Forcing in NFU and NF

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March 28, 2019

1 Introduction to *NFU* and *NF*

This section provides a brief introduction to the theories *NFU* and *NF*. *NF* was proposed by W. V. O. Quine in [10]. *NFU* was proposed as a variant of *NF* by R. B. Jensen in [9].

These theories are based on the simple theory of types of Russell, as simplified by Ramsey. This is a sorted first-order theory with sorts indexed by the natural numbers. The intended semantics is that type 0 should be a type of individuals (about whose structure we make no assumptions), type 1 a type of sets of individuals, type 2 a type of sets of sets of individuals, and in general type $n+1$ will consist of sets of type n individuals. The primitive predicates of the theory of types are equality and membership. Atomic sentences are well-formed when and only when they are typed thus: $x^n = y^n$; $x^n \in y^{n+1}$.

The axioms of the theory of types are extensionality (each type $n+1$ object is determined exactly by its elements) and comprehension (for any formula ϕ, $\{x^n \mid \phi\}^{n+1}$ exists). Axioms of infinity and choice may be added.

Russell and others (including Quine, of course) observed that the theory of types $0, 1, 2 \ldots$ looks exactly like the theory of types $1, 2, 3 \ldots$. Quine was mad enough to propose (contrary to sensible intuition) that the reason that the theories look the same is that the types are in fact all one and the same type. The resulting theory is called "New Foundations" (*NF*) and is a one-sorted first order theory with the following axioms:

Extensionality: Objects with the same elements are the same, i.e., $A = B \equiv (\forall x.x \in A \equiv x \in B)$.

Stratified Comprehension: For any formula ϕ such that an assignment of types to its variables can be made which makes sense in the theory of types (such formulas are said to be "stratified") $\{x \mid \phi\}$ exists, i.e., $(\exists A.(\forall x.x \in A \equiv \phi))$ (where A is not free in ϕ).

It is traditional to rephrase the definition of stratification in such a way that no reference to the theory of types is required. Such a definition follows below. It is also possible to replace the axiom scheme of stratified comprehension with a finite set of instances of the scheme, which also has the effect of eliminating the need to refer to types (the original reference for this is [4]).

Definition: Let ϕ be a formula in the language of *NF*. A function σ from the set of variables (considered as syntactical items) to the natural numbers (or, equivalently to the integers) is said to be a *stratification* of ϕ if for each atomic subformula $\ulcorner \alpha = \beta \urcorner$ of ϕ we have $\sigma(\alpha) = \sigma(\beta)$ and for each atomic subformula $\ulcorner \alpha \in \beta \urcorner$ of ϕ we have $\sigma(\alpha) + 1 = \sigma(\beta)$. (Thanks to the referee for pointing out that Quine's quasi-quotes ([12]) make it easier to express this!)

Definition: Let ϕ be a formula in the language of *NF*. ϕ will be said to be *stratified* iff there is a stratification of ϕ. An effective algorithm for determining whether a formula is stratified is easily developed.

Stratified Comprehension (restated): For any stratified formula ϕ, $(\exists A.(\forall x.x \in A \equiv \phi))$ (where A is not free in ϕ).

It is worth giving an indication of how stratification is handled in a richer language (we really do not conduct the business of set theory strictly in the first-order language with equality and membership!) Term constructions can be handled formally by adding the definite description operator $(\iota x.\phi)$, denoting the unique x such that ϕ if there is such an object and we care not what otherwise. The definition of stratification would be widened so that a stratification function would assign natural number (or integer) values to every term, not just to variables, with the additional constraint that the value assigned to $(\iota x.\phi)$ would be the same as the value assigned to x. It is straightforward to verify that the stratification one gets for a formula involving $(\iota x.\phi)$ will be the same as the stratification of the formula obtained by eliminating occurrences of the term $(\iota x.\phi)$ in the usual way. It is also worth observing that it is not necessary to require that variables free in an instance of comprehension be typed, and of course many superficial problems with stratification can be eliminated by renaming bound variables.

NF is not known to be consistent. It is known to disprove the axiom of choice (Specker showed this in 1953 ([14])), and thus proves the "axiom" of infinity as a theorem (if the universe were finite, it could be well-ordered).

