Review of "Finsler Set Theory: Platonism and Circularity", David Booth and Renatus Ziegler, eds.

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1 Introduction

This book contains translations of papers on set theory by Paul Finsler with supporting materials prepared by the editors.

The book is divided into three parts, a philosophical part, a foundational part, and a combinatorial part.

We summarize our conclusions at the outset. Finsler has interesting things to say about the philosophy of set theory and of mathematics in general. He has an interesting analysis of the notion of set, and of the reasons why the paradoxes of self-reference present themselves and the nature of an appropriate response to the paradoxes. He defends a Platonist view of mathematics using arguments which we find congenial.

Finsler's contributions to foundations are interesting, but ultimately unsatisfactory in the form in which he presents them. We agree with the apparent consensus of modern set theorists that the Finsler set theory is incoherent as presented. In asserting that Finsler's full set theory is incoherent, we are taking issue not only with Finsler himself, but with the editors of the volume. However, Finsler's concept of "circle-free" sets can be successfully implemented; this is seen in the set theory of Ackermann, a complete description of which can be extracted from Finsler's papers presented in this book; discovering this fact made reading the book worthwhile for the reviewer. Finsler's notion of allowing non-well-founded sets but strengthening the notion of extensionality on non-well-founded sets has been carried further by Scott and Aczel, among others, and systems with "anti-foundation axioms" resembling Axiom II of Finsler's system are now popular and are finding applications.

Finally, the combinatorial aspects of Finsler's set theoretical work issue from a specific line of investigation into the properties of small non-well-founded sets which can be carried out in the various set theories defined in Aczel's book on non-well-founded set theory as well as in the Finsler set theory. No profoundly interesting combinatorial results are given in this book, but it seems that there might be potentially interesting problems in finite combinatorics in the investigation of hereditarily finite non-well-founded sets. There is a philosophically interesting idea to this section: Finsler enunciates a view of sets as "generalized numbers", regarding a set as having its elements as "predecessors". This idea may be of some use in trying to understand Finsler's intuitions about set theory. Finsler goes on to define analogues of arithmetic operations on small sets. We will have nothing further to say about this aspect of the book in this review, but the article by Booth introducing this part and at least the first article by Finsler are worthwhile.

2 Finsler's Platonism

There are many things which we find extremely problematic in Finsler's work, but his defense of mathematical Platonism is not one of them. It is Finsler's Platonism which leads him to the conviction that the paradoxes of set theory must (and should be able to be) *solved*, not endured (as somehow unavoidable) or merely avoided (as by schemes of type theory). Finsler believed that the paradoxes arose from errors in reasoning which it should be possible to identify clearly. He resists the idea that the mathematical world is "constructed" by human thought. We quote from p. 79: "Now an antinomy would result if ... we were to see ourselves being compelled to 'form' mathematical objects (for example, the set of all transfinite ordinal numbers) that is, to *affirm* their existence. This idea confuses the *class* of the ordinal numbers and the differently defined *set*. In an exact set theory sets do not arise through an act of collecting, but are mathematical objects with definite properties, in exactly the same way as the natural numbers are objects with definite properties. From p. 80: "In order to be able to know what is true and what is false in set theory, one has to have won back the certainty lost through the antinomies, however, nothing stands in the way. This does not mean that one could then solve every single question; this is not the case elsewhere in mathematics, either. ... By "Platonism" one can of course understand various things. In the case before us this expression only means that in the realm of set theory too, objective relationships do exist; it is not meant that sets would have to be given to us in some way other than our knowledge of their existence. Consequently, quite another question is how we know these objective relationships and how are we to base classical mathematics on them?" Later, same page, referring to some disagreement with Bernays about the significance of Ackermann's formalization of part of Finsler's theory (which was probably independent of Finsler's work): "The actual existence of infinitely many things cannot be guaranteed in this way. This turns mathematics into a 'doing as if', pretending that there are infinitely many things. I cannot accept this." Finsler does state a criterion for the existence of mathematical objects: on p. 169: "consistent things can always be taken to exist; in pure mathematics, existence means nothing more than freedom from contradiction". The reviewer finds nothing here to disagree with.

3 Paradoxes of Self-Reference

Paradoxes of self-reference are addressed by Renatus Ziegler, one of the editors, in an introductory article, "Intrinsic Analysis of Antinomies and Self-Reference". We found this article not to be helpful. We summarize our objections to its approach by considering an example and then considering the consequences of this approach for an algebraic example.

