M. Randall Holmes

Paradoxes in double extension set theories

Abstract. Three systems of "double extension set theory" have been proposed by Andrzej Kisielewicz in two papers. In this paper, it is shown that the two stronger systems are inconsistent, and that the third, weakest system does not admit extensionality for general sets or the use of general sets as parameters in its comprehension scheme. The parameter-free version of the comprehension principle of double extension set theory is also shown to be inconsistent with extensionality. The definitions of the systems and a self-contained exposition of their properties is given, sufficient to develop the inconsistency proofs.

Keywords: double extension set theory, Quine ordered pair, universal set

1. Introduction

In [1], Andrzej Kisielewicz proposed a system of "double extension set theory", based on an interesting approach to weakening the unrestricted (and inconsistent) axiom scheme of comprehension of so-called "naive set theory", and gave an argument for the claim that this system implements all mathematical constructions of ZF. In [2], he presented two weaker versions of this system.

In this note, we will show that the two stronger systems of the three are inconsistent and the weakest system does not admit extensionality for general objects or the use of general objects as parameters in instances of comprehension. The mathematical development here will be self-contained; one needs to do a fair bit of work to derive the paradoxes.

Double extension set theory (in all three forms) is a first-order theory with three primitive predicates, equality and two membership relations $\in$ and ϵ.

The axiom of extensionality of double extension set theory, as stated in [1], stipulates that if two sets have the same extension under either membership relation, the two sets are equal:

$$((\forall x.x \in A \equiv x \in B) \vee (\forall x.x \,\epsilon\, A \equiv x \,\epsilon\, B)) \rightarrow A = B.$$

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The two systems of [2] assume a "mixed" extensionality axiom involving both membership relations:

$$(\forall x. x \in A \equiv x \,\epsilon\, B) \rightarrow A = B$$

and do not assume the extensionality axiom of the system of [1].

We call a formula *uniform* if it contains no instance of ϵ. For any formula ϕ, we define ϕ^* as the formula obtained by replacing any occurrence of $\in$ in ϕ with ϵ and any occurrence of ϵ with $\in$.

With the aid of these concepts, we can present a preliminary form of the axiom scheme of comprehension of double extension set theory:

Axiom 1.1. (Axiom Scheme of Comprehension (parameter-free)) For each uniform formula ϕ in which no variable other than x is free, the following is an axiom:

$$(\exists A.(x \,\epsilon\, A \equiv \phi) \wedge (x \in A \equiv \phi^*)).$$

We refer to this set A as $\{x \mid \phi\}$. Sets witnessing instances of comprehension are called "set abstracts". In English, any formula (without parameters) which does not mention ϵ determines an extension for $\in$ and any formula (without parameters) which does not mention $\in$ determines an extension for ϵ and (crucially) the set with ϵ extension determined by ϕ is the same object as the set with $\in$-extension determined by ϕ^*. (It should be noted that in the absence of additional axioms there may be other sets with the same extension under just one of the membership relations).

The Russell paradox is thwarted: let $R = \{x \mid x \notin x\}$. Comprehension gives us $x \,\epsilon\, R \equiv x \notin x$, so $R \,\epsilon\, R \equiv R \notin R$, and dually $R \in R \equiv R \not\epsilon R$. This is not a contradiction: in one sense R belongs to R and in the other it does not.

This is very appealing, as far as it goes.

2. Formal development of the systems

The absence of parameters is a serious weakness in the system. Kisielewicz gives two different approaches to remedying this, one of which subsumes the other (though this is not obvious without a little work).

Parameters cannot be added in an unrestricted way. This is easy to show. Consider the uniform formula $x \in a$. The set $\{x \mid x \in a\}$, if we were permitted to form it, would have as its ϵ-members exactly the $\in$-members of a, and extensionality would imply further that $a = \{x \mid x \in a\}$. This would be true for any a, so the two memberships would coincide for all sets and we would recover the Russell paradox immediately.

Definition 2.1. We say that a set A is *regular* iff $(\forall x.x \in A \equiv x \epsilon A)$: regular sets are those which have the same extension under both membership relations.

We introduce a weak comprehension scheme (there will be a "strong comprehension scheme" described below). Further, we assume the mixed case of extensionality, and do not assume the extensionality axiom of [1]. Note that the mixed case of extensionality does imply those cases of the extensionality axiom of [1] in which one or both of the sets compared is regular.

Axiom 2.2. (Weak Axiom Scheme of Comprehension) For each uniform formula ϕ in which each free variable other than x refers to a regular set (and A is not free), the following is an axiom:

$$(\exists A.(x \epsilon A \equiv \phi) \wedge (x \in A \equiv \phi^*)).$$

Kisielewicz makes an additional assumption, which we present as a separate axiom:

Definition 2.3. We say that a set A "has regular elements" iff $(\forall x.(x \in A \vee x \epsilon A) \rightarrow x$ is regular): any element of A in either sense is regular. We say that a set A is "partially contained in a set B" iff $(\forall x.x \in A \rightarrow x \in B) \vee (\forall x.x \epsilon A \rightarrow x \epsilon B)$.

Axiom 2.4. (Supplemental Regularity Axiom) Any set which is partially contained in a set with regular elements is regular.

The system we have now presented is the weaker of the two systems described in Kisielwicz's second paper [2], though it is not formulated in precisely the same way.