In his 1969 paper [9], R. B. Jensen proposed weakening the axiom of extensionality to allow atoms or *urelements*. The resulting theory is called *NFU*. *NFU* does not prove Infinity or disprove Choice: Jensen showed that *NFU* + Infinity + Choice is consistent (it has the same strength as the theory of types with infinity; *NF* is not known to be any stronger than this).

NFU can formalized by modifying extensionality to take the form

Weak Extensionality: $z \in A \rightarrow (A = B \equiv (\forall x.x \in A \equiv x \in B))$

which asserts that objects with elements are equal iff they have the same elements. Another common way to formalize *NFU* is to adjoin a sethood predicate, stipulating that anything with an element is a set, that sets with the same elements are equal, and that the objects provided by the axiom scheme of stratified comprehension are sets. It is equivalent to provide a constant $\emptyset$ representing the empty set (as opposed to the other objects with no elements): the sethood predicate can be defined thus: $\texttt{set}(x) \equiv_{def} (\exists y.y \in x) \vee x = \emptyset$.

It is important to notice that the comprehension scheme of *NFU* is the same as that of *NF*.

There are other fragments of *NF* known to be consistent, but they are not relevant here.

2 Mathematics in *NF(U)*

NF is noted (notorious?) for allowing large sets. *NFU* provides the same large sets, because it has the same comprehension scheme. For example, the universe $V = \{x \mid x = x\}$ is provided by an instance of stratified comprehension. But notice that the Russell class $R = \{x \mid x \notin x\}$ is not provided by stratified comprehension, because there is no consistent way to type x in the formula $x \notin x$. This is not because $x \in x$ is ill-formed, as it would be in the theory of types: $V \in V$, for example, is both well-formed and true (the universe is an element of the universe).

2.1 Definitions of familiar concepts in *NF(U)*; Cantor's theorem

It is important to note that we use a type-level ordered pair ((x, y) has the same type as x or y for stratification purposes: strictly speaking, we introduce the relations π_1 and π_2 which a pair has to its projections, with a suitable axiom ensuring that they really are projection relations and with the same stratification requirements as equality, and then the term (x, y) defined as $(\iota z.z\pi_1 x \wedge z\pi_2 y)$ can be seen to have the same value under any stratification as x or y). The usual definition ((x, y) defined as $\{\{x\}, \{x, y\}\}$) gives a pair two types higher than its projections. An important advantage of using a type-level ordered pair is that a function is one type higher than its arguments and values, rather than three types higher. A type-level ordered pair is definable in *NF* (see [11]) and can be proved to exist in *NFU* + Infinity + Choice.

Cardinal numbers are defined as equivalence classes of sets under the usual relationship of equinumerousness. So, for example, 1 is the set of all one-element sets, 3 is the set of all 3 element sets, and $\aleph_0$ is the set of all countably infinite sets. These very large collections are proper classes in the usual set theory, but they are provided by instances of stratified comprehension without difficulty. Ordinal numbers, similarly, are defined as equivalence classes of well-orderings under similarity.

Cantor's theorem, which in *ZFC* asserts that $|A| < |\mathcal{P}(A)|$, takes a different form in *NF*. Note that in type theory $|A| < |\mathcal{P}(A)|$ is not even a well-formed assertion ($|\mathcal{P}(A)|$ is one type higher than $|A|$). The theorem one can prove in type theory is $|\mathcal{P}_1(A)| < |\mathcal{P}(A)|$, where $\mathcal{P}_1(A)$ is the set of one-element subsets of A. This is also the theorem one can prove in *NF(U)*. Cantor's paradox arises from applying Cantor's theorem to the cardinality of the universal set.

In *NF(U)*, the result we obtain is $|\mathcal{P}_1(V)| < |\mathcal{P}(V)|$: the cardinality of the set of all singletons is less than the cardinality of the universe. While this is not as bad as the paradoxical $|V| < |\mathcal{P}(V)|$, it is still counterintuitive (after all, we can "see" the bijection $(x \mapsto \{x\})$); but it is also obvious that the definition of $(x \mapsto \{x\}) = \{(x, y) \mid y = \{x\}\}$ is unstratified.

2.2 Strongly cantorian sets and subversion of stratification

The form of Cantor's theorem motivates some definitions important in our technical development.

Definition: A set which satisfies the condition $|A| = |\mathcal{P}_1(A)|$ is said to be *cantorian*.