The examples that Ziegler discusses are of the form

a**:** b is true

b**:** a is false

This pair of sentences is circularly defined and certainly paradoxical. Ziegler alleges that the reason that the paradox arises is that the two occurrences of a and b respectively are identified; if the identification were shown to be unsound, the paradoxical character of the pair of sentences would be resolved. We actually agree with this assertion. But Ziegler goes on to find the difference between the two occurrences of a (for example) in the fact that the second occurrence of a is presented to us as the subject of a proposition while the first occurrence of a is presented to us as an entire proposition. On this basis, he tells us that the two occurrences of a (and likewise the two occurrences of b) should be differnetiated (say by adding superscripts) producing a situation something like this:

$a^{(1)}$**:** $b^{(1)}$ is true

$b^{(2)}$**:** $a^{(2)}$ is false

Of course, there is no longer any contradiction in this set of sentences. Ziegler says that this is the result of "strictly following the *principle of identity*, which says that only objects possessing identical properties may be identified" (p. 17).

We present a *reductio ad absurdum* of this line of argument. Consider the familiar identity

$$x + x = 2x$$

. The three x's here are presented to us in quite different ways: the first is the left term of a sum, the second is the right term of a sum, and the third is not a term at all but a factor. Thus, they must be distinct: the correct way of presenting this identity is

$$x^{(1)} + x^{(2)} = 2x^{(3)}$$

. Of course, this is entirely absurd: the whole point of the identity (and any other algebraic equation) is that a letter such as x *can* refer to the same object in different contexts. Indeed, it must refer to the same object in our algebraic context; the modified identity is no longer valid (it is the same as asserting that

$$x + y = 2z$$

for all x, y, and z)! The differences between the contexts in which different occurrences of a letter like x occur do not make the referents of the different tokens different.

The sentences must be understood as intended: the two occurrences of a and the two occurrences of b are indeed to be understood as having identical referents if they can be regarded as intelligible at all. If we cannot understand different occurrences of the same symbol as having the same referent, we cannot engage in mathematical discourse of even the simplest kind.

Something can be made of what Ziegler is trying to say if we explicitly take into account the fact that the constructions involved in paradoxes of self-reference are "token-reflexive", so where a token (of certain specific kinds) occurs can affect its meaning; Finsler does something like this with his idea of "implicit content" below.

Finsler himself has much more reasonable things to say about paradoxes of self-reference; the reader should look at Finsler's articles in the book and disregard Ziegler's treatment.

Finsler states (we believe correctly) that the difficulty with examples like the one above is that they involve *circular definition*, and that a circular definition does not need to be satisfied. It is more usual to assert that a "circular definition" is not a *definition* at all, but given that Finsler allows such things to be called definitions, what he has to say about them is unexceptionable. A trio of examples taken from his paper "Are there contradictions in mathematics" (presented in this volume) should make this clear:

$$x = a - b,$$

where a and b are previously given numbers not depending on x, defines a number x.

$$x = a - x$$

is "circular", and is not a definition in the usual sense, but it is nonetheless successful in specifying a unique value $\frac{a}{2}$ for x. In the final example

$$x = a + x,$$

in which a is understood to be nonzero, we see a "circular definition" which is not satisfiable. Notice that Ziegler's treatment would entirely destroy the sense of either of the last two equations.

To see how Finsler approaches an example more similar to Ziegler's, of a circularly defined *proposition*, consider his treatment of the Liar in the paper "Are there undecidable propositions?" His conclusion is that the Liar sentence "This sentence is false", which in Ziegler's notation might be presented as

s: The sentence s is false

is actually non-paradoxically false. For he notes that s has not only an explicit content "sentence s is false" but an implicit content "sentence s is true" deriving from the fact that it *is* sentence s. The assertion "sentence s is false" is true, if it is written anywhere but next to the label s, where it has an implicit conjunct "s is true" which renders it contradictory. We are actually paraphrasing his argument, because we are using the labelling idiom from Ziegler, but we believe, and the reader may check to his own satisfaction, that we are much closer to Finsler's own assertion.