Kisielewicz folded what we call the supplemental regularity axiom into the axiom of comprehension, by weakening the condition on parameters to allow any set partially contained in a regular set to be a parameter. But this is equivalent to stipulating our second and third axioms, as well as being a quite awkward formulation. To see the equivalence, it is sufficient to observe that if a is allowed to be a parameter, $\{x \mid x \in a\}$ has as its ϵ-extension the $\in$-extension of a, from which it follows by mixed extensionality that $a = \{x \mid x \in a\}$ and this is a regular set (it has the same extension under both memberships). So allowing any class of objects to be parameters is equivalent to asserting that objects of that class are regular.

This way of introducing parameters, which seems quite arbitrary when it is presented in [2], is in fact a natural outgrowth of the basic comprehension scheme. It is a restriction of a much more general way in which parameters can be introduced which is described in Kisielewicz's first paper [1].

Consider that any formula ϕ in which ϵ does not occur and parameters do not occur is the ϵ-extension of some set. This suggests that formulas $y \,\epsilon\, A$, where A is an arbitrary parameter, might be taken as standing in for arbitrary $\in$-formulas (this would be legitimate, for example, if we supposed that all objects were set abstracts), which would motivate the following definition and stronger comprehension scheme:

Definition 2.5. A formula ϕ is said to be "uniform in x", if every occurrence of ϵ in ϕ is in the context $y \,\epsilon\, a$, where a is a free variable other than x, and moreover, every occurrence of a free variable other than x in ϕ is to the right of ϵ.

Axiom 2.6. (Strong Axiom Scheme of Comprehension) For each formula ϕ which is uniform in x and in which A is not free, the following is an axiom:

$$(\exists A.(x \,\epsilon\, A \equiv \phi) \wedge (x \in A \equiv \phi^*)).$$

From this axiom scheme, it is easy to derive the comprehension scheme for ϵ-free formulas with regular parameters (our official scheme), because any ϵ-free formula ϕ with regular parameters other than x can be transformed into a formula uniform in x. A subformula $y = z$ of ϕ, where y and z are both parameters, can be replaced with an ϵ-free tautology or contradiction (depending on its truth value). If $y = z$ is an atomic subformula of ϕ in which y is not a parameter and z is a (regular) parameter other than x, we can replace it with the equivalent $(\forall u.u \in y \equiv u \,\epsilon\, z)$, in which u is a new variable and the occurrence of z now satisfies the requirements for uniformity in x. If $y \in z$ is a subformula of ϕ in which y is not a parameter and z is a regular parameter other than x, it is equivalent to $y \,\epsilon\, z$ by the definition of regularity. If $z \in y$ is a subformula of ϕ with the same conditions on y and z, $(\exists u.u \in y \wedge u = z)$, u a new variable, is an equivalent formula in which the subformula $u = z$ can be transformed as indicated above.

In an alternative formulation of double extension set theory given in [2], apparently intermediate in strength between the systems already described, Kisielewicz proposes that what we call the weak comprehension scheme be adopted along with the axioms

$$(\exists B.(\forall x.x \in B \equiv x \notin A) \wedge (\forall x.x \,\epsilon\, B \equiv x \not\epsilon\, A))$$

and

$$(\exists C.(\forall x.x \in C \equiv (x \in A \vee x \in B)) \wedge (\forall x.x \,\epsilon\, C \equiv (x \,\epsilon\, A \vee x \,\epsilon\, B))),$$

which assert that the universe is a boolean algebra in both senses. These axioms are both instances of the strong comprehension scheme. They are sufficient to prove that any set which is partially contained in a set with regular elements is regular: we present this proof (found in [2]) to (almost) complete the motivation of Kisielewicz's weaker axiom set.

Under the boolean algebra axioms, the boolean operations on the universe under either membership relation are exactly the same operation (this is obvious from the form of the axioms). Moreover, the inclusion relation on the universe is the same relation in either sense. If $(\forall x.x \in A \rightarrow x \in B)$, then the set $x - y$ (which can be defined using the boolean operations given by the axioms above) is empty, but this means that $x - y$ is empty in both senses by extensionality (it must be equal to the usual empty set $\{x \mid x \neq x\}$, which is empty with respect to both $\in$ and ϵ), which means that $(\forall x.x \,\epsilon\, A \rightarrow x \,\epsilon\, B)$ must hold as well.

It is important to observe that one consequence of this is that the boolean algebra axioms imply that any two sets which have the same $\in$-extension also have the same ϵ-extension: this is equivalent to two sets including each other. The relation of coextensionality is self-dual.

This implies that a set partially contained in a set with regular elements is itself a set with regular elements (it must be a subset of the same set with regular elements in both senses, though we have not yet shown that it must have the same extension in each sense). Now we show that a set with regular elements must be regular. Suppose A has regular elements. Let $a \in A$. Comprehension with regular parameters gives us the set $\{a\} = \{x \mid x = a\}$, whose sole member is a under either membership. We see that $\{a\}$ is included in A in the sense of $\in$; we have seen that this implies that it is also included in A in the sense of ϵ, from which it follows that $a \,\epsilon\, A$. A dual argument shows that any ϵ-element of A is an $\in$ element as well. A has the same elements under both membership relations, which is what it means for A to be regular.

At this point we have established that the set comprehension axiom of [1] is stronger than the comprehension axioms of the intermediate system of [2], which is in turn stronger than the weaker system of [2]. The intermediate system of [2] is stronger than the weaker system of [2], but we cannot conclude that the system of [1] is strictly stronger than these systems because mixed extensionality does not seem to be a consequence of the axioms of [1].

