via the originally intended interpretation. It will turn out that the system has other quite natural interpretations as well.

We begin by describing the language of CC. CC is a strictly equational theory; sentences of CC are of the form $T = U$, where T and U are terms of the language of CC.

Atomic terms of the language of CC are taken from a countable supply of free variables. Atomic constants will be introduced as well. Atomic terms of CC are terms of CC; if T and U are terms, then T^c, $T \times U$, T/U, and ΔT are terms. Other operations on terms may be introduced by definition. All terms are built from atomic terms by the primitive operations and operations defined in terms of the primitive operations below. Unary operations are always considered to have higher precedence than binary operations; no other precedence convention is adopted.

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We now present the intended interpretation of the notion of "concept".

Definition: If $\mathcal{D}$ is our universe of discourse and $n \in \mathcal{N}$, we define a *concept of degree* n as a pair (S, n) with $S \subseteq \mathcal{D}^n$. (The pairing with n is needed because empty concepts of different degrees need to be distinct.) We identify 1-tuples of elements of $\mathcal{D}$ with elements of $\mathcal{D}$ itself, and we postulate a unique 0-tuple, so concepts of degrees 0 and 1 make sense.

Any formula in the language of first-order logic with variables indexed by the positive integers can be associated in a natural way with a concept (actually, with a concept of each sufficiently large degree):

Definition: A concept (A, n) is said to represent a formula ϕ just in case ϕ has no free variable other than variables x_i for $i \leq n$ and $\{(x_1, \ldots, x_n) \mid \phi\} = A$. Notice that concepts of degree 0 represent sentences with no free variables (the two concepts of degree 0 represent the truth values).

Each term of the language of CC is intended to represent a concept. Note that there is no explicit indication of numerical degree in CC; the explicit indications of degree are a feature of the interpretation being given.

We now introduce the operations of CC (in their intended interpretation)

We first define $x.y$, for $x \in \mathcal{D}^m, y \in \mathcal{D}^n$, as the element of $\mathcal{D}^{m+n}$ obtained by concatenating x and y.

complement: The *complement* of a concept (A, n), written $(A, n)^c$ is

$$(\mathcal{D}^n - A, n).$$

(Quine would write $-(A, n)$).

product: The *product* of concepts (A, m) and (B, n), written $(A, m) \times (B, n)$ is

$$(\{x.y \mid x \in A \wedge y \in B\}, m + n).$$

Quine calls this the "Cartesian product", but this is somewhat misleading, since the underlying operation is concatenation rather than pairing; for example, this operation is associative and Cartesian products properly so-called are not.

quotient: The *quotient* of a concept (A, m) by a concept (B, n), written $(A, m)/(B, n)$, is

$$(\{x \mid \exists y \in B.(x.y \in A)\}, \max(m - n, 0)\}).$$

Notice that the subtraction of degrees natural in this theory is a subtraction on natural numbers. Quine called this the "image" operation and used the double-quote notation for image to represent it. The definition of quotient when the degree of B exceeds the degree of A is a "don't care" case, and in fact is different under this definition than it will be under our axioms.

diagonalization: The *diagonalization* of a concept (A, n), written $\Delta(A, n)$ is the concept

$$(\{x.x \mid x \in A\}, 2n).$$

Quine uses I as the name of this operation.

It is quite appealing that this set of operations has the expressive power of first-order logic. The operations might be thought to have some of the same intuitive appeal as the basic operations of Boolean algebra.

3 The expressive power of CC

Theorem (Quine): The expressive power of the calculus of concepts is at least that of first-order logic with equality (if concepts are understood via the interpretation given above).

Proof: We use V^1 as the notation for the concept $(\mathcal{D}, 1)$ whose extension is the whole domain of the theory being interpreted. We can define V^n for each positive n as the product of n copies of V^1 and V^0 as V^1/V^1; V^0, the universal concept of degree 0, is an absolute notion of the calculus, unlike the V^i's with $i > 0$.

The sentence $x_i = x_j$ (for $i < j$) is represented by the concept $V^{i-1} \times (\Delta V^{j-i}/V^{j-i-1}) \times V^k$ (for any k).

If A and B are concepts of the same degree, $\Delta A/B$ is the intersection $A \cap B$ of A and B.

We proceed to represent a formula ϕ whose variables (free and bound) have indices $\leq n$ using a concept of degree n. We proceed by induction on the structure of ϕ.