Another example, taken from the paper "Are there contradictions in mathematics", is that of a blackboard on which are written the symbols 1, 2, 3, and "the smallest natural number not represented on this blackboard". If we call the blackboard B, Finsler's conclusion is that the smallest natural number not represented on blackboard B is 4, but that the occurrence of the string "the smallest natural number not represented on this blackboard" actually written on B does not represent any natural number; the naive argument that it represents 4 (which leads to a paradox!) disregards the implicit content "I am a number written on B" of the string, which contradicts the explicit content and causes the string to fail to refer.

The idea of implicit content which Finsler uses here seems to be supplementary to his argument that circular definitions are not necessarily satisfied; the notion that circular definitions are not necessarily satisfied can be used by itself to conclude that the labels a and b in Ziegler's example do not necessarily refer to any sentences, and the same for the Liar sentence and the purported number on Finsler's blackboard. The occurrence of paradoxes if they are understood as referring can be used to draw the stronger conclusion that they actually do not refer. Notice that our conclusion with regard to the Liar differs from Finsler's: we would prefer to regard the Liar sentence as not being a sentence rather than as being false.

Now we consider Finsler's application of these ideas to the paradoxes of set theory. He asserts that the Russell paradox of "the set of all sets which are not elements of themselves" is in fact a circular definition. The reason for this is that Cantor's original definition of the notion of a set is understood by Finsler to be a circular definition. He says that there would be no such problem with Cantor's definition if it spoke of collections of e.g., concrete material things, which are themselves understood without reference to the notion of "set", but as soon as we allow sets themselves and the relation of membership to enter into the definitions of further sets, we are in the realm of circular definition.

We clarify this point by considering two examples. We consider a set A, the set of all stars in the Milky Way Galaxy which are brighter than our Sun. The stars in our Galaxy are objects which we understand without any reference to sets. We may consider sets of stars (collections of stars considered as one thing) without fear of circularity. For any star s, we have $s \in A$ (s is an element of A) defined as "s is a star in the Milky Way brighter than the Sun"; note that the definition of $s \in A$ eliminates all reference to set theoretical concepts. Now consider a set R, defined as "the set of all sets which are not elements of themselves": for any set x, $x \in R$ is defined as "x is not an element of x", or, more briefly, $x \notin x$. Notice that the set theoretical relation $\in$ is not eliminated here; membership in the set R is defined in terms of further information about membership in sets. Now observe that the sentence $R \in R$ is defined as $R \notin R$, a contradiction!

Finsler does not believe that the solution to the paradoxes is to be found in an attempt to eliminate the circular definitions; he thinks, and mathematical experience reveals as well, that the "circularity" here (the dependence of some assertions about membership in sets on further assertions about membership in sets) is *essential*. For example, the set $\mathcal{N}$ of natural numbers is defined as the intersection of all sets which contain 0 and are closed under the successor operation (in the usual set theory ZFC, 0 is defined as the empty set and the successor operation applied to a general set x is $x \cup \{x\}$). The assertion that a given object n is a natural number is expanded via the definition of $\mathcal{N}$ into the assertion that n belongs to each of a very large class of further sets (and, even worse, this class includes $\mathcal{N}$ itself!) The viewpoint adopted in modern set theory is that comprehension axioms which assert the existence of sets defined by properties of their elements are not definitions at all; Finsler's view was that they are definitions, but, since they are circular, only some of them will succeed; this may ultimately be no more than a difference in terminology.

We have now summarized Finsler's view of the nature of the paradoxes of self-reference. Equally worthy of note is his attitude toward the paradoxes. He is confident that the paradoxes represent, not some essential deficiency in human understanding, but an *error* in reasoning that can be identified and corrected. He dislikes solutions to the paradoxes along the lines of type theory, which seem to him only to avoid the contradictions, not to explain them. The reviewer finds this attitude entirely congenial. His confidence that the paradoxes represent mistakes which can be corrected rather than fundamental limitations of reason goes along with his Platonist philosophy of mathematics. We also think that he correctly identifies the nature of the mistake, in the case of the paradoxes of set theory, as having to do with the distinction between a class (an arbitrary collection of objects) and a set (a particular single object which we have associated with some collection); in fact, the reviewer has independently articulated an explanation of the paradoxes resting on the same distinction. A further discussion of this distinction belongs to the "foundational part" of the review.