However, we can conclude that the contradiction in the intermediate system of [2] which we derive below affects the system of [1], because the only consequence of mixed extensionality which is used in the argument is the assertion that the set of natural numbers is regular, and this is proved in [1] using an additional axiom we do not introduce here. It should also be noted that the results of section 6 below show that the system of [1] is inconsistent using only axioms we have introduced here.

3. Terms and natural numbers

From this point on, we restrict ourselves to the axioms of mixed extensionality, comprehension with regular parameters, and regularity of sets partially contained in sets with regular elements (the weaker system of [2]), and attempt the derivation of a paradox. The derivation does not seem to succeed, but it does succeed if we add as additional assumptions either the extensionality axiom of the system of [1] or the boolean algebra axioms of the intermediate system of [2].

We note that any instance of the strong comprehension scheme in which the parameters are either regular sets or set abstracts (sets of the form $\{x \mid \phi\}$ (ϕ uniform with all parameters regular)) is true, in one case simply because the parameter is regular, and in the other case because the formula $u \epsilon \{x \mid \phi\}$ can be replaced by $\phi[u/x]$, which is uniform with regular parameters by hypothesis. This will be important in our development: we originally derived the paradox for the theory with the strong comprehension scheme, then observed that all the occurrences of irregular parameters were inessential to the argument.

We discuss the use of terms $\{x \mid \phi\}$ in instances of comprehension. There are two obstructions to the use of such terms. The first is that in general the description of $\{x \mid \phi\}$ involves both membership relations: in terms of a definite description operator (and in the presence of extensionality assumptions), $\{x \mid \phi\} = (\iota A.x \epsilon A \equiv \phi) = (\iota A.x \in A \equiv \phi^*)$: a formula containing one of these descriptions is in general neither an $\in$-formula nor an ϵ-formula. An exception to this is a term like $A \cup B = \{x \mid x \epsilon A \vee x \epsilon B\} = (\iota C.x \in C \equiv x \in A \vee x \in B)$. Such terms, in which only ϵ appears in ϕ (at least in effect) because all terms to the right of a membership relation are either parameters or regular sets, are permitted by the strong comprehension scheme, and also permitted under our official comprehension scheme when the parameters are set abstracts. The second obstruction is the fact that we only assume the mixed extensionality axiom. This means that we cannot assume, even when an object with the extension appropriate to $\{x \mid \phi\}$ is provided by extensionality, that there is a unique such object. We nonetheless will need the ability to use term constructions in set definitions, and we adopt a uniform convention for this purpose.

CONVENTION 3.1. We state the intended meanings of formulas involving set builder notations $\{x \mid \phi\}$ and terms defined as abbreviating such notations. The formula $a = \{x \mid \phi\}$ is taken to mean $(\forall y.y \in a \equiv \phi^*)$. The formula $\{x \mid \phi\} \in a$ is taken to mean $(\forall y.(\forall z.z \in y \equiv \phi^*) \rightarrow y \in a)$. Note that this asserts that *all* sets with the extension appropriate to $\{x \mid \phi\}$ belong to a. The formula $\{x \mid \phi\} = \{x \mid \psi\}$ means $(\phi^* \equiv \psi^*) \wedge (\exists A.(\forall x.x \in A \equiv \psi^*))$. The formula $a \in \{x \mid \phi\}$ means $\phi[x/a]^* \wedge (\exists A.(\forall x.x \in A \equiv \phi^*))$. To translate formulas involving ϵ, dualize the forms given here (the "equality" formulas will not change their surface form when dualized, but the correct membership relation to use will always be deducible from context).

These conventions will normally be applied in cases where $\{x \mid \phi\}$ is a set permitted by the strong axiom scheme of comprehension in which ϕ^* turns out to be an $\in$-formula or readily translatable to an $\in$-formula.

Kisielewicz claims in [2] that double extension set theory supports all mathematical constructions which can be carried out in *ZF*. We need one of the results of his development of this claim. The set $\mathcal{N}$ of all natural numbers is defined as $\{x \mid (\forall I.(\emptyset \in I \wedge (\forall y \in I.y^+ \in I))) \rightarrow x \in I\}$, where y^+ denotes $y \cup \{y\} = \{z \mid z \,\epsilon\, y \vee z = y\}$, as usual, and the assertion $y^+ \in I$ is defined using our convention 3.1 above (note that the translated formula is an $\in$-formula). Kisielewicz proves that $\mathcal{N}$ is regular, that all of its elements are regular, and that all of its elements actually have successors (so it is an infinite set).

For the sake of self-containedness, we reproduce this proof here. We follow [2] very closely.

LEMMA 3.2. *Let $S(x)$ represent the formula $(\exists y.(\forall z.z \in y \equiv z = x))$. Then x is regular iff both $S(x)$ and $S^*(x)$ hold.*

PROOF. If a is regular, then the set $\{a\} = \{x \mid x = a\}$ exists and witnesses both $S(x)$ and $S^*(x)$. If $S(x)$ and $S^*(x)$ both hold, then the witnesses to the two statements must be equal, by mixed extensionality. Let A be the object that witnesses both statements, whose only element in either sense is a. Note that A is regular, so we can define the set $a' = \{x \mid (\exists y.x \in y \wedge y \in A)\}$

(in which A is a parameter). Note that $x \epsilon a' \equiv x \in a$, because a is the only element of A. Note then that $a' = a$ (by mixed extensionality) and $a = a'$ is regular (because its ϵ- and $\in$-extensions are seen to be the same). ■

DEFINITION 3.3. Let $\mathtt{Ind}(x)$ abbreviate $\emptyset \in x \wedge (\forall y.(S(y) \wedge y \in x) \to y^+ \in x)$. Recall that formula $y^+ \in x$ is to be interpreted using our convention 3.1 above. Define the set N as $\{x \mid S(x) \wedge (\forall y.\mathtt{Ind}(y) \to x \in y)\}$. The definition of N looks just like the definition of $\mathcal{N}$, except for occurrences of $S(x)$.