If ϕ is an atomic sentence $R[x_{a_1} \ldots x_{a_k}]$, we let A be a concept representing the formula $R[x_1, \ldots, x_k]$. Let E be the intersection of the predicates of degree $n + k$ representing the equality assertions $x_{a_i} = x_{n+i}$ for each $i \leq k$. The sentence $R[x_{a_1} \ldots x_{a_k}]$ is represented by the degree n concept $((V^n \times A) \cap E)/V^k$ If ϕ is of the form $\sim \psi$, where ψ is represented by a concept A of degree n, ϕ is represented by A^c. If ϕ is of the form $\psi \wedge \chi$, where ψ and χ are represented by concepts A and B of degree n, ϕ is represented by $A \cap B$. All propositional logic operations can be defined in terms of negation and conjunction.

Let ϕ be of the form $(\exists x_i.\psi)$, where ψ is represented by a concept A of degree n. Let E be the intersection of the concepts of degree $2n$ representing the assertions $x_j = x_{n+j}$ for j less than or equal to n and not equal to i. ψ is represented by the concept $((V^n \times A) \cap E)/V^n$. The universal quantifier can be defined in terms of the existential quantifier.

The proof of the theorem is complete.

4 Rules of inference

The rules of inference of CC are those appropriate for any purely equational theory. We present them explicitly for the sake of completeness. In this section only, capital letters represent general terms of CC, while lower-case letters represent atomic terms of CC (concpet variables).

Any theorem of CC is of the form $T = U$, where T and U are terms of CC as defined above. We use the notation $A[T/x]$ to represent the result of substituting the term T for the variable x throughout the term A.

(Axiom) Axioms of CC are theorems of CC. (The axioms are enumerated below).

(Sub1) If $A = B$ is a theorem of CC and $C[A/x] = D[A/x]$ is a theorem of CC, then $C[B/x] = D[B/x]$ is a theorem of CC. This rule allows substitution of equals for equals. It also supports symmetry and transitivity of equality.

(Sub2) If $A[y/x] = B[y/x]$ is a theorem of CC (note that y as well as x is a concept variable here), then $A[T/x] = B[T/x]$ is a theorem of CC, for any term T. This rule supports the intuition that variables appearing in theorems represent arbitary concepts.

We introduce a single axiom in this section, to complete the implementation of the basic properties of equality:

(Reflex) $A = A$

5 Axioms for the calculus of degree

Developing a set of axioms for CC turns out to be an interesting exercise. The first point to observe is that, since uninterpreted terms of CC do not have explicit degree (as do the atomic concepts of the intended interpretation), it is necessary to axiomatize the properties which degree is expected to have.

We rely on the intended interpretation to develop the basic approach. In the intended interpretation, there is a canonical object in each degree which we may as well use to represent that degree: this is the universal concept $(\mathcal{D}^n, n)$. In the intended interpretation, the universal concept of the same degree as a concept A can be expressed as $(\Delta A/A^c)^c$; it is straightforward to check that this works. In uninterpreted CC, we introduce the following

Definition: V^A, called the *degree of* A, is defined as $(\Delta A/A^c)^c$. $\emptyset^A$ is defined as $(V^A)^c$.

The operations on degrees which are needed in the intended interpretation are addition and subtraction of natural numbers; our intention is that the degree of $A \times B$ will be the sum of the degrees of A and B, while the degree of A/B will be the difference of the degree of A and the degree of B. In the intended interpretation, we decided that the degree of A/B would be 0 when the degree of B exceeded the degree of A; this corresponds to a natural decision as to how to define subtraction as a complete operation on the natural numbers.

These are the axioms which we adopt for addition of degrees:

D1: $V^{V^A} = V^A$

D2: $V^{A^c} = V^A$

D3: $V^{\Delta A} = V^A \times V^A$

D4: $V^{A \times B} = V^A \times V^B$

D5: $(A \times B) \times C = A \times (B \times C)$

Axiom **D5** is given in more generality than is needed for the calculus of degrees alone.

D6: $V^A \times V^B = V^B \times V^A$

D7: $V^{A/A} = V^{B/B}$

Axiom **D7** motivates the following

Definition: We define V^0 as $V^{A/A}$ and $\emptyset^0$ as $(V^0)^c$.

Degree 0 is the only degree which can be defined in absolute terms. In the intended interpretation, degree 0 has the two inhabitants V^0 and $\emptyset^0$, which are natural representatives of the truth values.