4 Formally Undecidable Propositions

Two papers by Finsler on undecidable propositions are included. One of these predates Gödel's first paper on the subject. We briefly describe Finsler's construction of a sentence which is formally undecidable. We are given a fixed language L consisting of finitely (or countably) many symbols. A fixed alphabetical order on these symbols is also given. A fixed dictionary D giving meanings for finite combinations of the symbols taken from L is also given. An object will be called "finitely definable" if there is a finite collection of symbols from L with which it is associated in D. Finsler now reasons about binary sequences (countable sequences of 0's and 1's); he points out that diagonalization over the finitely definable binary sequences gives a binary sequence which can have no definition in D.

Finsler defines a *formal proof* as a finite combination of symbols of L whose translation via D is a correct proof. We consider all formal proofs for the fact that the number 0 occurs infinitely often in a given binary sequence or for the fact that it does not occur infinitely often. We list these proofs lexicographically. Each such proof determines a binary sequence (the one in which it proves that there are infinitely many 0's); we form the sequence whose nth term is different from the nth term of the binary sequence associated with the nth proof for each n. Call this sequence s. We now consider the statement "the number 0 does not occur infinitely often in s" (or its negation!) This sentence cannot be decided by any formal proof (by construction). It is, though, clearly the case that 0 occurs infinitely many times in the sequence s (there are, for example, infinitely many proofs that the sequence consistning entirely of 1's does not contain infinitely many 0's), so the statement we have given is false, though formally undecidable – and we have given a "proof" that it is false!

Finsler has not anticipated Gödel here; the contribution that Gödel made was to provide a completely formal definition of the notion of a proof, and to see what really could be done formally with this notion. We can formally define the predicate of strings "is a proof of a sentence of the form "b is a binary sequence and there are infinitely many n such that $b(n) = 0$" or of the negation of such a sentence". Given this, we can try formally defining the notion "if the nth string satisfying the predicate above has $b(n) = 0$, where b is the sequence to which the nth proof refers, then 1 else 0". This would be Finsler's s. But this definition does not succeed, because it requires a truth predicate: we need to be able to determine whether the sentence $b(n) = 0$ holds. Tarski's theorem tells us that it is not possible to define the truth predicate of a language inside that language. Finsler has not achieved what Gödel achieved, because the sequence s cannot actually be defined in his language, and so his "formally undecidable sentence" is not a sentence expressible in his language at all.

We outline a correct development of a formally undecidable sentence along similar lines. Define P_n as the nth proof deciding a sentence "b is a binary sequence and there are infinitely many n such that $b(n) = 0$". Define b_n as the name of the sequence referred to in P_n. Define S as the sequence whose nth term is 1 if we can prove that $b_n(n) = 0$ and 0 otherwise. Certainly S is a binary sequence (this can be proven). Further, we can prove that there are infinitely many n such that $S(n) = 0$; consider n such that P_n proves that the constant sequence 1 does not have infinitely many 0's: there are infinitely many such n, in each case we can prove that $b_n(n) = 1$, so $S(n) = 0$. This proof can be carried out (details omitted). This proof P_m has $b_m = S$. Now consider $S(m)$: this is 1 if we can prove that $b_m(m) = 0$ and 0 otherwise; but $b_m = S$! We can see that the sentence $S(m) = 0$ is formally undecidable; $S(m) = 0$ exactly if we cannot prove $S(m) = 0$! We can also see that $S(m) = 0$ is true; if it were false, that is, if $S(m) = 1$, we would be able to prove $S(m) = 0$, which is absurd. But $S(m) = 0$ is not the analogue to Finsler's sentence: the analogue to Finsler's sentence is "$S(n) = 0$ does not hold for infinitely many n", which is both false and provably false.

The editors of the book are clearly aware that the distinction between Finsler's and Gödel's approach is that Gödel has expended more effort on the "arithmetization of syntax". They do not appreciate the fact that Finsler's approach, though more cursory in the area of formal syntax, does require a correct understanding of what can be referred to in the dictionary D. Finsler is *wrong* when he asserts (p. 54) that the formal proof he gives there can be expressed in words taken from D; the truth predicate for his language is required in a full definition of his "anti-diagonal sequence", and this predicate cannot be defined in D. It *is* necessary, even from Finsler's more philosophical standpoint, to concern oneself with what one really can express in one's language and what one cannot. A modern editorial treatment should have included a formal analysis of what Finsler actually did prove.

