

# Relation algebra and finite axiomatization of $NF(U)$

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That stratified comprehension can be finitely axiomatized is an old result, due to Theodore Hailperin. His construction is analogous to the finite axiomatization of predicate class theory. His axiomatization is notably awful.

Other finite axiomatizations have been presented since.

The one described here was used in the first chapters of my book *Elementary Set Theory with a Universal Set*. It is adapted from the axiomatization of relation algebra combined with the interpretation of first order logic in relation algebra with ordered pair relations, due to Tarski and Givant and previously discussed in this seminar.

The primitive notions of the theory we describe are membership, equality, the projection relations  $\pi_1$  and  $\pi_2$  of the ordered pair [taken as a primitive notion] and the empty set  $\emptyset$ .

The first axiom, which plays no essential role in our development, is weak extensionality. Strong extensionality could be used if desired, and one would obtain NF instead of NFU.

The weak extensionality axiom asserts that nonempty sets are determined exactly by their extensions.

$$(\forall A B z : z \in A \rightarrow (A = B \leftrightarrow (\forall x : x \in A \leftrightarrow x \in B)))$$

We now begin the axiomatization of stratified comprehension. The intention is that the axioms we present seem simply to be natural hypotheses about what sets exist.

**Axiom of the Empty Set:** We provide a primitive constant  $\emptyset$  with axiom  $(\forall x : \neg x \in \emptyset)$ .

**Definition:** We say that  $x$  is a set iff

$$(\exists y : y \in x) \vee x = \emptyset.$$

We can then define  $\{x \mid \phi\}$  as the unique set  $A$  such that  $(\forall x \mid x \in A \leftrightarrow \phi)$ , for any formula  $\phi$ . This is an inessential but very useful strengthening of the original conceptual kit of NFU.

**Axiom of Singletons:** For every  $x$ ,

$$\{x\} = \{y \mid y = x\}$$

exists. We also use the notation  $\iota(x)$  for  $\{x\}$ , which has the advantage that  $\iota^n(x)$  can be used to represent repeated application.

To an audience of philosophers, I should emphasize that no existence predicate is mentioned here. “ $\{x \mid \phi\}$  exists” is a convenient way to say

$$(\exists A : (\forall x : x \in A \leftrightarrow \phi))$$

( $A$  not free in  $\phi$ ).

It is worth noticing that the axiom of singletons and the axiom of the empty set are together enough to show that there are infinitely many objects in any model of NFU [the fact which Quine persisted in saying is evidence for the axiom of infinity in NF, which it does not, as Infinity is not provable in NFU].

**Axiom of Complements:** For each set  $A$ ,  $A^c = \{x \mid x \notin A\}$  exists.

**Axiom of Binary Union:** For any sets  $A, B$ ,  $A \cup B = \{x \mid x \in A \vee x \in B\}$  exists.

**Definition:**  $V$  is defined as  $\emptyset^c = \{x \mid x = x\}$ .

**Definition:**  $A \cap B$  is defined as  $(A^c \cup B^c)^c$  and is equal to  $\{x \mid x \in A \wedge x \in B\}$ .  $A - B$  is defined as  $A \cap B^c = \{x \in A \wedge \neg x \in B\}$ .

**Definition:**  $\{x_1, x_2\}$  is defined as  $\{x_1\} \cup \{x_2\}$ ,  $\{x_1, x_2, \dots, x_n\}$  is defined as  $\{x_1\} \cup \{x_2, \dots, x_n\}$ .

This gives us the Boolean algebra of all concrete finite sets (of whatever objects there are) and their complements.

We did mention relations.

**Axiom of Ordered Pairs:** For each  $x, y$  there is a unique  $z$  such that  $z\pi_1x$  and  $z\pi_2y$ , which we call  $(x, y)$ . For all  $x, y, z, w$ ,  $(x, y) = (z, w)$  implies  $x = z$  and  $y = w$ . Objects  $(x, y)$  are called ordered pairs.

**Definition:** A relation is a set of ordered pairs. We write  $\{(x, y) \mid \phi\}$  for

$$\{z \mid (\exists xy : z = (x, y) \wedge \phi)\},$$

where  $z$  is a fresh variable.

We write  $xRy$  for  $(x, y) \in R$ , when  $R$  is a relation.

We comment that in our book we introduced ordered pairs as a primitive in the context of NFU [the intention being that the pair be type level, which cannot be achieved by definition in NFU].

**Axiom of Cartesian Products:** For any sets  $A, B$ ,  $A \times B = \{(a, b) : a \in A \wedge b \in B\}$  exists. We define  $V^2$  as  $V \times V$  and  $V^{n+1}$  as  $V \times V^n$ .

**Definition:** When  $R$  is a relation, we write  $R^c$  for  $(V \times V) - R$ . The overloading should be harmless as long as we are clear about the intentions with which letters are used.

**Axiom of Converses:** For any relation  $R$ ,  $R^{-1} = \{(y, x) \mid (x, y) \in R\}$  exists.

**Axiom of Relative Products:** For any relations  $R, S$ ,  $R|S = \{(x, y) \mid (\exists z : x R z \wedge z S y)\}$  exists.