D8: $A \times V^0 = V^0 \times A = A/V^0 = A$

D9: $A \times \emptyset^0 = \emptyset^0 \times A = \emptyset^A$

A more difficult proceeding was the selection of the axioms for subtraction of degrees. We recall having seen axiomatizations for natural number subtraction, but we were not able to discover a reference to such a definition while preparing this paper. We adopted the following axioms:

D10: $(V^A \times V^B)/V^B = V^A$

D11: $(V^A \times V^{B/A})/V^{A/B} = (V^A/V^{A/B}) \times V^{B/A} = V^B$

D12: $(V^A/V^{A/B}) \times V^{A/B} = V^A$

D13: $A/(B \times C) = (A/C)/B$

D14: $A/B = A/(B/V^{B/A})$

These axioms are motivated by the fact that $m - n = m - \min(m,n)$ under the definition of subtraction given above. Note that $m - (m - n) = \min(m,n)$ and $m + (n - m) = \max(m,n)$ can be defined on the natural numbers using addition and natural number subtraction. If the equations just given relating natural number subtraction to maximum and minimum are interpreted instead as relating a "subtraction" operation on a lattice to least upper bound and greatest lower bound, the axioms (suitably restricted to degrees) will hold on more general lattices with an addition operation; this will be exploited below.

Axiom **D13** is stated in more generality than is needed for the calculus of degrees alone, just as was the case for axiom **D5**.

Axiom **D14** implements a different approach to the problem of "bad quotients" than the one adopted in the description above of the intended interpretation. Above, we simply declared such quotients empty; the axiom given here turns out to be more convenient.

6 Boolean operations and axioms

Of the operations of Boolean algebra which we might expect to have on each degree, only complement is given to us as a primitive, and we state its axiom

B1: $(P^c)^c = P$

We define the other boolean operations by first defining *intersection* using the observation that $\Delta A/B = A \cap B$ when A and B have the same degree, along with a trick.

Definition: $A \cap B$, the intersection of concepts A and B, is defined as $\Delta(A \times V^{B/A})/(B \times V^{A/B})$.

The trick is observing that $A \times V^{B/A}$ and $B \times V^{A/B}$ have the same degree. In the intended interpretation, this will be the maximum of the degrees of A and B. From the axioms, one can only prove that the degree of each of the expressions will be equal to the least upper bound of the degrees of A and B in a suitable order (and it does happen in interpretations in many-sorted logic that this may be strictly "larger" than both A and B).

The rest of the boolean axioms follow:

B2 (a definition): $P \cup Q = (P^c \cap Q^c)^c$

B3 (a definition): $(V^A)^c = \emptyset^A$

B4: $P \cap Q = Q \cap P$

B5: $(P \cap Q) \cap R = P \cap (Q \cap R)$

B6: $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$

B7: $P \cap \emptyset^Q = \emptyset^{P \times (Q/P)}$

B8: $P \cap V^Q = P \times V^{Q/P}$

B9: $P \cap P^c = \emptyset^P$

B10: $P \cap P = P$

7 Axioms of padding

The introduction of products with universal concepts (which corresponds to "padding" formulas with dummy variables in the intended interpretation) would obstruct such simple procedures as dualizing axioms B1-10 if it were not for these (obvious) axioms:

P1: $(V^A \times B \times V^C)^c = V^A \times B^c \times V^C$

P2: $V^A \times (B \cap C) \times V^D = (V^A \times (B \times V^{C/B}) \times V^D) \cap (V^A \times (C \times V^{B/C}) \times V^D)$

P3: $A \times B = (V^A \times B) \cap (A \times V^B)$

Axioms P1 and P2 tell us that "padding" commutes with Boolean operations (an earlier axiom tells us that V^0 is left and right identity for product, so we can convert P1 and P2 to forms suitable for considering padding on one side).

Axiom P3 shows us that product reduces to padding and Boolean operations.

8 Axioms of quantification

Quotients by arbitrary concepts can be reduced to quotients by universal concepts and padding (similar to axiom P3). The appearance of $V^{B/A}$ (which will be V^0 in all situations we care about) is a degree arithmetic fix for bad quotients.

Q1: $A/B = (A \cap ((V^{A/B} \times B)/V^{B/A}))/V^B$

Q2: $(A/V^B)^c = (A/V^B)^c \cap (A^c/V^B)$

Q3: $(A \cap B)/V^C = (A \cap B)/V^C \cap ((A \times V^{B/A})/V^C)$

Q4: $(A \cup B)/V^C = ((A \times V^{B/A})/V^C) \cup ((B \times V^{A/B})/V^C)$

Q5: $(A \times B)/V^{B/(B/C)} = A \times (B/V^{B/(B/C)})$

Q6: $A = A \cap ((A/V^{A/(A/B)}) \times V^{A/(A/B)})$

To understand the forms of axioms Q5 and Q6, note that $V^{B/(B/C)}$ will be the minimum degree of V^B and V^C; we are ensuring that we do not take a quotient by a universal concept with degree greater than that of B.