A set is cantorian just in case there is a bijection between the set and the set of its one-element subsets. A cantorian set will satisfy the "naive" form of Cantor's theorem. The condition we are interested in here is the even stronger condition that the restriction of the bijection $(x \mapsto \{x\})$ to A is a set: not only are A and $\mathcal{P}_1(A)$ the same size, but this fact is witnessed by the obvious map.

Definition: A set A is said to be *strongly cantorian*, which we abbreviate *s.c.*, iff the restriction of the bijection $(x \mapsto \{x\})$ to A is a set.

Suppose that A is a strongly cantorian set, ϕ is a formula, and a is a variable restricted to A in the formula ϕ. Let k be the restriction of the singleton map to A. Let ψ be a stratified subformula of ϕ. We can raise the type of a in ψ by one by replacing ψ with the equivalent formula $(\exists bc.\psi[b/a] \wedge b \in c \wedge (a, c) \in k)$, in which b can be assigned the original type of a and the only occurrence of a is assigned a type one higher. We can lower the type of a in ψ by replacing ψ with the equivalent formula $(\exists bc.\psi[b/a] \wedge (b, c) \in k \wedge a \in c)$. (In both of these contexts, $\psi[b/a]$ is the result of substituting b for a in ψ). A less formal way of putting this is that any reference to a in the formula ψ can be replaced with a reference to the element of $k(a)$, and such a replacement raises the type of the occurrence of a by one. Similarly, a reference to a can be replaced by a reference to $k^{-1}(\{a\})$, in which the type of a is lowered by one. In this way, the types of all occurrences of a in the original formula ϕ can be adjusted to achieve stratification. In other words, stratification restrictions can be subverted for variables restricted to strongly cantorian sets. This will be vital in our development of forcing in *NFU*.

2.3 The axiom of counting

Rosser proposed the following axiom (for *NF*) which Orey showed essentially strengthens *NF* if *NF* is consistent, and which Jensen showed to be consistent with *NFU*.

Rosser's Axiom of Counting: $\{1, \ldots, n\}$ has n elements.

This axiom is equivalent to any of the following statements:

1. All finite sets are s.c.
2. The set of natural numbers is s.c.
3. There is an infinite s.c. set.

For proofs see [3], p. 30-31.

The (proper) class of s.c. sets is closed under power set, union, cartesian product, subsets, etc., so many familiar mathematical structures become s.c. once the set of natural numbers is taken to be s.c.

3 Forcing in *NFU*

3.1 Prehistory of forcing in *NFU*

The set of isomorphism types of well-founded extensional relations with "top" elements can be interpreted as the set of "pictures" of sets of a Zermelo-style set theory, and the membership relation suggested by this interpretation is a set relation. This was fully developed by Hinnion in [5]; it is also described in [7], pp. 165-177.

One can do forcing in *NFU* by exploiting this fact: build isomorphism types of "Boolean-valued" well-founded extensional relations and emulate the usual development of Boolean-valued models. There is a technical trick which allows one to recover a model of *NFU* based on the Boolean-valued model of an initial segment of the cumulative hierarchy that one obtains, as long as the Boolean algebra used is s.c. (this is basically the same as the method of interpreting *NFU* in the set of isomorphism types of well-founded extensional relations with "top" element described in [7], p 176-177; the Boolean algebra needs to be s.c. in order for the definitions of equality and membership in the forcing model to be stratified, much as is the case in the construction given below).

This approach creates lots of new urelements (this is evident just from consideration of the "two-valued" version of the construction), so it cannot be used to prove independence results from *NF*: if one starts with a model of *NF* and a Boolean algebra, one will obtain a Boolean-valued model of *NFU* with many urelements (even if the Boolean algebra is the trivial one with two elements!)

We give a brief description of what happens in the two-valued construction. One obtains "pictures" of ranks $V_{\alpha_1} \subset V_{\alpha_2}$ of the cumulative hierarchy which are externally isomorphic (there is an isomorphism between them, but it is a proper class in *NFU*). V_{α_1} and V_{α_2} are very "big" ranks, and the isomorphism between them is induced by taking images of elements of V_{α_2} under the singleton map in a suitable sense to get elements of the "smaller" rank V_{α_1}. One can then exploit the fact that every set in V_{α_1+1} is an element of V_{α_2} and thus can be coded into V_{α_1} using the external isomorphism. But α_2 is necessarily much larger than α_1+1 (from the internal standpoint): the elements of V_{α_1} that correspond to sets in V_{α_1+1} in this way are a small initial segment of V_{α_1}, because V_{α_1+1} is a small initial segment of V_{α_2}, and the rest of the elements of V_{α_1} must be interpreted as urelements. If one has a more complex Boolean algebra, there are additional considerations which generate even more urelements.