LEMMA 3.4. $\mathtt{Ind}(N)$.

PROOF. Clearly $\emptyset \in N$. Now suppose that $S(x)$ holds and $x \in N$. $x \in N \to S^*(x)$ by comprehension and the definition of N, whence x is regular by Lemma 1. From regularity of x the existence of $x^+ = \{y \mid y = x \vee y \epsilon x\} = \{y \mid y = x \vee y \in x\}$ follows. Moreover, it is clear from the instance of comprehension defining x^+ and the regularity of x that x^+ is regular and so is the only set with its extension (in either sense). Since x^+ is regular $S^*(x^+)$ holds. Since $x \in N$, it follows by comprehension that $x \epsilon y$ for every y such that $\mathtt{Ind}^*(y)$, which implies further that $x^+ \epsilon y$ for every such y (by definition of $\mathtt{Ind}^*$) . Now $S^*(x^+) \wedge (\forall y.\mathtt{Ind}^*(y) \to x^+ \epsilon y)$ is precisely what is needed for $x^+ \in N$ to hold. Thus we have shown that $\mathtt{Ind}(N)$ holds. ■

LEMMA 3.5. *N is regular, and all elements of N are regular.*

PROOF. Suppose $x \epsilon N$. Then by comprehension and the definition of N, $x \in y$ for every y such that $\mathtt{Ind(y)}$, so by Lemma 3.4 $x \in N$. Dually, $x \in N \to x \epsilon N$, so N is regular. Since N is regular, all elements x of N satisfy both $S(x)$ and $S^*(x)$ and so are regular. ■

THEOREM 3.6. *$N = \mathcal{N}$, so $\mathcal{N}$ is regular and has all elements regular.*

PROOF. $\mathcal{N}$ is defined as the intersection of all sets which contain $\emptyset$ and are inductive in the usual sense. Since N contains $\emptyset$ and is inductive in the usual sense (every element of N satisfies $S(x)$ and has a unique successor, so the additional conditions in the definition of $\mathtt{Ind}$ can be ignored), it follows that every element of $\mathcal{N}$ belongs (in both senses, since the argument can be dualized) to N. Now it is clear that $\mathcal{N}$, a subset of N containing $\emptyset$ and closed under successor, must contain all of N (because we know that $S(x)$ will hold of all elements of N, so any subset A of N which contains $\emptyset$ and is inductive in the usual sense satisfies $\mathtt{Ind}(A)$ and so will contain all of N).

■

The ability to prove the Axiom of Infinity is one of the most appealing aspects of Kisielewicz's theory.

We very briefly outline the approach taken by Kisielewicz to proving the result claimed in [2] that a certain class of sets (the "hereditarily regular" sets) support all mathematical constructions of *ZF*. If x is regular, we define $TC(x)$ as $\{y \mid (\forall A.(x \subseteq A \wedge (\forall zw.z \in w \wedge w \in A \rightarrow z \in A)) \rightarrow y \in A)\}$. We say that a regular set x is *hereditarily regular* if all elements of $TC(x)$ are regular (if this is true in either sense it will of course be true in both senses).

It is straightforward to show that pairs of hereditarily regular sets are regular, that power sets of hereditarily regular sets are hereditarily regular, and that unions of hereditarily regular sets are hereditarily regular. Analogues of the axioms of replacement and separation are hardly more difficult to verify, but see our comments on this in the next paragraph. The set $\mathcal{N}$ whose existence has just been proved can easily be shown to be hereditarily regular, so the analogue of the axiom of infinity holds. The collection of hereditarily regular sets cannot be used directly to develop a paradox, because, being defined in terms of both membership relations, the predicate "hereditarily regular" cannot be mentioned in an instance of comprehension.

The fact that hereditary regularity cannot be mentioned in instances of comprehension casts some doubt on Kisielewicz's claim that the hereditarily regular sets support all the mathematical constructions possible in *ZF*. He proves, for example, that for any uniform formula ϕ and hereditarily regular set A, the set $\{x \in A \mid \phi\}$ is hereditarily regular. But this does not give the full capability of construction of sets by separation found in *ZF*. Instances of separation of *ZF* might involve unbounded quantifiers over the universe of *ZF*, which would translate to unbounded quantifiers over the class of hereditarily regular sets, which cannot in any obvious way be expressed in a uniform formula of double extension set theory. Bounded instances of separation (in which each quantifier is restricted to a hereditarily regular set) will be successfully implemented. The treatment of replacement has the same weakness. The discussion in [2] does show that the class of hereditarily regular sets satisfies bounded *ZF* (without foundation, which can be restored by considering well-founded hereditarily regular sets), which is strong enough for almost all mathematical purposes, but it is not immediately clear how to prove that it satisfies full *ZF*. We derive this observation from a personal communication of Olivier Esser.