5 Finsler set theory

5.1 Naive set theory; paradoxes; sets and classes

We now turn to Finsler's development of the foundations of mathematics. We review his development at the beginning of the paper "On the foundations of set theory, part 1". He begins by remarking that the assumption that arbitrarily specified things can always be combined into a set is the foundation of naive set theory, and that it inevitably leads to contradiction. He gives the definition of the Russell class as an example of why the naive assumption is false.

He gives two reasons why the naive approach is in error. The first is the same consideration as above: the assumption that we can collect arbitrary objects together is safe as long as it is made in a "circle-free" context, that is, as long as we do not admit (or at least somehow restrict) the formation of collections whose elements are collections in their turn. Definitions of collections like the universe, which must contain themselves, and of other collections whose definitions depend in more complicated ways on their own presence, are "circular definitions" which can sometimes be satisfied and sometimes cannot be satisfied.

The second reason he gives is that the naive approach fails to distinguish between "sets" and "classes". In the definition of a set, there are two components: the specification of a collection of objects and the association of a unique single object with that collection. What Finsler suggests (quite in line with modern thinking) is that it is always admissable to discuss a collection of objects (a "class") of objects of our universe of discourse defined in whatever manner (as long as it is defined precisely) but that it is a further step to assert that there is a unique object in our universe of discourse (a "set") which we can identify with this collection, and which can itself participate in further collections. Finsler says "...sets are things which *correspond* to collections, in so far as this is consistent. It is in general better not to refer to collections as 'things'". We can then say that the error of the naive approach is that it confuses class and set: we can define a collection of sets (or other objects) freely (this much of the naive approach is sound) but we cannot then freely assume that the collection (a class) is associated with an object in our universe of discourse (a set). As in Russell's paradox, we can define a collection of sets in such a way as to frustrate the possibility of this collection being identified with any of the sets we have available.

Finsler suggests (again in line with modern thinking) that we should consider only "pure sets", those whose elements, elements of elements, etc. include only sets. He observes (in line with the discussion of the previous paragraph) that we can freely construct classes or "systems" of sets, but that we cannot assume without restriction that these will be sets in their turn. Systems are not really objects of our universe of discourse, so we do not investigate the possibility of forming systems of systems, or higher iterations.

5.2 Finsler's axioms

We now state Finsler's axioms. We must at this point note Finsler's preference for the converse of the usual membership relation: he writes $x\beta y$ where we would usually write $y \in x$, and β is primitive for him.

"We consider a system of things, which we call *sets*, and a relation, which we symbolize by β. The exact and complete description is achieved by means of the following axioms.

I. Axiom of Relation: For arbitrary sets M and N it is always uniquely determined whether M possesses the relation β to N or not.

II. Axiom of Identity: Isomorphic sets are identical.

III. Axiom of Completeness: The sets form a system of things which, by strict adherence to the axioms I and II, is no longer capable of extension.

That is, it is not possible to adjoin further things in such a way that the axioms I and II are satisfied."

Finsler's axiom I asserts that sets are well-defined collections: everything is either in a given set or not in it.

Finsler's axiom II (which appears in various forms) needs explanation. To understand its place in his theory, it is sufficient to understand it as saying that the identity of a set is determined by the isomorphism type of the relation β restricted to the transitive closure of the set under the relation. One of the consequences of axiom II is extensionality: sets with the same elements are the same. But it is stronger than that: it is easy to see, for example, that any two sets which are their own sole elements will have isomorphic transitive closures and so will be the same. Axiom II is an "anti-foundation axiom" in the sense of Aczel. The precise form of isomorphism needed is a subject for technical adjustment, and Finsler did have occasion to change it.

The reviewer (and many earlier workers) have serious difficulties with axiom III. We have struggled to understand what is meant by this axiom and how Finsler could draw his stated conclusions from it, and we have been unable to come up with a coherent explanation.

Finsler claims that the following are consequences of axiom III:

Proposition 6: For any well-defined class of sets, there exists a set which contains each member of the class, if and only if the assumption that such a set exists does not contradict axiom I.

Finsler asserts that proposition 6 is a consequence of axiom III but not equivalent to it; he points out that a set which is its own sole element is not provided by proposition 6 alone.

Proposition 7: An arbitrarily defined set M exists…if the assumption that such a set M exists does not contradict the first two axioms.

Finsler asserts that proposition 7 is equivalent to axiom III.