There is also a philosophical objection: this approach allows one to do independence proofs in *NFU* only by constructing an interpretation of some fragment of *ZFC* in *NFU* and emulating the familiar development of independence proofs in that context. It is certainly not an advertisement for *NFU* as an independent approach to set theory!

3.2 A natural forcing construction in *NFU*

We now describe a technique of forcing natural for *NFU*. It will really be a development of Boolean-valued models, as the discussion above might already have suggested.

Let $\leq$ be a strongly cantorian partial order with domain P (P will also be s.c.) We will indulge ourselves by stipulating that when $p \leq q$, the condition q represents a state with *more* information (this is the reverse of the usual convention).

We are interested in a certain collection $\texttt{RO}(P)$ of subsets of P, which can be thought of as representing "truth values". The elements of $\texttt{RO}(P)$ are those subsets A of P which satisfy:

1. if $p \in A$ and $p \leq q$ then $q \in A$
2. if p satisfies the condition $(\forall q \geq p.(\exists r \geq q.r \in A))$ (the set of r stronger than p which are in A is dense) then $p \in A$.

These sets are the "regular open" sets of a certain topology. Because *NFU* does not satisfy the axiom scheme of separation found in familiar set theories, it is possible that P may have subclasses which are not sets. Subclasses of P which are not sets but which do otherwise satisfy the conditions for membership in $\texttt{RO}(P)$ will be referred to as "regular open" classes.

We insert a definition and an observation which will be useful below.

Definition: If $A \in \texttt{RO}(P)$, we define $A^{\perp}$ as $\{p \in P \mid (\forall q \geq p.q \notin A)\}$.

Observation: For any element $A \in \texttt{RO}(P)$, it is the case that $A^{\perp} \in \texttt{RO}(P)$. For any $\mathcal{A} \subseteq \texttt{RO}(P)$, it is the case that $\bigcap \mathcal{A} \in \texttt{RO}(P)$. This is even true (in a sense) if $\mathcal{A}$ is a proper class: in this case $\bigcap \mathcal{A}$ may not be eligible for membership in $\texttt{RO}(P)$, because it may not be a set, but it will still be a "regular open" class. Note that if $\mathcal{A}$ is a subclass of P, the subclass $\mathcal{A}^{\perp} = \{p \in P \mid (\forall q \geq p.q \notin \mathcal{A})\}$ will also be "regular open", though it may fail to be a set.

We now set out to interpret all sets in our model of *NFU* as "names". First, we need a way to interpret any set as a function $V \to \texttt{RO}(P)$: for any sets A

and x, define $A[x]$ as the unique element v of $\mathtt{RO}(P)$ such that $(x, v) \in A$, or as $\emptyset$ if there is no such uniquely determined v. Observe that $\emptyset \in \mathtt{RO}(P)$. Note that the relative type of x in $A[x]$ is one lower than the type of A (this depends on the use of a type-level ordered pair).

Intuitively, we would like to say, for any condition p, that p forces $x \in A$ just in case $p \in A[x]$. This will not work – it will need to be adjusted a bit, but it is the basis of our development.

3.3 Defining equality

We use the intuitive motivation just given to formulate our definition of equality in the forcing interpretation:

We say that $p \vdash A = B$ just in case $(\forall q \geq p.(\forall x.q \in A[x] \equiv q \in B[x]))$.

A little reflection will reveal the problem we now face. The criterion of identity on "elements" x in this definition of equality is the identity criterion of the underlying model of NFU, not the new identity criterion we are trying to define.

The standard approach would be to define equality by induction on membership, but definitions by induction on membership do not work in *NF(U)*: they are horribly unstratified (and the membership relation is not well-founded, anyway).

Instead, we use an idea introduced by Marcel Crabbé in his proof (in [2]) that *NF* without any extensionality interprets *NFU* (weak extensional collapse): we will reinterpret all those names which do not respect the new identity criterion as urelements.