9 Axioms of equality

There now remains the diagonalization operation. Like the padding and quotient operations, it can be simplified using Boolean operations.

E1: $\Delta A = \Delta V^A \cap (A \times V^A) = \Delta V^A \cap (V^A \times A)$

It is an easy consequence of E1 and the distribution axioms for padding that diagonalization distributes over Boolean operations relative to ΔV^A.

We introduce a definition to make it easier to talk about general equality concepts:

Definition: The concept $\mathrm{Eq}(A, B, C)$ is defined as $V^A \times (\Delta V^{B \times C}/V^C)$.

Note that this definition depends only on the degrees of its arguments. $\mathrm{Eq}(A, B, C)$ contains all those sequences of variables with an initial segment in V^A, a second segment in V^B, a third segment in V^C and a fourth segment equal to the second segment. $\mathrm{Eq}(A, B, C)$ is the general form of a concept capturing all those sequences with a final segment in V^B equal to an earlier non-overlapping segment in a given position. We now state the further axioms of equality:

E2: $(\mathrm{Eq}(A, B, C) \times V^{D \times B}) \cap \mathrm{Eq}(A, B, C \times B \times D) \cap \mathrm{Eq}(A \times B \times C, B, D) = (\mathrm{Eq}(A, B, C) \times V^{D \times B}) \cap \mathrm{Eq}(A, B, C \times B \times D) = \mathrm{Eq}(A, B, C \times B \times D) \cap \mathrm{Eq}(A \times B \times C, B, D) = (\mathrm{Eq}(A, B, C) \times V^{D \times B}) \cap \mathrm{Eq}(A \times B \times C, B, D)$

E3: $(\mathrm{Eq}(A, B, C) \cap D^c)/V^{B/(B/E)} = ((\mathrm{Eq}(A, B, C) \cap D)/V^{B/(B/E)})^c$

E4: $(\mathrm{Eq}(A, B, C) \cap (D \cap E))/V^{B/(B/F)} = ((\mathrm{Eq}(A, B, C) \cap (D \times V^{E/D}))/V^{B/(B/F)}) \cap ((\mathrm{Eq}(A, B, C) \cap (E \times V^{D/E}))/V^{B/(B/F)})$

E5: $\mathrm{Eq}(A, B \times C, D) = (\mathrm{Eq}(A, B, C \times D) \times V^C) \cap \mathrm{Eq}(A \times B, C, D \times B)$

Axiom E2 is the transitive law of equality. In this context, where "variables" (positions in argument lists of concepts) cannot be permuted or duplicated (these effects are simulated using diagonalization), we should not expect to find analogues of the reflexive or symmetric properties of equality.

Axioms E3 and E4 assert that existential quantification distributes over all Boolean operations when the variable quantified over is asserted to be equal to a variable which remains free. (It also applies to blocks of variables). Axiom E5 allows equality between blocks of variables to be analyzed into equalities between sub-blocks.

10 Why these axioms? Showing that they work!

Reasoning in the calculus of concepts with these axioms corresponds to a regimented style of reasoning in which any primitive predicate appears applied to a consecutive block of variables (shifting blocks of variables causes no problems because of the degree arithmetic and padding axioms) and in which all quantifications are applied to the last free variable available (which means that we need to reason with labelled formulas; this is also handled by the degree and padding axioms and related features of the boolean axioms). These features together eliminate the need for bound (or any) variables.

Modulo these restrictions, the boolean and quantification axioms are easily seen to be adequate mod adequacy of our extended boolean algebra.

The factor which needs to be considered is the possibility of permuting or identifying variables. This is handled via the expedient of introducing new variables, declaring them equal to earlier variables (using concepts $\mathrm{Eq}(A, B, C)$), asserting desired properties of the sequence of variables, then quantifying them out (as we did in the interpretation of first-order logic in CC). Axioms E3 and E4 ensure that this process commutes with Boolean operations (as it should). Axiom E1 allows facts about equal blocks of variables to be transferred from one block to another. Axiom E5 ensures that equality between blocks is related correctly to equality between corresponding sub-blocks. Axiom E2 allows us to deduce any additional equalities between blocks from given equalities between blocks (where equalities between overlapping blocks are involved, the use of E5 may be needed). As noted earlier, no analogue of symmetry or reflexivity is needed (or possible!)