4. The paradox foreshadowed; notions of pairing

Now we consider the motivation for the paradox. This is an idea we had immediately when we saw the definition of this theory, but the execution is quite difficult.

Russell's paradox can be considered as a special case of the application of a "fixed point combinator" in untyped λ-calculus. For any function f, we can define C_f as $(\lambda x.f(x(x)))$ and discover that $C_f(C_f) = f(C_f(C_f))$. Let f be negation, where the application operation $x(y)$ is interpreted as $y \in x$ on sets (viewing a set as a function from sets to truth values), and we have precisely Russell's paradox. What we noticed is that it is crucial that the period of the negation operation is 2 $(f(f(x)) = x)$ because there are just two membership operations in use. This might seem a silly observation, since the domain of operation of this particular f (truth values) has just two elements, but there are of course ways to code operations on larger sets into set theory.

An obstruction presents itself immediately. In order to code functions into set theory in a general way, it appears necessary (and certainly it is usual) to use ordered pairs. The standard ordered pair is not defined on the universe of double extension set theory. One can prove, in fact, that the existence of $\{\{x\}, \{x, y\}\}$ in the sense of both memberships implies that x is regular (this follows from Lemma 3.2 above: if the Kuratowski pair exists in both senses, then the singleton of x exists in both senses, and x at least is regular). This put our doubts to rest for a bit.

Then we considered a different definition of the ordered pair, due to Quine in [3], where it was developed for use with the set theory *NF* (New Foundations).

To understand the development which follows, it is necessary to give the construction of the Quine pair (formulated in the way appropriate to double extension set theory).

Definition 4.1. We define x^1 as the set $\{y \mid (y \epsilon x \wedge y \not\epsilon \mathcal{N}) \vee (\exists z. z \epsilon x \wedge z \in \mathcal{N} \wedge y = z^+)\}$, and x^2 as $x^1 \cup \{0\}$. The Quine pair (A, B) is defined as $\{a^1 \mid a \epsilon A\} \cup \{b^2 \mid b \epsilon B\}$.

The sets x^1 and x^2 are obtained by first replacing each element of x which is a natural number with its successor, then (in the case of x^2) adding 0 as an additional element). It should be clear that $x^1 = y^1$ or $x^2 = y^2$ imply that x and y have the same extension, and also that $x^1 = y^2$ is impossible. The Quine pair is a kind of disjoint union construction.

It is important to notice that in the context of double extension set theory we expect $y \in z^1 \equiv (y \in z \wedge y \notin \mathcal{N}) \vee (\exists u. u \in \mathcal{N} \wedge u \in z \wedge u^+ = y)$ (and dually for ϵ): this is really a notion defined in terms of a single membership relation (similarly for the definition of z^2). In this respect, these operators are like the boolean algebra operations on sets in the two stronger systems. The existence and uniqueness of such sets for general z follows easily from the strong comprehension scheme and extensionality axiom of [1] The existence of objects with the desired extensions also follows from regularity of $\mathcal{N}$ and its elements combined with the boolean algebra axioms of the intermediate system of [2] (we do not give details, but they are not difficult). There is no obvious argument for the existence of such objects for general x in the official weak version of the theory, though objects with the extensions expected of x^1 and x^2 do exist for each set abstract x, which turns out to be all that we need. Similarly, $x \in (A, B) \equiv (\exists z. z \in A \wedge z^1 = x) \vee (\exists z. z \in B \wedge z^2 = x)$: the strong comprehension scheme allows us to prove that there is a uniquely determined object satisfying this for any A and B, but neither of the weaker theories allow us to do this.

5. The paradox implemented

In this section, we use the weak system of [2] with the additional assumption that sets which have the same $\in$-extension also have the same ϵ-extension (coextensionality is self-dual).

Note that the additional assumption holds in [1], because it follows from the extensionality axiom of [1]. It also holds in the intermediate system of [2], because it is a consequence of the boolean algebra axioms: inclusion is self-dual in the presence of the boolean algebra axioms, and two sets are coextensional if they are included in each other.

In spite of all apparent obstacles, we proceed to construct a fixed point for an operator F with period 3, which is fatal to this system. The operator acts on pairs of propositions. Without any real philosophical commitment to such a position, we adopt a convention of Frege and identify the truth values with the numbers 1 and 2 (though we use the notation t and f for them qua truth values), and regard each proposition as a name for its truth value. This allows us to treat pairs of propositions as objects. This is not in any way essential to our argument, but it is advantageous as a matter of notational convenience. A pair (p, q) is sent by the operation to the pair $(q, \neg p \wedge \neg q)$. This operation sends (t, t) to (t, f) to (f, f) to (f, t) back to (t, f); if one starts with any pair of truth values and applies this operation F repeatedly, one will settle down into the indicated cycle of length 3.

Suppose temporarily that we have strong extensionality and we have a well-defined pair and projection operators definable in terms of either membership relation (so that they can appear in instances of comprehension). Then the paradoxical set is readily defined.

Define X_1 as $\{y \mid y \in \pi_2(y)\}$.

Define X_2 as $\{y \mid y \notin \pi_1(y) \wedge y \notin \pi_2(y)\}$.

Let $X = (X_1, X_2)$. Note that $\pi_1(X) = X_1$ and $\pi_2(X) = X_2$.

Now $X \epsilon X_1 \equiv X \in X_2$ (by the instance of comprehension defining X_1 and the definition of X) and $X \epsilon X_2 \equiv X \notin X_1 \wedge X \notin X_2$.