It is not hard to show that for any A and B, the class of all $p \in P$ such that $p \vdash A = B$ is a set and belongs to $\mathtt{RO}(P)$.

3.4 Defining sethood

We define conditions for a name to be the name of a *set* under a condition p:

$p \vdash \mathtt{set(A)}$ just in case $(\forall q \geq p.(\forall xy.((q \in A[x]) \wedge (q \vdash x = y)) \rightarrow q \in A[y]))$.

This captures precisely the idea that a name is a set if it respects the new identity criterion: it says that if x is (intuitively) an "element" of A and $x = y$, then y is an "element" of A, under any condition stronger than p.

This definition requires careful attention to stratification. In the definition of equality, A and B each have the same type (just as in $A = B$), x is one type lower, and p and q are two types lower. Here, we have q being one type lower than x (and two types lower than A) in $q \in A[x]$, but q is two types lower than x in $q \vdash x = y$. This is not a problem, but only because P is a strongly cantorian set: the types of variables p and q restricted to P can be adjusted as needed to restore stratification.

With this remark on stratification, it is not hard to show that the class of all $p \in P$ such that $p \vdash \mathtt{set}(A)$ is a set and belongs to $\mathtt{RO}(P)$.

3.5 Defining membership

Now that we have sethood, it is easy to define membership.

$p \vdash x \in A$ just in case $p \in A[x] \wedge p \vdash \mathtt{set}(A)$

Observe that the type of x is one lower than the type of A.

It is important to observe that the relation between the types of A and B in $p \vdash A = B$ is the same as in $A = B$, and the relation between the types of x and A in $p \vdash x \in A$ is the same as in $x \in A$, while the type of the condition can be ignored, because it is an element of the s.c. set P.

It is not hard to see that the class of all $p \in P$ such that $p \vdash x \in A$ is a set and a member of $\mathtt{RO}(P)$.

3.6 Logical considerations

We define forcing of complex sentences.

$p \vdash \phi \wedge \psi$ just in case $(p \vdash \phi) \wedge (p \vdash \psi)$

$p \vdash \neg\phi$ just in case $(\forall q \geq p.\neg(p \vdash \phi))$

$p \vdash (\forall x.\phi)$ just in case $(\forall a.(p \vdash \phi[a/x]))$

There is no problem with representing sentences with arbitrary sets replacing variables (as is required in the definition of forcing of universal sentences). It should be noted that this definition only succeeds for concrete sentences; we do not succeed in defining a set relation $\vdash$ between conditions and sentences.

It is the case, however, that $p \vdash \phi$ will be stratified if ϕ is stratified; this is true of atomic sentences and nothing in the definition of logical operators will interfere with it.

Further, the class of $p \in P$ such that $p \vdash \phi$ will be "regular open" for any ϕ (though it may fail to be a set if ϕ is unstratified): we have noted that this class is a set and an element of $\mathtt{RO}(P)$ for each atomic sentence ϕ, and the Observation following the definition of $\mathtt{RO}(P)$ allows us to prove by induction on the complexity of sentences that the class of p such that $p \vdash \phi$ will be "regular open" (and so be an element of $\mathtt{RO}(P)$ if it is a set). If ϕ is stratified, the class of $p \in P$ such that $p \vdash \phi$ will be a set (and so a member of $\mathtt{RO}(P)$), since $p \vdash \phi$ will also be stratified.

It is worth noting that in some extensions of *NFU* (notably the system of [7]), it is possible to arrange for all subclasses of strongly cantorian sets to be sets, which would eliminate the need to worry about proper subclasses of P. Solovay claims (personal communication) that *NFU* + Counting + "all subclasses of s.c. sets are sets" is strictly weaker than *ZFC* (and we believe it); the system of [7] is shown (in our paper [8]) to have the same strength as Morse-Kelley set theory with the proper class ordinal measurable (this is weaker than *ZFC* + a measurable, but stronger than the familiar large cardinal hypotheses short of a measurable).

The other connectives are defined using their definitions in classical logic. We now prove a basic theorem. The method of proof is essentially standard.

Forcing Theorem: If $p \vdash \phi$ and ψ is a classical logical consequence of ϕ, then $p \vdash \psi$.