**Axiom of the Diagonal:**  $\{(x, x) \mid x = x\}$  exists.

At this point we have defined the primitives of relation algebra. To support the strategy of Tarski and Givant, we need the projection relations. To support set definitions involving membership, we need the relation of inclusion.

**Axiom of Projections:**  $[\pi_1] = \{(x, y) \mid x\pi_1y\}$  exists.  $[\pi_2] = \{(x, y) \mid x\pi_2y\}$  exists. I have always said that Ordered Pairs implies Infinity, but in this axiomatization it is actually this axiom that entails Infinity.

**Axiom of Inclusion:**

$[\subseteq] = \{(x, y) : (\forall z : z \in x \rightarrow z \in y)\}$   
exists.

It should be noted, though we will not appeal to this, that we now have everything needed for the interpretation of first order logic in relation algebra due to Tarski and Givant.

We need three additional axioms, one to translate from the domain of relations to the domain of sets, and one to establish parallelism between the domain of general objects and the domain of singleton sets. The last is the familiar axiom of set union.

**Axiom of Domains:** For any relation  $R$ ,  $\text{dom}(R) = \{x \mid (\exists y : x R y)\}$  exists.

**Axiom of Singleton Images:** For any relation  $R$ , the relation  $R^\iota = \{(\{x\}, \{y\}) : x R y\}$  exists. Note that  $\text{dom}((A \times V)^\iota)$  implements  $\iota^{\iota}A = \{\{x\} \mid x \in A\}$ .

**Axiom of Set Union:** For any set  $A$ ,  $\bigcup A = \{x \mid (\exists a \in A : x \in a \wedge a \in A)\}$  exists. If set union is omitted, we get predicative NFU.

This completes our list of axioms. We introduced these as the axioms of the theory in our book for rhetorical purposes. These are all quite natural hypotheses about the existence of sets and relations, and not only believable but motivated by the need to reason about relations. We do not give stratified comprehension (a quite esoteric scheme) as the motivation of these axioms. But we proceed to prove that the scheme of stratified comprehension follows from the axioms, and so is supported by any intuition which finds the axioms reasonable

In what follows, we will show that the scheme of stratified comprehension follows from our axioms. In the process, we will in effect establish that first order logic can be interpreted via relation algebra with projection relations, though we are not exactly giving a proof of the results of Tarski and Givant, because we allow ourselves to freely use the interplay between sets and relations in our axiomatics.

We say that a sentence  $P(x_1, \dots, x_n)$  with the  $x_i$ 's all its free variables is interpreted by a set  $A$  just in case  $A = \{(x_1, \dots, x_n) : P(x_1, \dots, x_n)\}$ . We stipulate that we define  $(x_1, x_2, \dots, x_n)$  as  $(x_1, (x_2 \dots, x_n))$ . We could conveniently define  $(x_1)$  as  $x_1$  and  $()$  as a convenient fixed object such as the empty set.

Some formulas (for example  $x_1 \notin x_1$ ) are well known not to be interpreted by any set.

If  $P(x_1, \dots, x_n)$  and  $Q(x_1, \dots, x_n)$  have the same free variables (which we leave out of the notation for the rest of this slide) then  $\neg P$  is interpreted by  $\{(x_1, \dots, x_n) \mid P\}^c$  and  $P \vee Q$  is interpreted by  $\{(x_1, \dots, x_n) \mid P\} \cup \{(x_1, \dots, x_n) \mid Q\}$ . This generalizes to all operations of propositional logic.

If  $P(x_1, \dots, x_n)$  is interpreted, then  $(\exists x_1 : P)$  is interpreted by  $\text{dom}(\{(x_1, \dots, x_n) \mid P\}^{-1})$ , which has free variables  $(x_2, \dots, x_n)$ . The universal quantifier is of course definable in terms of the existential quantifier and negation.

For any relation  $R$ , we define  $R^0$  as  $[=]$  and  $R^{n+1}$  as  $R^n|R$ .

We define  $\Pi_n$  as  $[\pi_2]^{n-1}|[\pi_1]$ .  $(x_1, \dots, x_i, \dots, x_n)$  has the relation  $\Pi_i$  to each  $x_i$  for  $i < n$  and the relation  $\Pi_n^* = [\pi_2]^{n-1}$  to  $x_n$ . We define  $\Pi_n^m$  ( $m \geq n$ ) as  $\Pi_n$  if  $m > n$  and  $\Pi_n^*$  if  $m = n$ .

We can then define the relation holding between a tuple  $(x_1, \dots, x_n)$  and a tuple  $(y_1, \dots, y_m)$  where  $x_i = y_{f(i)}$  for each  $i$  as the intersection of all relations  $\Pi_i^n | (\Pi_{f(i)}^m)^{-1}$ , for  $1 \leq i \leq n$ , starring the  $\Pi$  operators where appropriate.

We can represent the image  $R^{\ast}A$  of a set  $A$  under a relation  $R$  as  $\text{dom}((R \cap (A \times V))^{-1})$ .