Dually, $X \in X_1 \equiv X \epsilon X_2$ (by the instance of comprehension defining X_1 and the definition of X) and $X \in X_2 \equiv X \not\epsilon X_1 \wedge X \not\epsilon X_2$.

Thus we have $(X \in X_1, X \in X_2) = F(X \epsilon X_1, X \epsilon X_2)$, and dually we have $(X \epsilon X_1, X \epsilon X_2) = F(X \in X_1, X \in X_2)$

From this we draw the conclusion that $(X \in X_1, X \in X_2) = F^2(X \in X_1, X \in X_2)$, and this is easily seen to be impossible by the periodicity structure of F exhibited above.

But of course we are not finished yet, because we have not shown that the pair and its projections can be defined as above. Indeed, we cannot (though we can in the system with the strong comprehension scheme and extensionality axiom). But we do not need to. If we assume that the pair above is the Quine pair, we can transform the definition of X into a form involving no explicit mention of the pairing or projection operators.

The trick is to observe that the bad sentences $y \in \pi_i(y)$ (for $i = 1, 2$) in the definitions of X_1 and X_2 can be expressed as $y^i \in y$ if the pair is supposed to be the Quine pair. It then becomes straightforward to define X as a set directly. Further, though general computation of projections and pairs does not work (in general, computation of x^1 and x^2 isn't even guaranteed to work) we only need to compute any of these in cases where the parameters are set abstracts, in which case the weaker comprehension scheme is sufficient to ensure that they work, in order to show the contradiction. We explicitly give conventions for the use of terms u^1, u^2.

Definition 5.1. Where u and v are arbitrary objects, we interpret $u^1 = v$ as signifying $(\forall x . x \in v \equiv (x \notin \mathcal{N} \wedge x \in u) \vee (x \in \mathcal{N} \wedge x - 1 \in u))$. We interpret $u^2 = v$ as signifying $(\forall x . x \in v \equiv (x \notin \mathcal{N} \wedge x \in u) \vee (x \in \mathcal{N} \wedge x - 1 \in u) \vee x = 0)$ Note that these are $\in$-formulas: the dual ϵ-formulas are written $u^i =^* v$, where $i = 1, 2$.

Definition 5.2. Where v is an arbitrary object, we interpret $v^i \in v$ (where $i = 1, 2$) as meaning "there is a w such that $v^i = w$ and $w \in v$, and for all w such that $v^i = w$, we have $w \in v$". The dual formulas are written $v^i \epsilon v$.

Definition 5.3. Define X as $\{u \mid (\exists v.v^1 = u \wedge v^2 \in v) \vee (\exists w.w^2 = u \wedge w^1 \notin w \wedge w^2 \notin w)\}$. Please note that in the argument which follows any statement we make about X is in fact a statement about all witnesses to the truth of this comprehension axiom, and so about all objects with the extensions for both memberships appropriate to this set abstract (if extensionality is not assumed, there may be many such objects; under Convention 3.1 as stated above, we would consider only one of the memberships (determined by context)).

Observation 5.4. Notice that any two sets with the same $\in$-extension will either both ϵ-belong to X or both not ϵ-belong to X, and, dually, any two sets with the same ϵ-extension will either both $\in$-belong to X or both not $\in$-belong to X, even if we assume only the mixed case of extensionality. To see this it is necessary to look at the meanings of the atomic formulas $v^i = u$ and $v^i \in v$ under our conventions just given: it is straightforward to verify that the truth values of these formulas depend only on the extension of v under the appropriate membership relation.

Sets with the extensions expected of X^1 and X^2 can be shown to exist by comprehension: $X^1 = \{x \mid (x \,\epsilon\, X \wedge x \notin \mathcal{N}) \vee (\exists y.y \,\epsilon\, X \wedge y \in \mathcal{N} \wedge x = y^+)\}$ is provided by comprehension because each formula "$u \,\epsilon\, X$" can be replaced with a formula containing no instance of ϵ using comprehension and the definition of X, and X^2 is equally easy. More briefly, these objects exist because X^1 and X^2 are defined by instances of strong comprehension in which the parameter X is a set abstract.

Now we consider the status of the sentences $X^1 \in X$ and $X^2 \in X$. These sentences are to be understood as asserting that all witnesses to the comprehension axiom defining X^i (for $i = 1, 2$) $\in$-belong to X; note that this is apparently a stronger assertion than $v^i \in v$ for general objects v, because the comprehension axiom places conditions on both extensions of X^i, not just the $\in$-extension. Our intention is to verify, following the argument already given above, that $(X^1 \in X, X^2 \in X) = F(X^1 \,\epsilon\, X, X^2 \,\epsilon\, X) = F^2(X^1 \in X, X^2 \in X)$, which is impossible.

We give full details for the equation

$$(X^1 \in X, X^2 \in X) = F(X^1 \,\epsilon\, X, X^2 \,\epsilon\, X).$$

The second equation is proved using the dual of this result.