Lemma: If it is not the case that $p \vdash \phi$, then there is $q \geq p$ such that $q \vdash \neg\phi$.

Proof of Lemma: This follows immediately from the fact that the class of $p \in P$ such that $p \vdash \phi$ is "regular open".

Proof of Forcing Theorem: We work with a set model M of *NFU* + Counting.

For the sake of a contradiction, in the model M fix an infinite s.c. partial order $\leq$ with domain P (the result is trivial if $\leq$ is finite), a condition $p \in P$ and sentences ϕ_0 and ψ_0 such that it is the case that $p \vdash \phi_0$, ψ_0 is a logical consequence of ϕ_0, but it is not the case that $p \vdash \psi_0$.

We extend the language of *NFU* with constants for the empty set (recall from above that this allows us to define the sethood predicate), the partial order $\leq$ (from which we can define its domain P), the specific condition $p \in P$, and arbitrarily chosen values from M for any parameters appearing in the sentences ϕ_0 and ψ_0.

We construct a countably infinite model M_0 which satisfies the same sentences of this language that the original model M satisfies.

We then define a two-valued logical structure for this language which satisfies ϕ_0 and does not satisfy ψ_0, which suffices for a contradiction. We use the standard construction of a generic ultrafilter in the partial order $\leq$.

We provide an enumeration $\{v_i\}_{i \in \mathcal{N}}$ of the definable "regular open" subclasses of P (not just the elements of $\mathtt{RO}(P)$!) in M_0 (this is of course external to the model M_0: M_0 believes that $\mathtt{RO}(P)$ is uncountable, and some terms of this sequence may be proper classes for M_0). We select a sequence of elements $\{p_i\}_{i \in \mathcal{N}}$ of P in M_0, in the following way. $p_0 = p$. Once p_i has been chosen, we choose $p_{i+1} \geq p_i$ so that $p_{i+1} \vdash \neg\psi_0$ (this obligation is fully discharged by the choice of p_1, using the Lemma) and either $p_{i+1} \in v_i$ or p_{i+1} is not dominated in the order $\leq$ by any element of v_i (this is possible because v_i is "regular open"). Choices of p_i at each stage involve no appeal to the Axiom of Choice, since M_0 is countable.

We then define $p_\infty \vdash \phi$, for each sentence ϕ of the extended language, as $(\exists n.p_n \vdash \phi)$. The following are easy to show:

1. For any sentence ϕ of the extended language (even with parameters representing arbitrarily chosen elements of M_0) either $p_\infty \vdash \phi$ or $p_\infty \vdash \neg\phi$: thus $p_\infty \vdash \neg\phi$ iff it is not the case that $p_\infty \vdash \phi$.
2. $p_\infty \vdash \phi \wedge \psi$ iff $p_\infty \vdash \phi$ and $p_\infty \vdash \psi$.
3. $p_\infty \vdash (\forall x.\phi)$ iff for all $a \in M_0$, $p_\infty \vdash \phi[a/x]$.

Further, we need to verify that our defined equality has the correct logical properties in this context. It is sufficient to establish that substitutions of equals for equals in atomic sentences behave correctly.

Suppose that $p_\infty \vdash A = B$. We then need to show that $p_\infty \vdash A = C$ is equivalent to $p_\infty \vdash B = C$ and that $p_\infty \vdash C = A$ is equivalent to $p_\infty \vdash C = B$. It is sufficient to observe that $p \vdash A = B$ iff $p \vdash B = A$ for all $p \in P$ and that $p \vdash A = B$ and $p \vdash B = C$ together imply $p \vdash A = C$ for any $p \in P$.

Further, we need to show that $p_\infty \vdash A \in C$ iff $p_\infty \vdash B \in C$. If $p_\infty \vdash A \in C$, then for some n, $p_n \vdash \texttt{set}(C)$ and $p_n \in C[A]$. Further, we may choose n large enough that $p_n \vdash A = B$ (because $p_\infty \vdash A = B$). From this we have $p_n \in C[A]$ iff $p_n \in C[B]$, whence we have $p_n \vdash \texttt{set}(C)$ and $p_n \in C[B]$, from which $p_n \vdash B \in C$, thus $p_\infty \vdash B \in C$ as desired. The other direction is symmetrical.