An objection to Finsler's theory expressed by Specker is that proposition 7 can clearly be seen to be false if "arbitrarily defined" really means what it says. It is possible to have the universe, the set of all x such that $x = x$, in a model of Finsler's axioms I and II (consider a system considering only of a set and its own sole element). In fact, Finsler claims that the existence of the universe follows from his axioms. It is also possible to have the set of all x such that x is an element of some set y and x is not an element of x: consider the system of all sets of hereditarily finite sets–the set of all hereditarily finite sets is the collection of all elements which are not elements of themselves in this system. No model of axioms I and II can contain both of these sets, so proposition 7, naively understood, cannot be true in any system. Finsler and the editors of this volume articulate objections to Specker's counterexample, but I do not understand the objections. They can only be understood if there is some kind of restriction to be placed on acceptable definitions of sets M which has not been made clear.

We must observe in this context that the example the editors give on p. 97, the set $W = \{x \mid x = 0 \text{ and } x = y \text{ for every set } y\}$ does not have the properties they ascribe to it (they claim that it is paradoxical). W is the empty set, pure and simple (if it exists); there can be an x with the properties ascribed to an element of W only if 0 is the only set (we assume that the authors follow the usual convention that 0 is the empty set), in which case W could not exist, but nothing precludes W existing, being empty, and there being other sets than the empty set (which we think is the actual situation). Note that the definition of W clearly succeeds in defining a set in Zermelo set theory, since it is a subset of the natural numbers defined by a first-order formula!

5.3 Model theory of Finsler's axioms

Finsler seems to believe (reading the preamble to his axioms) that they are categorical. They are not. We state some definitions of our own:

Definition: A *Finsler premodel* is a (class) relation R the union of whose domain and range is a class X with the property that for all x and y in X, if the restriction of R to the transitive closure of x under R and the restriction of R to the transitive closure of y under R are isomorphic relations (in a suitable sense whose details do not matter here) then $x = y$.

A Finsler premodel is a model of axioms I and II.

Theorem (Baer): If the union X of the domain and range of the defining relation R of a Finsler premodel is not the universal class, then the Finsler premodel can be properly extended.

Proof of Baer's Theorem: Choose an object y not an element of X and extend R to a relation R' by stipulating that $yR'x$ holds precisely if xRx does not hold, for each x in X. If the restriction of R' to the transitive closure of y were isomorphic to the restriction of R to any x in X, that element x would be the Russell class in the original Finsler premodel, which is impossible.

Finsler (and the editors) object to Baer's assumption that he can introduce a new object. This is fine; Baer's theorem is understood by us as well as by them to require that a maximal Finsler premodel must have the union of the domain and range of its membership relation be the universal class. It is quite easy to believe that there are Finsler premodels in which the entire universe participates; but these fail to be extendible only in the quite trivial sense that there is no new object which can be adjoined to them.

Theorem (ours): Given a maximal Finsler premodel in which the universal set does exist, it is possible to construct a maximal Finsler premodel in which the universe does not exist.

Proof: Take the object v such that vRx holds for all x, and modify the definition of R to R': "$xR'y$ iff either $x \neq v$ and xRy or $x = v$ and R restricted to the transitive closure of y with respect to R is well-founded". This amounts to replacing the extension of the erstwhile universal set with the extension of the erstwhile class of well-founded sets (which cannot be a set). It is straightforward to check that this is still a maximal Finsler premodel but no longer has a universal set.

The last theorem is sufficient to establish that Propositions 6 and 7 do *not* follow from axioms I-III. This establishes that Finsler's treatment of set theory is basically incoherent.

Having given our own development of the model theory of Finsler's axioms I-III, we turn to Finsler's own.

Finlser proposes to construct a model of his theory in the following way: given any collection (however large) of models of axioms I-II, we can take their union, identifying those elements of distinct models which have isomorphic transitive closures under the local membership relation. He proposes to construct a maximal model of axioms I-III by carrying this out for *all* models of axioms I-II. An objection which he notes to this is that he is not restricting his models of axioms I-II to be sets, so he is considering the union of a system of proper classes! I will allow him this construction for the sake of argument, however. The punchline is that this construction can "almost" be carried out in standard set theory with proper classes (as, for example, by Aczel), and it gives what we would call a maximal Finsler premodel with rather nice properties, but without a universal set (and so not satisfying his propositions 6 or 7). The object produced by this construction is an extension of the model of *ZFC-* with Finsler's anti-foundation axiom that Aczel constructs; it is an extension because the class of relations that are used is larger. The reason I say "almost" above is that there is a technical problem: the construction of this model appears to require superclasses – it will be larger than the original model of the theory of sets and classes one started with.