$X^1 \in X$ iff $(\exists v.v^1 =^* X^1 \wedge v^2 \,\epsilon\, v) \vee (\exists w.w^2 =^* X^1 \wedge w^1 \not\epsilon\, w \wedge w^2 \not\epsilon\, w)$, which is in turn true iff $X^2 \,\epsilon\, X$. This is because $w^2 =^* X^1$ is impossible, and $v^1 =^* X^1$ implies that X has the same ϵ-extension as v, (the formula abbreviated $v^1 = X^1$ by our convention is dualized to an ϵ-formula $v^1 =^* X^1$ when comprehension is applied), and the truth-value of $v^2 \,\epsilon\, v$ depends only on the ϵ-extension of v (because the formula $v^2 \,\epsilon\, v$ asserts that all objects (if any) with a certain extension defined in terms of the ϵ-extension of v, are in fact ϵ-elements of v) , so is equivalent to that of $X^2 \,\epsilon\, X$. There is a subtlety here which requires the application of our additional assumption that coextensionality is self-dual (and appears to save the weak system of [2] from this paradox). The implication from $v^2 \,\epsilon\, v$ to $X^2 \,\epsilon\, X$ is direct; the converse implication is not. If $X^2 \,\epsilon\, X$, (X^2 being any witness to the appropriate comprehension axiom) this implies that all objects with the same $\in$-extension (not ϵ-extension!) as X^2 ϵ-belong to X, because the ϵ-membership of X^2 in X depends on properties of its $\in$-extension. We apply the additional assumption that coextensionality is self-dual to conclude that all objects with the same ϵ-extension as X^2 are ϵ-members of X. We are indebted to an anonymous referee for pointing out this subtle issue and saving us from making the mistaken claim made in a draft version of this paper that this argument shows the inconsistency of the weak system of [2].

$X^2 \in X$ iff $(\exists v.v^1 =^* X^2 \wedge v^2 \,\epsilon\, v) \vee (\exists w.w^2 =^* X^2 \wedge w^1 \not\epsilon\, w \wedge w^2 \not\epsilon\, w)$, which is true if and only if $X^1 \not\epsilon\, X$ and $X^2 \not\epsilon\, X$. This is because $v^1 =^* X^2$ is impossible, and $w^2 =^* X^2$ implies that X has the same ϵ-extension as w (the $\in$-formula which "$w^2 = X^2$" abbreviates by convention is dualized to an ϵ-formula $w^2 =^* X^2$ when comprehension is applied), and the truth value of $w^1 \not\epsilon\, w \wedge w^2 \not\epsilon\, w$ depends only on the ϵ-extension of w, so is equivalent to that of $X^1 \not\epsilon\, X \wedge X^2 \not\epsilon\, X$. The proof of this last equivalence requires the additional assumption that coextensionality is self-dual for the same reasons spelled out explicitly in the previous paragraph.

We have verified $(X^1 \in X, X^2 \in X) = F(X^1 \,\epsilon\, X, X^2 \,\epsilon\, X)$.

We have shown at this point that the system of [1] and the intermediate system of [2] are inconsistent, and that the weak system of [2] is inconsistent with the extensionality axiom of [1] (so there must be irregular sets which are coextensional with respect to one membership relation but not the other) and with the boolean algebra axioms (so that even very mild attempts to introduce irregular sets as parameters lead to paradox).

6. Inconsistency of the parameter-free axiom with extensionality or boolean algebra axioms

Further, we can adapt the argument to show that the parameter-free axiom scheme of comprehension for double extension set theory is inconsistent with the self-duality of coextensionality (the proof as given above depends on the use of regular parameters in the proof that $\mathcal{N}$ is infinite). To show this, we abandon the quest for a perfect ordered pair altogether, and define x^1 as $x - \{0\}$ and x^2 as $x \cup \{0\}$, for any x. Then define X exactly as above.

The argument will go in the same way if we can show that $v^1 = X^1$ still implies that $v^2 \in v \equiv X^2 \in X$, and $w^2 = X^2$ still implies that $w^1 \notin w \wedge w^2 \notin w \equiv X^1 \notin X \wedge X^2 \notin X$ (with dual results for ϵ). This is no longer immediately obvious because $v^1 = X^1$ no longer implies that v and X have the same extension – but it still implies that 0 is the only possible element of their symmetric difference. Now suppose $v^1 = X^1$. We have $v^1 = X^1$ iff $v^2 = X^2$. This means that $v^2 \in v$ is equivalent to $X^2 \in v$ (it is necessary to recall here that $v^2 = X^2$, according to the convention for use of terms given above, merely asserts that v^2 and X^2 are coextensional, not that they are actually equal – $X^2 \in v$ follows not by substitution of equals for equals but by the observation that membership in X (and so in v, whether or not it differs from X with respect to 0) depends only on extension ($\in$-membership in X depends only on ϵ-extension, but sets with the same ϵ-extension also have the same $\in$-extension by self-duality of coextensionality; this holds for ϵ as well by duality), which is equivalent to $X^2 \in X$ because obviously $X^2 \neq 0$. The argument that $v^2 = X^2$ implies $w^1 \notin w \wedge w^2 \notin w \equiv X^1 \notin X \wedge X^2 \notin X$ is precisely similar (we show that $w^1 \in w \equiv X^1 \in X$ and $w^2 \in w \equiv X^2 \in X$ under the assumption $w^2 = X^2$ just as we showed $v^2 \in v \equiv X^2 \in X$ under the assumption $v^1 = X^1$ above).

So even the parameter-free version of Kisielewicz's comprehension scheme is inconsistent with the assumption that coextensionality is self-dual, and so with extensionality and with the boolean algebra axioms. Of course this last argument by itself proves our main thesis, but we believe it is better to present it in the context of its motivation in terms of the Quine pair, so that the reader can determine how we came up with it.