Finally, we need to show that $p_\infty \vdash C \in A$ iff $p_\infty \vdash C \in B$. It is first necessary to establish that $p_\infty \vdash \texttt{set}(A)$ iff $p_\infty \vdash \texttt{set}(B)$ Suppose $p_\infty \vdash \texttt{set}(A)$. Thus for some n $p_n \vdash \texttt{set(A)}$, which means that for all $q \geq p_n$, $q \vdash x = y$ and $q \in A[x]$ implies $q \in A[y]$. We can suppose also that $p_n \vdash A = B$, by choosing n large enough. If $q \geq p_n$ and $q \vdash x = y$ and $q \in B[x]$, then $q \in A[x]$ because $p_n \vdash A = B$, $q \in A[y]$ because $q \vdash x = y$, and $q \vdash B[y]$ because $p_n \vdash A = B$, which shows that $p_n \vdash \texttt{set}(B)$. Since the situation is symmetrical, we have $p_\infty \vdash \texttt{set}(A)$ iff $p_\infty \vdash \texttt{set}(B)$.

Now suppose that $p_\infty \vdash C \in A$, i.e., for some n, $p_n \vdash C \in A$. We may suppose further that $p_n \vdash A = B$, by choosing n large enough. It follows from this that $p_n \vdash \texttt{set}(A)$ and thus $p_n \vdash \texttt{set}(B)$. Further, we have $p_n \in A[C]$, which implies that we also have $p_n \in B[C]$, since $p_n \vdash A = B$. Since we have $p_n \in B[C]$ and $p_n \vdash \texttt{set}(B)$, we have $p_n \vdash C \in B$, and so $p_\infty \vdash C \in B$ as desired, and symmetry of the situation completes the argument.

It follows that we have constructed a two-valued logical structure for the (extended) language of *NFU* (upon identifying elements A and B of M_0 iff $p_\infty \vdash A = B$), in which ϕ_0 holds and ψ_0 does not. But we see now that if $p_\infty \vdash \phi_0$ and ψ_0 is a logical consequence of ϕ_0, we should have $p_\infty \vdash \psi_0$ as well, which is a contradiction.

This completes the proof of the Forcing Theorem.

In the absence of choice (as in *NF*) we may have $p \vdash (\exists x.\phi)$ without having any name a such that $p \vdash \phi[a/x]$. This appears to be harmless: there will be a dense set of conditions q stronger than p such that there is a name a (depending on q) such that $q \vdash \phi[a/x]$, and the Forcing Theorem still holds (notice that we did not assume that the model M satisfied Choice in the proof above). It is possible that the same thing might happen with witnesses to unstratified sentences $(\exists x.\phi)$ in the presence of choice.

3.7 The axioms of *NFU* hold

The definitions of sethood and membership were engineered to make weak extensionality hold; we will not present details.

We will exhibit a name for the set $\{x \mid \phi\}$ which will witness the truth of comprehension under any condition: the set of all pairs (a, A) where A is $\{p \in P \mid (p \vdash \phi[a/x])\}$ does the trick.

If ϕ is stratified, the set A will exist for each a and be "regular open". It is then easy to see that $\texttt{set}(\mathcal{A})$ holds (this follows from the fact shown in the proof of the theorem above that the defined equality has the correct logical properties, so that $p \vdash a = b$ and $p \vdash \phi[a/x]$ entail $p \vdash \phi[b/x]$) and that $p \vdash (\forall x.x \in \mathcal{A} \equiv \phi)$ for any $p \in P$, which is what is wanted.

The Axiom of Counting continues to hold for the standard reasons that forcing models have the same natural numbers as the models they are built from. Choice continues to hold (if it held in the original model) for the same reason that choice is preserved by the usual forcing constructions.

3.8 Familiar results in *NFU*

Familiar results such as the independence of the continuum hypothesis can be established in *NFU* using this forcing technology. The proofs go essentially the same way as in *ZFC*. The fact that *NFU* uses different definitions for ordinal and cardinal numbers makes for some superficial differences in the way a full development looks; we do not give details here.

We have used a "Boolean-valued" approach here (actually, we really think in terms of the "possible world semantics" for intuitionistic logic). It is possible to get 2-valued interpretations using ultrafilters if one has choice, by means strictly internal to one's model of *NFU* (not by appeal to external countability of a model as in the proof above). The technology of countable models and generic ultrafilters seems less natural in *NFU*, though it can be adapted to this context (and in fact, we use it in our proof of the Forcing Theorem). It is generally less natural to think of (set) models of *NFU* inside *NFU* than it is to think of set models of *ZFC* inside *ZFC*.