5.4 Finsler's mistake

We discuss the error in reasoning behind Finsler's axiom III. Finsler states that the criterion for mathematical existence is consistency, which is a reasonable criterion for a Platonist. On p. 169: "consistent things can always be taken to exist; in pure mathematics, existence means nothing more than freedom from contradiction". We agree with this criterion, on the whole; we agree that every structure that we can describe consistently is a legitimate object of mathematical study and must be taken to exist from a Platonist standpoint. But we do not agree that it follows from this that every consistently satisfiable set definition can be satisfied at once (Finsler's axiom III). Moreover, we believe that we can identify the mistake. The set of all sets is a satisfiable object; we can present a model with a relation β_1 of converse membership in which this object is found. The set of all elements which are not elements of themselves is also a satisfiable object; we can present a model with a relation β_2 of converse membership in which this object is found. The illicit further step which Finsler takes is to think that we can identify the membership relations on these two structures. The reason that we cannot identify them is that each of these set definitions places restrictions on what other sets there can be in the model which includes it. Each of these definitions has consequences for the kinds of sets there must be which preclude the satisfaction of the other definition. Each "set" must be possible to discover in the Platonist universe, but they will not be found in the same set theory. Another point against the intelligibility of axiom III is that we can at least entertain doubts that the totality of all consistent definitions is a consistent totality.

A corrective to the reasoning behind axiom III would be to say that all those sets can be taken to exist simultaneously whose definitions do not depend on the question of what sets exist in general; it is reasonable to suppose that such definitions would be compatible with one another. This is a vague idea, but it is made much more precise in the motivation for the set theory of Ackermann which I will quote later. In Ackermann's theory, there are sets and classes–a class of sets is defined by any condition, and a class all of whose elements are sets is itself a set if it can be defined by a condition which *does not depend on the notion of sethood*. Curiously, the set theory of Ackermann can be understood as an implementation of Finsler's notion of "circle-free" sets, our next topic!

5.5 Circle-free sets and the set theory of Ackermann

We now discuss Finsler's concept of "circle-free" sets. Certainly a set which is included in its own transitive closure is "circular"; all circle-free sets must be well-founded. Finsler assumes further that any set whose definition leads to paradox is "circular" (in the sense of being "circularly defined"). The set of all well-founded sets, for example, is a paradoxical object; since it is a collection of sets none of which are contained in their own transitive closures, it should not appear in its own transitive closure – but then it should belong to itself, and so to its own transitive closure!

Finsler is led to the conclusion that the correctly defined class of circle-free sets is itself a circular set (in the sense of "circularly defined").

Definition: A set M which is said to be *circle-free* if M together with every set in the transitive closure of M is independent of the concept "circle-free".

This definition is somewhat paraphrased from the original paper.

Sets not in their own transitive closures which are not circle-free are said to be "circular".

Finsler asserts the following propositions about circle-free sets:

Proposition 9: If M is circle-free then every set in the transitive closure of M is circle-free and distinct from M.

Proposition 10: Every well-defined class of circle-free sets forms a set. This can be either cirlce-free or circular, but it is disinct from every set which is contained in it.

Proposition 12: A well-defined class of circle-free sets forms a circle-free set iff it is independent of the concept "circle-free", i.e., iff it can be so defined that the definition always yields the same class regardless of which sets are classified as being circle-free.

Proposition 16: Each well-defined class of elements of a circle-free set M forms a circle-free subsetof M.

The editors seem to believe that the axiom that every subclass of a circle-free set is a set is not found in Finsler (p. 101)! It is found, as we indicate here, and essential use of it is made (as in Ackermann's later work) to prove the existence of power sets.

It should be noted that Finsler merely asserts the basic propositions about circle-free sets; he does not attempt to deduce them from axioms I-III.

Compare this to a subtheory of the set theory of Ackermann, which we present as a theory with sets and classes. We warn the reader that "set" does not coincide with "element" in Ackermann's system; it is a theorem of Ackermann's system that there are non-sets which are elements of classes.