7. Variations?

Attempts to save the strong Kisielewicz comprehension criterion by introducing a larger finite number of membership relations with an analogous cycle of comprehension schemes will be defeated by essentially the same argument, involving construction of a (nonexistent) fixed point for a propositional logic operation on a sufficiently long finite vector of truth values.

The comprehension scheme for ω such membership relations is consistent and in fact quite weak. We introduce membership relations $\in_i$ for each $i \in \mathcal{N}$. We define ϕ_i for any formula ϕ as the formula obtained by adding i

to the index of each membership relation appearing in ϕ. We call a formula ϕ "uniform in x" if it contains only equality, $\in_0$ and $\in_1$, with $\in_0$ appearing only in contexts $y \in_0 z$, with z a free variable other than x, and with free variables other than x appearing in no other context.

Axiom 7.1. (Axiom Scheme of Comprehension) Let ϕ be a formula which is uniform in x. Then there is an object A_ϕ such that $(\forall x.x \in_n A_\phi \equiv \phi_n)$ is an axiom for each n. (We need to use the explicit name A_ϕ, more conveniently written $\{x \mid \phi\}$, because we cannot express an infinitary conjunction over all the membership relations!)

Note that ϕ_n will involve only equality and the membership relations $\in_n$ and $\in_{n+1}$, with $\in_n$ appearing only to the left of a parameter.

Note also that a precise formulation of the versions of Kisielewicz set theory with any finite number n of membership relations alluded to above can be obtained by identifying $\in_i$ and $\in_j$ in this scheme whenever $i \equiv j \bmod n$. This works for the case $n = 2$, recovering the strong comprehension scheme for double extension set theory.

This scheme, with all membership relations distinct, is readily seen to be consistent by a compactness argument. We describe a model of the subscheme consisting of the instances $(\forall x.x \in_n A_\phi \equiv \phi_n)$ of the comprehension scheme with $n < N$. The elements of our model will be the set abstracts $A_\phi = \{x \mid \phi\}$, ϕ uniform in x, considered as bits of syntax. We evaluate every sentence $A \in_N B$ (A and B set abstracts) as true. It is then clear how to determine the truth value of every sentence $A \in_{N-1} B$ by consulting the comprehension scheme (this will reduce by applying comprehension to a sentence involving $\in_N$, which we have already interpreted, and sentences $C \in_{N-1} D$, where D is a parameter appearing in B, and these can be evaluated recursively because there cannot be an infinite regress of parameters), and thence it is clear how to determine the truth value of every sentence $A \in_{N-2} B$, and so forth. So the subscheme is consistent for each N, and the full axiom scheme is consistent by compactness.

We do not believe that this "ω-extension set theory" will support a development of set theory along the lines proposed by Kisielewicz. It is clear from the proof that the consistency strength of the comprehension scheme is very weak. The concept of regularity cannot be defined in this language, though one could introduce a primitive notion $\texttt{regular}(x)$ satisfying axioms $\texttt{regular}(x) \rightarrow (\forall y.y \in_m x \equiv y \in_n x)$, and there are sets which would seem to be regular in this sense (though how to frame axioms which would allow one to prove this is unclear). The form of extensionality which we would adopt for the best analogy with Kisielewicz's system is

$(\forall x.x \in_m A \equiv x \in_n B) \rightarrow A = B$. Note that we have included the case $m = n$ of strong extensionality. The proof of infinity given by Kisielewicz does not work here: we assume infinity as an axiom. With strong extensionality and infinity, we have the ability to define the Quine pair using a single membership relation (without strong extensionality there are obstructions involving other objects with the same extension as x^1's and x^2's). We can then define $(\lambda x.T)$ for any term T such that $y = T$ is equivalent to a formula $\phi(x, y)$ in x, y and one membership relation as $\{(x, y) \mid \phi(x, y)\}$. $f(x)$ is a term with this property, once we indicate default values for the cases when f is not a function or x is not in its domain. If the function f is not regular, the value of $f(x)$ may differ depending on the membership relation being used: we use the notation $f(x)_n$ to signal the use of $\in_n$ in the definition of the term. Thus we can define $F = (\lambda x.f(x(x))$ for suitable functions f. Now consider the "fixed point" term $F(F)$. This is the unique y such that $(F, y) \in F$, that is such that $y = f(F(F))$. Note that we applied comprehension at the last step, so the index of the membership relation in use was incremented, and the $F(F)$ appearing in the last formula is not necessarily the $F(F)$ we started with: the calculation actually shows that $F(F)_n = f(F(F))_{n+1}$. However, if we take f to be the function which sends each natural number to its successor and every other object to 0, we obtain ω-inconsistency: $F(F)$ is a natural number (a different one for each choice of index for the membership relation) but it is not equal to any standard natural number, because $F(F)_n = 1 + F(F)_{n+1}$. We conjecture that the theory with infinity and extensionality is inconsistent, but we have not been able to prove this.

Of course, the weak system of [2] still may be consistent, and if it is, it is of some interest, since it combines sufficient strength for almost all mathematical purposes with admirable simplicity.

References

[1] Kisielewicz, Andrzej, 'Double extension set theory', *Reports on Mathematical Logic* 23:81–89, 1989.

[2] Kisielewicz, Andrzej, 'A very strong set theory?', *Studia Logica* 61:171–178, 1998.

[3] Quine, W. V. O, 'On ordered pairs', *Journal of Symbolic Logic* 10:95–96, 1945.

M. Randall Holmes
Department of Mathematics
Boise State University
1910 University Drive
Boise, Idaho, USA
holmes@math.boisestate.edu