4 What about *NF*?

If we began with a model of *NF*, we would end up with a forcing interpretation of *NFU*: the construction creates urelements (names which do not respect the equality of the forcing interpretation).

But the picture is different in this treatment of forcing than in the "prehistoric" treatment. The surprise is that this construction does not create very many urelements. It turns out that one can define an injection from the universe of the forcing interpretation into the sets, if the original model satisfied strong extensionality, which means that V and $\mathcal{P}(V)$ are the same size (by Schröder-Bernstein). One can then apply a construction due to Maurice Boffa (in [1]): in any model of *NFU* with a bijection f from V onto $\mathcal{P}(V)$, defining a new membership relation $x \in_{\texttt{new}} y \equiv x \in f(y)$ gives an interpretation of *NF*.

This is a variation of the permutation method introduced by Scott ([13]), though f is not actually a permutation. The kinds of results for which consistency and independence results are proved by forcing are invariant under "permutation methods", even as generalized here.

For example, if one forces on the inclusion order on well-orderings of subsets of the continuum in a model of *NF* + AxCount + DC, it is straightforward to show that one obtains an interpretation of *NFU* + AxCount + DC + "the continuum is well-ordered", which can be changed by a Boffa-style "permutation" to a model of *NF* + AxCount + DC + "the continuum is well-ordered". This is a new kind of independence result in the *NF* context. DC (dependent choices) is needed to make the forcing argument work correctly; stronger versions of dependent choices would be needed for relative consistency proofs of the existence of well-orderings of larger sets.

4.1 The Injection from the Universe into the Sets in a Forcing Extension of *NF*

We describe the injection from the universe into sets found in any forcing extension of a model of *NF*. It should be clear that the construction which follows will not work in the presence of urelements!

Recall that the reason that a set considered as a "name" becomes an urelement (under a condition) in the interpretation is that it does not respect the identity criteria of the forcing extension (under that condition).

We define, for any set x, $\texttt{char}(x)$ as the function taking each element of x to P and each non-element of x to $\emptyset$. The nice property of $\texttt{char}(x)$ is that $(\forall p.(x = y \equiv (p \vdash \texttt{char}(x) = \texttt{char}(y))))$. It is important to note that the type of $\texttt{char}(x)$ is the same as the type of x.

For each set A we define A^* as the function taking each x to the set of all conditions p such that $(\forall q \geq p.(\exists r \geq q.(\exists y.((r \vdash x = \texttt{char}(y)) \land y \in A[r]))))$. The idea is that each "element" y of A under any condition is replaced by $\texttt{char}(y)$, which eliminates conflicts between the identity conditions of "elements" of A^* in the original model of *NF* and their identity conditions in the forcing extension. To restore sethood in the forcing interpretation, one then adds (under each condition) all objects x equal to an appropriate $\texttt{char(y)}$ under that condition.

It is straightforward to see that $p \vdash A = B$ iff $p \vdash A^* = B^*$ and that $p \vdash \texttt{set}(A^*)$ for any condition p and sets A, B. We can thus construct a map in the forcing extension implementing $(A \mapsto A^*)$, which will be bijective and send every object to a set. We have already seen that this is enough to allow an interpretation of *NF* in the forcing extension.

5 Conclusion

We believe that the most important result in this paper is that there is a "native" approach to forcing in *NFU* which does not rely on coding a version of Zermelo set theory into *NFU* (as did the "prehistoric" approach). This strengthens our thesis (developed in [6] and [7]) that *NFU* can be the basis of an independent approach to the foundations of mathematics.

The fact that it becomes possible to do forcing in *NF* is appealing, but turns out to be of limited usefulness because of the failure of choice in *NF*, though it is impossible to discount the value of the ability to get *any* new relative consistency results with *NF*.

The same techniques can be used to build nonclassical models of intuitionistic *NFU*, by using general upward closed subsets of the domain of an s.c. partial order as "truth values" instead of "regular open" upward closed subsets. We are investigating the possibility that a similar "trick" will allow the construction of nonclassical models of intuitionistic *NF* from these nonclassical models of intuitionistic *NFU*.

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