1. Any condition whatever on sets defines a class (not necessarily a set). Note that we do *not* require that all elements of classes are sets.

3. Any element of a set is a set.

4. Any subclass of a set is a set.

5. Any class all of whose elements are sets is a set if the class can be defined without reference to the property of being a set or to any non-set parameter.

All of the axioms we give are found in Finsler, if we interpret the class of "sets" as Finsler's class of circle-free sets. It is known that Ackermann's theory is essentially equivalent to Zermelo-Frankel set theory (with all sets of *ZFC* being understood to be sets in Ackermann's sense, and classes of sets being understood to be proper classes). The mathematically interesting work in Finsler's paper is all derived from these axioms (not from Axioms I-III), and in essentially the same way that Ackermann later derived the same propositions from his axioms. The axiom of foundation is sometimes added to Ackermann's axioms (as for instance by Levy), and certainly holds for circle-free sets. The axiom of infinity is provable from these axioms; an analogue of Ackermann's proof was given earlier by Finsler (in the paper "The existence of the natural numbers and the continuum").

Just for fun, we prove that there must be an ordinal in Ackermann's system which contains a non-set ordinal as an element. The class of set ordinals exists by Ackermann's axioms; it is itself an ordinal, which we will call Ω, and cannot be a set. If all elements of ordinals were sets, then we could use the predicate "is an element of some ordinal", which does not mention sethood, to define Ω, and, since all of its elements are sets, it would then be a set by axiom 5, which is absurd. Thus, there must be an ordinal which has non-set elements, which implies in particular that $\Omega + 1$ must exist.

Finsler asserts that the theory of circle-free sets is adequate for applications (e.g., on pp. 49, 150). This is fortunate; for this means that Finsler's work in applied areas can be understood as work in a correct theory. One can then leave aside his claims that the universal set, the set of all singletons of ordinals, or a largest ordinal must exist; these assertions are based on an unsound intuition, whereas his proof of the axiom of infinity and the existence of the continuum, for example, can be regarded as being based on an intuition as sound as that on which the usual set theory is based, given what we now know about the set theory of Ackermann.

It is worth describing the motivation for axiom 5 of Ackermann's system as reported by Azriel Levy: "Let us consider the sets to be the "real" objects of set theory. Not all the sets are given at once when one starts to handle set theory– the sets are to be thought of as obtained in some constructive process. Thus at no moment during this process can one consider the predicate [of sethood] as a "well-defined"predicate, since the process of constructing the sets still goes on and it is not yet determined whether a given class X will eventually be constructed as a set or not. As a consequence, a condition... can be regarded as "well-defined" only if it avoids using the predicate [of sethood]. Also, parameters are allowed in such a definition only to the extent that they stand for "well-defined" objects, i.e. sets." Finsler would not agree with the idea that the sets are "constructed" in quite the sense that Ackermann thinks they are, but there seems to be some relation between Ackermann's idea that the sethood predicate is not "well-defined" and Finsler's assertion that it is "circular". It is interesting to see how quite different intuitions on the surface can lead to an axiomatization of basically the same form.

6 Conclusion

In conclusion, I find Finsler's papers extremely interesting. Finsler's intuition is clearly profound, though the notion behind Axiom III does not work out. A presentation of Finsler's papers is a contribution at least to historical scholarship about set theory. The editorial apparatus with which the papers are encumbered has serious deficiencies. The papers could have used the services of editors who understood the relevant mathematics better themselves. This is painfully evident in the treatment of Finsler's axioms, which cannot be defended as the editors have attempted here, and it is unfortunate that the editors were not able to bring out the actual nature of Finsler's derivation of a "formally undecidable proposition".

Aczel, Peter, *Non-well-founded sets*, CSLI, Stanford, 1988.

Booth, David and Ziegler, Renatus, *Finsler Set Theory: Platonism and Circularity*, Birkäuser-Verlag, Basel, 1996.

Holmes, M. Randall, "Review of "Finsler Set Theory: Platonism and Circularity", David Booth and Renatus Ziegler, eds.", unpublished, available at `http://math.idbsu.edu/faculty/holmes.html`

Levy, Azriel, "The role of classes in set theory", in Müller, Gert, ed., *Sets and Classes*, North Holland, Amsterdam, 1976. See pp. 207-212 on Ackermann's theory.