We can then represent  $R(y_{f(1)}, \dots, y_{f(n)})$  as a predicate of  $(y_1, \dots, y_m)$ , as

$$\left[ \bigcap_{1 \leq i \leq n} (\prod_i^n | (\prod_{f(i)}^m)^{-1}) \right] "R \cap V^m).$$

Note that this readily allows us to represent quantification over variables other than  $x_1$ , by reordering first and then undoing the reordering if desired).

In this way we can make every constituent formula of a complex formula be represented have the same argument list (it is also convenient to arrange for bound variables to be identified only with variables bound by the same binder). We can use the same construction to remove variables which in fact have no effect.

This has the effect that representable formulas are closed under all constructions of first order logic.

We note that the membership relation  $[\in]$  is not representable: if it were, we could construct  $\text{dom}([\in] \cap [=])^c$ , and enough said.

We transform any stratifiable formula  $\phi$  with type assignment  $\sigma$  into a representable formula, illustrating the use of the final elements of our toolkit.

Recall that we introduced the notation  $\iota(x)$  for  $\{x\}$  because we can conveniently write  $\iota^n(x)$  for iteration of this operation  $n$  times.

Where  $N$  is the maximum element of the range of  $\sigma$ , replace each variable  $x$  in the formula  $\phi$  with  $\iota^{N-\sigma(x)}(x)$ .

To preserve the meaning of the formula, we need to replace the relations in atomic formulas.

Any atomic formula  $x \in y$  becomes

$$\iota^{N-\sigma(x)}(x) \in \iota^{N-\sigma(y)}(y).$$

Note that this is  $\iota^{N-\sigma(x)}(x) \text{ ??? } \iota^{N-(\sigma(x)+1)}(y)$  and can be helpfully written  $\iota^M(x) \text{ ??? } \iota^{M-1}(y)$  (where  $M = N - \sigma(x)$ ) Now if  $M = 1$  (the smallest possible value) we want  $\{x\} \text{ ??? } y$  to be equivalent to  $x \in y$ . The relation to use is the inclusion relation with domain restricted to the collection of singletons. The collection of singletons is  $\iota \text{“} V = \text{dom}(V \times V) \text{”}$ , so the desired relation of “atomic inclusion” is  $E = (\iota \text{“} V \times V \text{”}) \cap [\subseteq]$ . Then ??? in the general formula  $\iota^M(x) \text{ ??? } \iota^{M-1}(y)$  is  $E^{\iota^{M-1}}$  (the notation indicating iteration of the singleton image operation on relations  $M - 1$  times). Other atomic relations are simply replaced by suitable iterated singleton images: the other relations are easier because they are representable and will be flanked by variables assigned the same type, so the same number of iterated applications of the singleton operation.

This immediately shows that stratified comprehension holds. A stratified formula  $\phi(x)$  converts to a representable formula  $\phi(\iota^M(x))$ . From its representation, it is straightforward to construct  $\{\iota^M(x) \mid \phi\}$  and then application of Set Union  $M$  times gives  $\{x \mid \phi\}$ .

Someone might think of parameters in  $\phi$  here. We note that  $x = a$  for any free variable  $a$  is represented by  $\{(x, a) \mid x = x\}$ , which exists by our axioms, being  $V \times \{a\}$ .

We do admit that we have given no more than a hint of the Tarski and Givant result, because we actually make free use of the interplay between sets and relations afforded by the domain operation. But I suspect it is a quite strong hint as to how their result is proved.

We think that we have entirely dispelled a common criticism of  $NF(U)$ . Stratified comprehension is not simply a syntactical trick. Here we see that it is a consequence of a finite list of principles for constructing sets which are quite credible, though not the same as those in the Zermelo toolkit we are used to.

A technical footnote: we axiomatized NFU with a type level ordered pair, which is an inessential strengthening of  $\text{NFU} + \text{Infinity}$ . NFU without Infinity can be axiomatized along these lines. Let  $(x, y)$  represent the usual Kuratowski pair  $\{\{x\}, \{x, y\}\}$ . The projections of the Kuratowski pair are provably not representable relations. So restate the axiom of projections to provide  $[\pi_i]$  as the set

$$\{(x, y) \mid y = \iota^2(\pi_i(x))\}$$

for  $i = 1, 2$ . Everything to do with projections and pairing is actually handled in a stratified formula  $y$  by atomic formulas  $x \pi_i y$  which can conveniently be read  $x[\pi_i]\iota^2(y)$  and then transformed into forms involving iterated singleton image operations as discussed above. But we do not give full details of this adaptation, as it also involves redefining general  $n$ -tuples in a really horrible way to get all their components to be of the same relative type. NFU with the type level ordered pair is a better foundational theory, anyway.

If we adopt strong extensionality we have an axiomatization of NF. NF proves that there is a type level ordered pair: I think it is most efficient to use the primitive one and identify it with the Quine pair after the fact. The reason for this is that the Quine pair definition is quite complicated, and we do not want to try to smuggle it in as a convincing basic axiomatic construction.